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# New Lower Bounds For Orthogonal Drawings

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**Abstract**

An orthogonal drawing of a graph is an embedding of the graph in the two-dimensional grid such that edges are routed along grid-lines. In this paper we explore lower bounds for orthogonal graph drawings. We prove lower bounds on the number of bends and, when crossings are not allowed, also lower bounds on the size of the grid.

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Some results of this paper were part of the author's diploma thesis at TU Berlin under the supervision of Prof. R. Möhring, and have been presented in an extended abstract at Graph Drawing '95, Passau, Germany.

# 1 Introduction

A graph $G = (V, E)$ is an abstract structure consisting of points (or *vertices*) $V$ and connections (or *edges*) $E$. Such a structure is found in many industrial applications, such as networks, production schedules and diagrams. With the aid of *graph drawing*, a graph is displayed in visual form, and the underlying information can be understood easily.

Many years of research have been spent on the development of graph drawing styles and graph drawing algorithms, see for example [8]. In this paper, we study *orthogonal drawings*, i.e., embeddings in the rectangular grid (see Section 2 for a precise definition). Many criteria are used to judge the quality of an orthogonal drawing, two of the most important ones are the area and the number of bends.

Orthogonal drawings with vertices drawn as points exist only if every vertex in the graph has at most four incident edges. Such a graph is called a *4-graph*, or more generally, a graph is called a $\Delta$-*graph* if the maximum degree of the graph is at most $\Delta$. In this paper we study only 4-graphs; lower bounds for graphs with larger degrees, specifically, lower bounds for the complete graph, have been studied in [6].

The question whether a graph can be embedded in a grid of prescribed size is NP-complete [9, 13]. Heuristics have been developed that are within a factor of $\mathcal{O}(\log n)$ of the minimal area (see [16] for an overview). With respect to planar drawings of planar graphs, minimizing the number of bends is NP-complete [10], but if the combinatorial embedding and the outer-face is fixed (the graph has a *fixed planar drawing*), then the orthogonal drawing with the minimum number of bends can be found in $\mathcal{O}(n^{7/4}\sqrt{\log n})$ time ([20] and [11]) and in linear time for 3-connected 3-graphs [18].

Another approach to orthogonal graph drawing is to develop simple heuristics and to prove worst-case bounds on the area and the number of bends. See farther below for an overview. The quality of such heuristics is measured by comparing them to lower bounds, i.e., to graphs that need at least a certain grid-size or at least a certain amount of bends in any orthogonal drawing. Some previous lower bounds have appeared in [12, 14, 19, 21, 22]. In this paper, we study, and in many cases improve, lower bounds for orthogonal drawings of 4-graphs and 3-graphs.

Many heuristics have been tailored to some particular graph class, for example 3-connected planar 3-graphs [12]. To measure the quality of such an algorithm, one should use a lower bound graph that also falls into this class. Thus we study many graph classes, distinguishing them by the following parameters (see Section 2 for formal definitions of technical terms):

- *Degree of planarity*: A graph can be planar or not. For a planar graph, there are three possibilities: An algorithm can draw the graph with crossings but using the fact that the graph is planar (*non-planar drawing*, see e.g. [15]), it can draw the graph without crossings (*planar drawing*), or it can draw the graph without crossings and exactly reflect the fixed planar drawing of the planar graph (*plane drawing*).

- *Connectivity*: Many heuristics are designed originally for graphs that are 2-connected or even 3-connected, and then extended to 1-connected graphs (e.g. [5, 12, 17]). In our study we include 4-connected graphs for the sake of completeness.
- *Degree of simplicity*: Most heuristics consider only simple graphs. However, some lower bounds are easier to obtain by proving a lower bound for a graph with multiedges or loops, and then converting this graph into a simple graph by subdividing edges. We will thus include multigraphs and graphs with loops in our discussion.
- *Maximum degree*: Some heuristics only work on graphs with maximum degree 3 (e.g. [1, 7, 12, 17]). We will thus study both 4-graphs and 3-graphs.

In Table 1 we list the (to our knowledge) best upper bounds on the grid-size and the number of bends for simple graphs. We contrast these upper bounds with the lower bounds, which, if given without citation, will be proved in this paper.

The paper is outlined as follows: After giving definitions in Section 2, we first prove lower bounds for non-planar drawings in Section 3 and for non-planar drawings of planar graphs in Section 4. We continue with lower bounds for plane drawings in Section 5. Using the same graphs, but considering many planar drawings, we then obtain lower bounds for planar drawings in Section 6. Some of the more tedious proofs are deferred to the appendix.

## 2 Definitions

Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. $G$ is called a $\Delta$*-graph* if its maximum degree is at most $\Delta$. By *subdividing* an edge $e$ we understand that we delete $e$, add a new vertex (the *subdivision vertex*), and connect it with the two endpoints of $e$.

$G$ is called *1-connected* if for any two vertices there exists a path between them. It is called *c-connected*, $c \geq 2$, if for any $c-1$ vertices $v_1, \ldots, v_{c-1}$ the graph remains 1-connected if these vertices are deleted. Menger's theorem states that a graph is $c$-connected if and only if for every pair of vertices there exist $c$ vertex-disjoint paths connecting them. The *connectivity* of a graph $G$ is the maximum number $c$ such that $G$ is $c$-connected. A $\Delta$-graph has connectivity at most $\Delta$.

Edges of the form $(v, v)$ are not necessarily forbidden; such edges are called *loops*. It is also not necessarily forbidden that two vertices are connected by more than one edge; such edges are called *multiple edges*. Edges with multiplicity two, three and four are called *double edge*, *triple edge*, and *quadruple edge*, respectively. A graph without loops and multiple edges is called *simple*. A graph without loops is called a *multigraph*. We use the expression *degree of simplicity* as categorizing term for "simple graph", "multigraph", and "graph with loops".

| | | | Non-planar graphs | Planar graphs: Non-planar drawing | Planar graphs: Planar drawing | Planar graphs: Plane drawing |
|---|---|---|---|---|---|---|
| 4-graphs | 3-connected | Upper bound | $0.76n^2$ area [17]<br>$2n{+}2$ bends [5] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $(\frac{2}{3}n{+}1)\times(\frac{2}{3}n{+}1)$-grid [3]<br>$\frac{4}{3}n{+}4$ bends [3, 12] | $(\frac{2}{3}n{+}1)\times(\frac{2}{3}n{+}1)$-grid [3]<br>$\frac{4}{3}n{+}4$ bends [3, 12] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{10}{7}n$ bends | $\Omega(n\log n)$ area [14]<br>$\frac{6}{5}n$ bends | $(\frac{n}{3}{+}1)\times(\frac{n}{3}{+}1)$-grid<br>$\frac{4}{3}n{-}2$ bends | $(\frac{2}{3}n{+}1)\times(\frac{2}{3}n{+}1)$-grid<br>$\frac{4}{3}n{+}4$ bends |
| | 2-connected | Upper bound | $0.76n^2$ area [17]<br>$2n{+}2$ bends [5] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $n\times n$-grid [5]<br>$2n{+}2$ bends [5] | $n\times n$-grid [5]<br>$2n{+}2$ bends [5] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{5}{3}n$ bends | $\Omega(n\log n)$ area [14]<br>$\frac{10}{7}n$ bends | $\frac{n}{2}\times\frac{n}{2}$-grid<br>$2n{-}6$ bends | $(n{-}1)\times(n{-}1)$-grid<br>$2n{-}2$ bends [21] |
| | 1-connected | Upper bound | $0.76n^2$ area [17]<br>$2n{+}2$ bends [5] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $n\times n$-grid [5]<br>$2n{+}2$ bends [5] | $\frac{6}{5}n\times\frac{6}{5}n$-grid [2]<br>$\frac{12}{5}n{+}2$ bends [2] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{11}{6}n$ bends | $\Omega(n\log n)$ area [14]<br>$\frac{11}{7}n$ bends | $\frac{n}{2}\times\frac{n}{2}$-grid<br>$2n{-}6$ bends | $(\frac{6}{5}n{-}\frac{11}{5})\times(\frac{6}{5}n{-}\frac{11}{5})$-grid<br>$\frac{12}{5}n{-}\frac{22}{5}$ bends |
| 3-graphs | 3-connected | Upper bound | $\lceil\frac{n+1}{2}\rceil\times\lceil\frac{n+1}{2}\rceil$-grid [1]<br>$\frac{n}{2}+2$ bends [1, 7, 17] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $0.17n^2$ area [3]<br>$\frac{n}{3}+3$ bends [3] | $0.17n^2$ area [3]<br>$\frac{n}{3}+3$ bends [3] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{n}{3}$ bends | $\Omega(n\log n)$ area [14]<br>$\frac{n}{3}$ bends | $(\frac{n}{6}{+}1)\times(\frac{n}{6}{+}1)$-grid<br>$\frac{n}{3}$ bends | $(\frac{n}{3}{+}1)\times(\frac{n}{3}{+}1)$-grid<br>$\frac{n}{3}$ bends |
| | 2-connected | Upper bound | $\lceil\frac{n+1}{2}\rceil\times\lceil\frac{n+1}{2}\rceil$-grid [1]<br>$\frac{n}{2}+2$ bends [1, 7, 17] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $\frac{n}{2}\times\frac{n}{2}$-grid [12]<br>$\frac{n}{2}+1$ bends [12] | $\frac{n}{2}\times\frac{n}{2}$-grid [12]<br>$\frac{n}{2}+1$ bends [12] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{n}{2}$ bends | $\Omega(n\log n)$ area [14]<br>$\frac{n}{2}$ bends | $(\frac{n}{4}{+}\frac{1}{2})\times(\frac{n}{4}{+}\frac{1}{2})$-grid<br>$\frac{n}{2}$ bends | $\frac{n}{2}\times\frac{n}{2}$-grid<br>$\frac{n}{2}+1$ bends |
| | 1-connected | Upper bound | $\lceil\frac{n+1}{2}\rceil\times\lceil\frac{n+1}{2}\rceil$-grid [1]<br>$\frac{n}{2}+2$ bends [1, 7] | $\mathcal{O}(n\log^2 n)$ area [15]<br>not analyzed | $\frac{n}{2}\times\frac{n}{2}$-grid [12]<br>$\frac{n}{2}+1$ bends [12] | $\frac{6}{5}n\times\frac{6}{5}n$-grid [2]<br>$\frac{12}{5}n{+}2$ bends [2] |
| | | Lower bound | $\Omega(n^2)$ area [22]<br>$\frac{n}{2}+1$ bends [19] | $\Omega(n\log n)$ area [14]<br>$\frac{n}{2}+1$ bends [19] | $(\frac{n}{4}{+}\frac{1}{2})\times(\frac{n}{4}{+}\frac{1}{2})$-grid<br>$\frac{n}{2}+1$ bends [19] | $(\frac{2}{3}n{+}1)\times(\frac{2}{3}n{+}1)$-grid<br>$\frac{5}{6}n-1$ bends |

Table 1: Overview of upper and lower bounds for simple graphs. All results without citation are proved in this paper.

$G$ is called *planar* if it has a planar drawing, i.e., a drawing in 2D without crossing. A planar drawing of a planar graph defines for every vertex $v$ a circular clockwise ordering of the incident edges of $v$; the collection of these orderings is called a *combinatorial embedding*. A planar drawing splits the plane into components called *faces*; the unbounded component is called the *outer-face*. A combinatorial embedding of a planar graph defines a planar drawing of it, which is topologically unique except for the choice of the outer-face. We say that a planar graph has a *fixed planar drawing* if both a combinatorial embedding and an outer-face have been specified.

An *orthogonal drawing* of $G$ is an embedding of $G$ in the two-dimensional rectangular grid. More precisely, every vertex is mapped to a grid-point, i.e., a point with integer coordinates. Every edge is mapped to a path of grid-segments connecting the two endpoints of the edge. A place where the route of an edge changes direction is called a *bend*. No two vertices may be mapped to the same point. No two edges may use the same grid-segment. No edge may pass through a grid-point of a non-incident vertex. No two bends may coincide.

An orthogonal drawing is called *planar* if no routes of edges intersect. It is called *plane* if it is planar and reflects the fixed planar drawing of the input graph. To facilitate notation, we use the term *non-planar drawing* for a drawing that may or may not have crossings.

If an orthogonal drawing can be enclosed by a box of *width* $n_1$ and *height* $n_2$ we call it a drawing with *grid-size* $n_1 \times n_2$ and *area* $n_1 \cdot n_2$. The width is one less than the number of columns and the height is one less than the number of rows.

Earlier, we proved the following theorem.

**Theorem 1** *[4] Let $\Gamma$ be an orthogonal drawing of a 4-graph $G = (V, E)$ which has $b$ bends, and uses $r$ rows and $c$ columns. Let $|V| = n$ and $|E| = m$. Then (a) $\max\{r, c\} \le \frac{1}{2}b + 2n - m$, and (b) $r + c \le b + 2n - m$.*

# 3 Non-planar drawings

In this section we study lower bounds for non-planar orthogonal drawings. Specifically, we present new lower bounds on the number of bends. The area is lower-bounded by $\Omega(n^2)$ [22], and we have not succeeded in improving the constant of this lower bound.

To prove lower bounds on the number of bends, we describe for each case (depending on connectivity, degree of simplicity, and maximum degree) a class of graphs. These graphs are built by combining many copies of a small graph through subdividing edges, identifying vertices and adding edges. We first introduce these small graphs and study their lower bounds. Then we investigate how subdividing edges affects lower bounds. Finally, we define the graph classes and prove lower bounds.

### 3.1 Lower bounds for small graphs

We use the following small graphs: the *loop* $L$, the *triple edge* $T$, the *quadruple edge* $Q$, the *complete graph on 4 vertices* $K_4$, the *4-wheel* $W$, the *complete graph on 5 vertices* $K_5$, and the *octahedron* $O$. See also Figure 1.

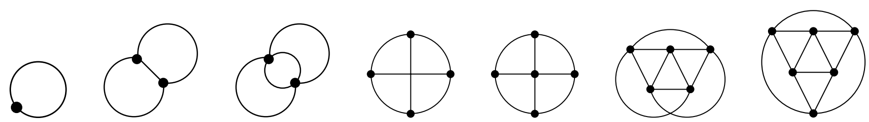


Figure 1: From left to right: $L$, $T$, $Q$, $K_4$, $W$, $K_5$, and $O$.

We now prove lower bounds on the number of bends of the small graphs. Most of these lower bounds were known before, but the argument for their proof was "by exhaustively checking cases" [19]. Such an argument is dangerous, for example in the above paper Storer claimed that the graph in his Figure 9 requires 10 bends, but in fact it can be drawn with 8 bends as shown in Figure 2. Thus, we provide a formal (and tedious) proof for each of these graphs, which can be safely skipped on first reading.

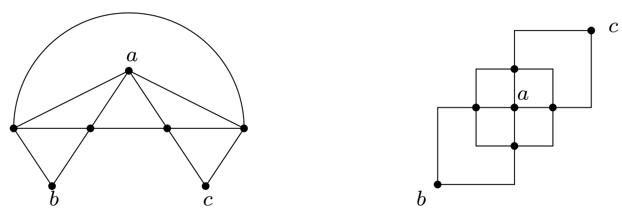



Figure 2: The graph by Storer [19], and an orthogonal drawing of it with 8 bends.

**Lemma 1** *The following number of bends is required in any orthogonal drawing:*
*$L$: 3 bends $T$: 4 bends $Q$: 8 bends $K_4$, $W$: 4 bends $K_5$, $O$: 12 bends*

**Proof:** In the following proofs, let $n$ and $m$ be the number of vertices and edges of the graph in question, and let $\Gamma$ be an arbitrary, but fixed, orthogonal drawing of the graph. Denote by $r$, $c$, and $b$ the number of rows, columns, and bends of $\Gamma$, respectively.

For each graph, we first show a lower bound on $r$, or on $r + c$, usually with the following "cut-argument": Let $C$ be a column, let $V_{\leq}(C)$ be the vertices placed in $C$ or in a column to the left of $C$, and let $V_{>}(C)$ be the vertices placed to the right of $C$. Any edge between $V_{\leq}(C)$ and $V_{>}(C)$ must cross the gap to the right of column $C$. Consequently, if the cut $(V_{\leq}(C), V_{>}(C))$ contains $k$ edges, then there are at least $k$ rows. Then we obtain a lower bound on $b$ by applying Theorem 1.

The individual claims are proved as follows:

$L$: The loop encloses at least one box of the grid, hence uses at least two rows and two columns. From Theorem 1(b), it follows that $b \geq r+c+m-2n \geq 2+2+1-2=3$.

$T$: After possible rotation of $\Gamma$, the two vertices are placed in different columns. Let $C$ be the left column containing a vertex. Then the cut $(V_{\leq}(C), V_{>}(C))$ contains 3 edges. So $r \geq 3$, and from Theorem 1(a), it follows that $b \geq 2r+2m-4n \geq 6+6-8=4$.

$Q$: After possible rotation of $\Gamma$, the two vertices are placed in different columns. Let $C$ be the left column containing a vertex. Then the cut $(V_{\leq}(C), V_{>}(C))$ contains 4 edges. So $r \geq 4$, and from Theorem 1(a), it follows that $b \geq 2r+2m-4n \geq 8+8-8=8$.

$K_4$: Let $C$ be the leftmost column containing vertices. If $|V_{\leq}(C)| \geq 3$, then there are at least three vertices placed in the column $C$, so $r \geq 3$. If $|V_{\leq}(C)| \leq 2$, then there are at least three edges in the cut $(V_{\leq}(C), V_{>}(C))$, and again $r \geq 3$. Applying the same argument to the topmost row containing vertices, we obtain $c \geq 3$. By $m = 2n-2$, it follows from Theorem 1(b) that $b \geq r+c+m-2n = 3+3-2=4$.

$W$: The proof is word by word the same as for $K_4$.

$O$: Assume first that each column contains at most three vertices. Then, by scanning columns from left to right, we can find a column $C$ such that $2 \leq |V_{\leq}(C)| \leq 4$. The cut $(V_{\leq}(C), V_{>}(C))$ contains at least six edges, so $r \geq 6$.

If there exists a column containing at least four vertices, then each row contains at most three vertices by $n=6$. Thus we obtain $c \geq 6$ with the same argument as before.

Either way, by Theorem 1(a), $b \geq 2\max\{r,c\}+2m-4n \geq 12+24-24=12$.

$K_5$: Since $K_5$ is not planar, $\Gamma$ has at least one crossing. Replace this crossing with a new vertex. We obtain an orthogonal drawing of some simple graph with six vertices where every vertex has degree 4. This drawing has the same number of bends as $\Gamma$. However, there exists only one simple graph with six vertices where every vertex has degree 4, namely, the octahedron. This graph needs 12 bends in any orthogonal drawing, so $b \geq 12$.

$\square$

## 3.2 Subdividing edges

To build large graphs, we subdivide edges of a small graph and then use the resulting vertices of degree 2 to connect many copies of this small graph by identifying vertices or adding edges. Thus, we now must study how subdividing edges affects a lower bound on the number of bends.

**Lemma 2** *If graph $G$ needs at least $b$ bends in any orthogonal drawing, and $G'$ results from subdividing one edge in $G$, then $G'$ needs at least $b-1$ bends in any orthogonal drawing.*

**Proof:** Let $\Gamma'$ be an arbitrary orthogonal drawing of $G'$, and assume that it has $b'$ bends. Removing the subdivision vertex, we obtain a drawing $\Gamma$ of $G$. This drawing inherits all bends of $\Gamma'$, and it may have one more bend at the point of the removed subdivision vertex. By the lower bound on $G$ therefore $b'+1 \geq b$, or $b' \geq b-1$. □

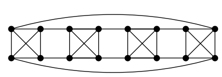
## 3.3 Building large graph classes

In this section we give the definitions of the graph classes for lower bounds. For easier orientation among the excessive number of cases, we use the following classification scheme: The graphs in graph class $N[\Delta, c, \alpha]$

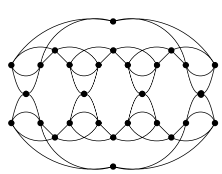
- have maximum degree $\Delta$, $\Delta \in \{3, 4\}$,
- are $c$-connected, $c \in \{1, 2, 3, 4\}$, and
- have degree of simplicity $\alpha$, i.e., they are simple, multigraphs, or may have loops if $\alpha = s, m$ and $l$, respectively.

When talking of one particular graph of class $N[\Delta, c, \alpha]$ we append a parameter $k$ which relates to the size, i.e., the number of vertices, of the graph. More precisely, the graph consists of $k$ or $2k$ copies of one of the small graphs defined before. To avoid trivial cases, we will consider only graphs with $k \geq 3$.

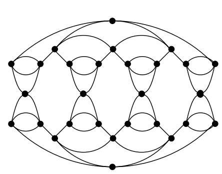
The details of how to build each graph class are given below. To facilitate the description, we use the following notations. A *$\delta$-vertex* is a vertex of degree $\delta$. If there are $k$ copies of a small graph, then they are numbered $1, \ldots, k$. If there are $2k$ copies, then they are numbered $(1,1), \ldots, (k,1), (1,2), \ldots, (k,2)$. All additions are modulo $k$.



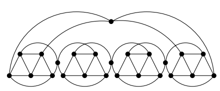
$N[4,4,s](k)$: Take $k$ copies of $K_4$. Connect two 3-vertices of copy $i$ to two 3-vertices of copy $i+1$, $i = 1, \ldots, k$.



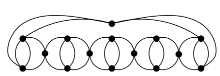
$N[4,3,s](k)$: Take $2k$ copies of $Q$. Subdivide three edges in each copy. Identify a 2-vertex of copy $(i,1)$ with a 2-vertex of copy $(i,2)$, $i = 1, \ldots, k$. Identify a 2-vertex of copy $(i,j)$ with a 2-vertex of copy $(i+1,j)$, $i = 1, \ldots, k$, $j = 1, 2$.



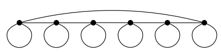
$N[4,3,m](k)$: Take $2k$ copies of $Q$. Subdivide one edge once and one edge twice in each copy. Identify a 2-vertex of copy $(i,1)$ with a 2-vertex of copy $(i,2)$, $i = 1, \ldots, k$. Identify a 2-vertex of copy $(i,j)$ with a 2-vertex of copy $(i+1,j)$, $i = 1, \ldots, k$, $j = 1, 2$.



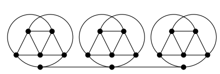
$N[4,2,s](k)$: Take $k$ copies of $K_5$. Subdivide two edges in each copy. Identify a 2-vertex of copy $i$ with a 2-vertex of copy $i+1$, $i = 1, \ldots, k$.



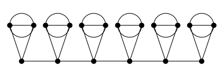
$N[4,2,m](k)$: Take $k$ copies of $Q$. Subdivide two edges in each copy. Identify a 2-vertex of copy $i$ with a 2-vertex of copy $i+1$, $i = 1, \ldots, k$.



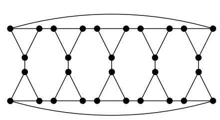
$N[4,2,l](k)$: Take $k$ copies of $L$. Connect the vertex of copy $i$ to the vertex of copy $i+1$, $i = 1, \ldots, k$.



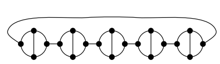
$N[4,1,s](k)$: Take $k$ copies of $K_5$. Subdivide one edge in each copy. Connect the 2-vertex of copy $i$ to the 2-vertex of copy $i+1$, $i = 1, \ldots, k-1$.



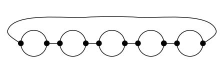
$N[4,1,m](k)$: Take $k$ copies of $Q$. Subdivide one edge in each copy. Connect the 2-vertex of copy $i$ to the 2-vertex of copy $i+1$, $i = 1, \ldots, k-1$.



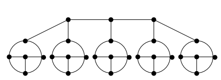
$N[3,3,s](k)$: Take $2k$ copies of $L$. Subdivide the edge twice in each copy. Connect a 2-vertex of copy $(i,1)$ to a 2-vertex of copy $(i,2)$, $i = 1, \ldots, k$. Connect a 2-vertex of copy $(i,j)$ to a 2-vertex of copy $(i+1,j)$, $i = 1, \ldots, k$, $j = 1, 2$.



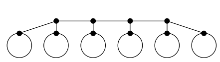
$N[3,2,s](k)$: Take $k$ copies of $T$. Subdivide two edges in each copy. Connect a 2-vertex of copy $i$ to a 2-vertex of copy $i+1$, $i = 1, \ldots, k$.



$N[3,2,m](k)$: Take $k$ copies of $L$. Subdivide the edge in each copy. Connect a 2-vertex of copy $i$ to a 2-vertex of copy $i+1$, $i = 1, \ldots, k$.



$N[3,1,s](k)$: Take $k$ copies of $K_4$. Subdivide one edge in each copy. Add $k-2$ vertices $w_2, \ldots, w_{k-1}$. Connect the 2-vertex of copy 1 to $w_2$. Connect the 2-vertex of copy $i$ to $w_i$, $i = 2, \ldots, k-1$. Connect the 2-vertex of copy $k$ to $w_{k-1}$. (This graph was presented first in [19].)



$N[3,1,l](k)$: Take $k$ copies of $L$. Add $k-2$ vertices $w_2, \ldots, w_{k-1}$. Connect the 2-vertex of copy 1 to $w_2$. Connect the 2-vertex of copy $i$ to $w_i$, $i = 2, \ldots, k-1$. Connect the 2-vertex of copy $k$ to $w_{k-1}$.

One immediately verifies the claims on the maximum degree and degree of simplicity. The claims on the connectivity will be proved for $c = 2, 3, 4$ in the appendix.

### 3.4 Lower bounds

**Theorem 2** *There exist lower bounds for the number of bends of non-planar orthogonal drawings as indicated below:*

| Non-planar graphs | | | | |
|---|---|---|---|---|
| Maximum degree $\Delta$ | Connectivity $c$ | Degree of simplicity $\alpha$ | | |
| | | Simple graph | Multigraph | Graph with loops |
| 4-graph | 4 | $n$ bends | – [1] | – |
| | 3 | $\frac{10}{7}n$ bends | $\frac{10}{7}n$ bends | – |
| | 2 | $\frac{5}{3}n$ bends [2] | $2n$ bends | $3n$ bends |
| | 1 | $\frac{11}{6}n$ bends | $\frac{7}{3}n$ bends | $3n$ bends |
| 3-graph | 3 | $\frac{n}{3}$ bends | – | – |
| | 2 | $\frac{n}{2}$ bends [3] | $n$ bends | – |
| | 1 | $\frac{n}{2}+1$ bends [19] | $n$ bends | $\frac{3}{2}n+3$ bends |

**Proof:** For each case of maximum degree $\Delta$, connectivity $c$ and degree of simplicity $\alpha$, we list in Table 2 the graph class used for the lower bound. (Note that for the cases $(4,1,l)$ and $(3,1,m)$ we use 2-connected graphs, which are also 1-connected.)

This table, which contains the proof of the lower bound for each case, should be read as follows: For each graph class, we start with some small graph which has, say, $n_t$ vertices and needs at least $b_t$ bends in any orthogonal drawing by Lemma 1. We do some number $d$ of subdivisions of edges in each copy; the small graph with subdivisions then has $n_d = n_t + d$ vertices and needs at least $b_d = b_t - d$ bends by Lemma 2.

To build larger classes, we take $q$ copies of the resulting small graph and add $n_a$ vertices. To connect these copies and vertices, we add edges or identify vertices. Both operations cannot decrease the lower bound on the number of bends. If we identified $i$ vertices per copy, then each copy now contributes $n_i = n_d - \frac{1}{2}i$ vertices. The total number $n$ of vertices therefore is $qn_i + n_a$, and the lower bound $b$ on the number of bends is $qb_d$. Reformulating the latter in terms of $n$ yields the desired lower bound. □

Remark: Note that we could have used class $N[4,3,s]$ to obtain the same lower bound for case $(4,3,m)$. We did not do this because $N[4,3,m](k)$ is planar while $N[4,3,s](k)$ is not; this will be exploited in the next section.

[4] A hyphen signifies that no such graphs exist, at least not for $n \geq 5$.

[5] Storer [19] reported a lower bound of $\frac{11}{6}n$ bends, but his proof is incorrect (see also Figure 2).

[6] A similar lower bound was proved by Papakostas and Tollis (private communication).

| Non-planar drawings | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | Graph | Small graph | | | Subdivide | | | Build | | Ident. | | Final graph | |
| $(\Delta, c, \alpha)$ | class | | $n_t$ | $b_t$ | $d$ | $n_d$ | $b_d$ | $q$ | $n_a$ | $i$ | $n_i$ | $n$ | $b$ |
| $(4,4,s)$ | $N[4,4,s]$ | $K_4$ | 4 | 4 | 0 | 4 | 4 | $k$ | 0 | 0 | 4 | $4k$ | $4k{=}n$ |
| $(4,3,s)$ | $N[4,3,s]$ | $Q$ | 2 | 8 | 3 | 5 | 5 | $2k$ | 0 | 3 | $\frac{7}{2}$ | $7k$ | $10k{=}\frac{10}{7}n$ |
| $(4,3,m)$ | $N[4,3,m]$ | $Q$ | 2 | 8 | 3 | 5 | 5 | $2k$ | 0 | 3 | $\frac{7}{2}$ | $7k$ | $10k{=}\frac{10}{7}n$ |
| $(4,2,s)$ | $N[4,2,s]$ | $K_5$ | 5 | 12 | 2 | 7 | 10 | $k$ | 0 | 2 | 6 | $6k$ | $10k{=}\frac{5}{3}n$ |
| $(4,2,m)$ | $N[4,2,m]$ | $Q$ | 2 | 8 | 2 | 4 | 6 | $k$ | 0 | 2 | 3 | $3k$ | $6k{=}2n$ |
| $(4,2,l)$ | $N[4,2,l]$ | $L$ | 1 | 3 | 0 | 1 | 3 | $k$ | 0 | 0 | 1 | $k$ | $3k{=}3n$ |
| $(4,1,s)$ | $N[4,1,s]$ | $K_5$ | 5 | 12 | 1 | 6 | 11 | $k$ | 0 | 0 | 6 | $6k$ | $11k{=}\frac{11}{6}n$ |
| $(4,1,m)$ | $N[4,1,m]$ | $Q$ | 2 | 8 | 1 | 3 | 7 | $k$ | 0 | 0 | 3 | $3k$ | $7k{=}\frac{7}{3}n$ |
| $(4,1,l)$ | $N[4,2,l]$ | $L$ | 1 | 3 | 0 | 1 | 3 | $k$ | 0 | 0 | 1 | $k$ | $3k{=}3n$ |
| $(3,3,s)$ | $N[3,3,s]$ | $L$ | 1 | 3 | 2 | 3 | 1 | $2k$ | 0 | 0 | 3 | $6k$ | $2k{=}\frac{n}{3}$ |
| $(3,2,s)$ | $N[3,2,s]$ | $T$ | 2 | 4 | 2 | 4 | 2 | $k$ | 0 | 0 | 4 | $4k$ | $2k{=}\frac{n}{2}$ |
| $(3,2,m)$ | $N[3,2,m]$ | $L$ | 1 | 3 | 1 | 2 | 2 | $k$ | 0 | 0 | 2 | $2k$ | $2k{=}n$ |
| $(3,1,s)$ | $N[3,1,s]$ | $K_4$ | 4 | 4 | 1 | 5 | 3 | $k$ | $k{-}2$ | 0 | 5 | $6k{-}2$ | $3k{=}\frac{n}{2}{+}1$ |
| $(3,1,m)$ | $N[3,2,m]$ | $L$ | 1 | 3 | 1 | 2 | 2 | $k$ | 0 | 0 | 2 | $2k$ | $2k{=}n$ |
| $(3,1,l)$ | $N[3,1,l]$ | $L$ | 1 | 3 | 0 | 1 | 3 | $k$ | $k{-}2$ | 0 | 1 | $2k{-}2$ | $3k{=}\frac{3}{2}n{+}3$ |
| Non-planar drawings of planar graphs | | | | | | | | | | | | | |
| $(4,4,s)$ | $N_{pl}[4,4,s]$ | $W$ | 5 | 4 | 0 | 4 | 4 | $k$ | 0 | 0 | 5 | $5k$ | $4k = \frac{4}{5}n$ |
| $(4,3,s)$ | $N_{pl}[4,3,s]$ | $O$ | 6 | 12 | 3 | 9 | 9 | $2k$ | 0 | 3 | $\frac{15}{2}$ | $15k$ | $18k = \frac{6}{5}n$ |
| $(4,2,s)$ | $N_{pl}[4,2,s]$ | $O$ | 6 | 12 | 2 | 8 | 10 | $k$ | 0 | 2 | 7 | $7k$ | $10k = \frac{10}{7}n$ |
| $(4,1,s)$ | $N_{pl}[4,1,s]$ | $O$ | 6 | 12 | 1 | 7 | 11 | $k$ | 0 | 0 | 7 | $7k$ | $11k = \frac{11}{7}n$ |

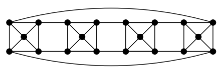
Table 2: For the proof of Theorem 2 and 3.

# 4 Non-planar drawings of planar graphs

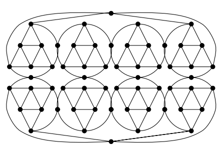
In this section we study lower bounds for non-planar drawings of planar graphs. The interest in such drawings arises from the fact that every planar graph can be drawn with $\mathcal{O}(n \log^2 n)$ area if crossings are allowed [15], whereas both non-planar graphs and planar drawings of planar graphs may require $\Omega(n^2)$ area ([22] and Section 6).

The lower bound for the area of non-planar drawings of planar graphs is $\Omega(n \log n)$ [14]. We did not improve on this lower bound, but study here lower bounds on the number of bends. For the most part, we use the graphs defined in Section 3.3. As can be seen from the drawings, these graphs are planar, with the exception of $N[4, c, s](k)$, $c = 1, 2, 3, 4$.

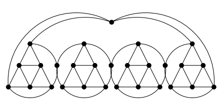
We now define four graph-classes $N_{pl}[4, c, s](k)$, of $c$-connected simple planar 4-graphs, using the 4-wheel $W$ and the octahedron $O$.



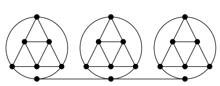
$N_{pl}[4, 4, s](k)$: Take $k$ copies of $W$. Connect two 3-vertices of copy $i$ to two 3-vertices of copy $i + 1$, $i = 1, \ldots, k$.



$N_{pl}[4, 3, s](k)$: Take $2k$ copies of $O$. Subdivide three edges on one face of each copy. Identify a 2-vertex of copy $(i, 1)$ with a 2-vertex of copy $(i, 2)$, $i = 1, \ldots, k$. Identify a 2-vertex of copy $(i, j)$ with a 2-vertex of copy $(i + 1, j)$, $i = 1, \ldots, k$, $j = 1, 2$.



$N_{pl}[4, 2, s](k)$: Take $k$ copies of $O$. Subdivide two edges on one face of each copy. Identify a 2-vertex of copy $i$ with a 2-vertex of copy $i + 1$, $i = 1, \ldots, k$.



$N_{pl}[4, 1, s](k)$: Take $k$ copies of $O$. Subdivide one edge of each copy. Connect the 2-vertex of copy $i$ with the 2-vertex of copy $i + 1$, $i = 1, \ldots, k - 1$.

One immediately verifies that these are indeed simple planar 4-graphs. The claims on the connectivity will be proved for $c = 2, 3, 4$ in the appendix.

**Theorem 3** *There exist lower bounds for the number of bends of non-planar orthogonal drawings of planar graphs as indicated below:*

| Non-planar drawings of planar graphs | | | | |
|---|---|---|---|---|
| Maximum degree $\Delta$ | Connectivity $c$ | Degree of simplicity $\alpha$ | | |
| | | Simple graph | Multigraph | Graph with loops |
| 4-graph | 4 | $\frac{4}{5}n$ bends | – | – |
| | 3 | $\frac{6}{5}n$ bends | $\frac{10}{7}n$ bends | – |
| | 2 | $\frac{10}{7}n$ bends | $2n$ bends | $3n$ bends |
| | 1 | $\frac{11}{7}n$ bends | $\frac{7}{3}n$ bends | $3n$ bends |
| 3-graph | 3 | $\frac{n}{3}$ bends | – | – |
| | 2 | $\frac{n}{2}$ bends | $n$ bends | – |
| | 1 | $\frac{n}{2} + 1$ bends [19] | $n$ bends | $\frac{3}{2}n + 3$ bends |

**Proof:** For all graphs except the simple 4-graphs, the lower bound is given by the same graph class as in Theorem 2 and has been proved there. For the simple 4-graphs, the proof is done in Table 2, which has to be read as explained in the proof of Theorem 2. □

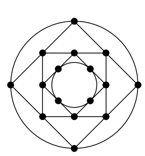
# 5 Plane drawings of planar graphs

In some algorithms for planar orthogonal drawings, e.g. [12, 20], the output planar orthogonal drawing exactly reflects the input planar drawing, i.e., the combinatorial embedding and the outer-face. In this section we prove lower bounds for such plane orthogonal drawings.

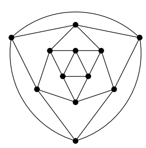
## 5.1 Definition of graph classes

We define new graphs for lower bounds for plane drawings, using the same classification scheme as before after replacing "$N$" by "$P$". Thus, $P[\Delta, c, \alpha]$ is a class of planar $c$-connected $\Delta$-graphs with degree of simplicity $\alpha$. We again use a parameter $k$ that relates to the size of the graph: $P[\Delta, c, \alpha](k)$ consists of $k$ edge-disjoint cycles $C_1, \ldots, C_k$, which are connected to each other by identifying vertices or adding edges. To explain the fixed planar drawing of the graph, we need some terminology. Let $e$ be an edge and let $C$ and $C'$ be circles. In a planar drawing, removing the closed Jordan curve defined by $C$ splits the plane into two open regions, one bounded and one unbounded. Edge $e$ is called *inside* circle $C$ if all interior points of the open Jordan curve representing $e$ are in the bounded open region thus defined by $C$. In particular, the endpoints of $e$, but not $e$ itself, may belong to $C$. Circle $C'$ is called *inside* circle $C$ if all edges of $C'$ are inside $C$. We say that $k$ cycles $C_1, \ldots, C_k$ are *stacked* if $C_i$ is inside $C_{i+1}$, $i = 1, \ldots, k-1$. We fix the combinatorial embedding and outer-face of $P[\Delta, c, \alpha](k)$ such that the cycles $C_1, \ldots, C_k$ are stacked.

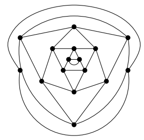
To facilitate the notations for the definitions, let an *$l$-cycle* be a cycle with $l$ vertices. If $C_i$ is an $l$-cycle, then denote its vertices as $v_1^i, \ldots, v_l^i$ in clockwise order around the cycle.



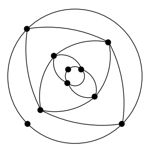
$P[4,4,s](k)$: $C_1$ and $C_k$ are 4-cycles. $C_2, \ldots, C_{k-1}$ are 8-cycles. Identify vertices $v_1^1, v_2^1, v_3^1, v_4^1$ with vertices $v_1^2, v_3^2, v_5^2, v_7^2$. Identify vertices $v_2^i, v_4^i, v_6^i, v_8^i$ with vertices $v_1^{i+1}, v_3^{i+1}, v_5^{i+1}, v_7^{i+1}$, $i = 2, \ldots, k-2$. Identify vertices $v_2^{k-1}, v_4^{k-1}, v_6^{k-1}, v_8^{k-1}$ with vertices $v_1^k, v_2^k, v_3^k, v_4^k$. We have $n = 4k-4$ and $m = 2n$.



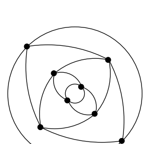
$P[4,3,s](k)$: $C_1$ and $C_k$ are 3-cycles. $C_2, \ldots, C_{k-1}$ are 6-cycles. Identify vertices $v_1^1, v_2^1, v_3^1$ with vertices $v_1^2, v_3^2, v_5^2$ . Identify vertices $v_2^i, v_4^i, v_6^i$ with vertices $v_1^{i+1}, v_3^{i+1}, v_5^{i+1}$, $i = 2, \ldots, k-2$. Identify vertices $v_2^{k-1}, v_4^{k-1}, v_6^{k-1}$ with vertices $v_1^k, v_2^k, v_3^k$. We have $n = 3k-3$ and $m = 2n$.



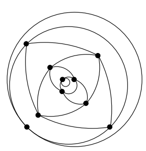
$P[4,3,m](k)$: Take a copy of $P[4,3,s](k-2)$. Subdivide two edges of $C_1$ and add a double edge inside $C_1$. Subdivide two edges of $C_{k-2}$ and add a double edge such that $C_{k-2}$ is inside it. We have $n = 3k-5$ and $m = 2n$.



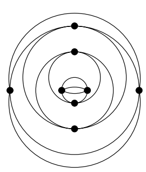
$P[4,2,s](k)$: Take a copy of $P[4,2,m](k)$ defined below. Subdivide one edge of $C_1$ and one edge of $C_k$. We have $n = 2k$ and $m = 2n-2$. (This graph was presented first in [21].)



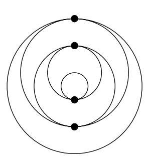
$P[4,2,m](k)$: $C_1$ and $C_k$ are 2-cycles. $C_2, \ldots, C_{k-2}$ are 4-cycles. Identify vertices $v_1^1, v_2^1$ with vertices $v_1^2, v_3^2$. Identify vertices $v_2^i, v_4^i$ with vertices $v_1^{i+1}, v_3^{i+1}$, $i = 2, \ldots, k-2$. Identify vertices $v_2^{k-1}, v_4^{k-1}$ with vertices $v_1^k, v_2^k$. We have $n = 2k-2$ and $m = 2n$. (This graph was presented first in [21].)



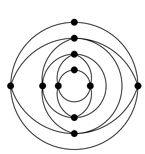
$P[4,2,l](k)$: Take a copy of $P[4,2,s](k-2)$. Add a loop inside $C_1$ at the 2-vertex of $C_1$. Add a loop at the 2-vertex of $C_{k-2}$ such that $C_{k-2}$ is inside it. We have $n = 2k-4$ and $m = 2n$.



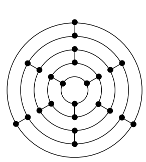
$P[4,1,m](k)$: Take a copy of $P[4,1,l](k-2)$ defined below. Subdivide $C_1$ twice and add a double edge inside $C_1$. Subdivide $C_{k-2}$ twice and add a double edge such that $C_{k-2}$ is inside it. We have $n = k+1$ and $m = 2n$.



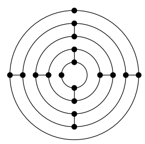
$P[4,1,l](k)$: $C_1$ and $C_k$ are 1-cycles. $C_2, \ldots, C_{k-2}$ are 2-cycles. Identify $v_1^1$ with vertex $v_1^2$. Identify $v_2^i$ with vertex $v_1^{i+1}$, $i = 2, \ldots, k-2$. Identify vertex $v_2^{k-1}$ with $v_1^k$. We have $n = k-1$ and $m = 2n$.



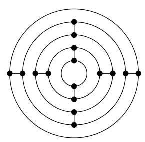
$P[4,1,s](k)$: This graph class is defined only for $k = 3l$. $C_1, C_4, \ldots, C_{3l-2}$ are 3-cycles. $C_2, C_5, \ldots, C_{3l-1}$ are 4-cycles. $C_3, C_6, \ldots, C_{3l}$ are 3-cycles. Identify vertices $v_1^{3i-2}, v_3^{3i-2}$ with vertices $v_1^{3i-1}, v_3^{3i-1}$, $i = 1, \ldots, l$. Identify vertices $v_2^{3i-1}, v_4^{3i-1}$ with vertices $v_1^{3i}, v_3^{3i}$, $i = 1, \ldots, l$. Identify vertex $v_2^{3i}$ with vertex $v_2^{3i+1}$, $i = 1, \ldots, l-1$. We have $n = \frac{5}{3}k + 1$ and $m = 2n-2$.



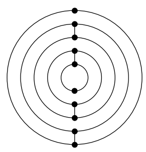
$P[3,3,s](k)$: $C_1$ and $C_k$ are 3-cycles. $C_2, \ldots, C_{k-1}$ are 6-cycles. Connect vertices $v_1^1, v_2^1, v_3^1$ with vertices $v_1^2, v_3^2, v_5^2$. Connect vertices $v_2^i, v_4^i, v_6^i$ with vertices $v_1^{i+1}, v_3^{i+1}, v_5^{i+1}$, $i = 2, \ldots, k-2$. Connect vertices $v_2^{k-1}, v_4^{k-1}, v_6^{k-1}$ with vertices $v_1^k, v_2^k, v_3^k$. We have $n = 6k-6$ and $m = \frac{3}{2}n$.



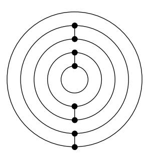
$P[3,2,s](k)$: Take a copy of $P[3,2,m](k)$ defined below. Subdivide one edge of $C_1$ and one edge of $C_k$. We have $n = 4k-2$ and $m = \frac{3}{2}n - 1$.



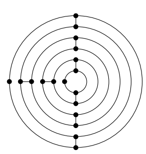
$P[3,2,m](k)$: $C_1$ and $C_k$ are 2-cycles. $C_2, \dots, C_{k-2}$ are 4-cycles. Connect vertices $v_1^1, v_2^1$ with vertices $v_1^2, v_3^2$. Connect vertices $v_2^i, v_4^i$ with vertices $v_1^{i+1}, v_3^{i+1}$, $i = 2, \dots, k-2$. Connect vertices $v_2^{k-1}, v_4^{k-1}$ with vertices $v_1^k, v_2^k$. We have $n = 4k-4$ and $m = \frac{3}{2}n$.



$P[3,1,m](k)$: Take a copy of $P[3,1,l](k)$ defined below. Subdivide the edge of $C_1$ and the edge of $C_k$. We have $n = 2k$ and $m = \frac{3}{2}n - 1$.



$P[3,1,l](k)$: $C_1$ and $C_k$ are 1-cycles. $C_2, \dots, C_{k-2}$ are 2-cycles. Connect $v_1^1$ with vertex $v_1^2$. Connect $v_2^i$ with vertex $v_1^{i+1}$, $i = 2, \dots, k-2$. Connect vertex $v_2^{k-1}$ with $v_1^k$. We have $n = 2k-2$ and $m = \frac{3}{2}n$.



$P[3,1,s](k)$: This graph class is defined only for $k = 2l$. $C_1, \dots, C_k$ are 3-cycles. Connect vertices $v_1^{2i-1}, v_3^{2i-1}$ with vertices $v_1^{2i}, v_3^{2i}$, $i = 1, \dots, l$. Connect vertex $v_2^{2i}$ with vertex $v_2^{2i+1}$, $i = 1, \dots, l-1$. We have $n = 3k$ and $m = \frac{3}{2}n - 1$.

One immediately verifies the claims on planarity, maximum degree and degree of simplicity. The claims on the connectivity will be proved for $c = 2, 3, 4$ in the appendix.

## 5.2 Lower bounds

**Lemma 3** *If an orthogonal planar drawing $\Gamma$ of a graph with $n$ vertices and $m$ edges contains $k$ stacked cycles, then it has at least $2k$ rows, $2k$ columns, and $4k + m - 2n$ bends.*

**Proof:** Let $C_1, \dots, C_k$ be the $k$ stacked cycles. Pick a point $p$ in $\Gamma$ which is not on a grid-line and inside cycle $C_1$. When traversing the horizontal ray starting at $p$ and proceeding towards $+\infty$, we must cross all $k$ stacked cycles. Because $p$ is not on a grid-line, the horizontal line through $p$ does not intersect any grid-points, therefore we must cross at least $k$ edges. To accommodate these edges, there must be at least $k$ columns to the right of $p$. See Figure 3.

Similarly, there must be at least $k$ columns to the left of $p$, and at least $k$ rows above and $k$ rows below $p$. This proves the claim on the rows and columns. The claim on the number of bends is then a reformulation of Theorem 1(b). $\square$

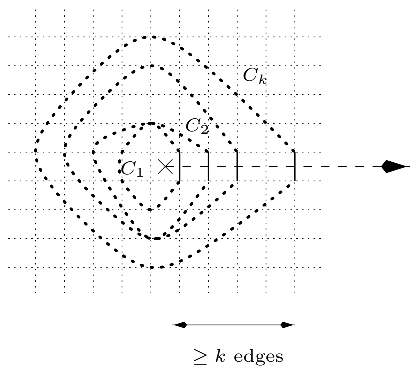



Figure 3: Going from inside $C_1$ to outside $C_k$ we cross all stacked cycles.

Since the fixed planar drawing of $P[\Delta, c, \alpha](k)$ contains $k$ stacked cycles, we obtain immediately the following corollary.

**Corollary 4** *For any combination of $\Delta, c, \alpha, k$ for which $P[\Delta, c, \alpha](k)$ is defined, $P[\Delta, c, \alpha](k)$ needs a $(2k-1) \times (2k-1)$-grid and $4k + m - 2n$ bends in any plane orthogonal drawing.*

**Theorem 4** *There exist lower bounds for plane orthogonal drawings as indicated below, with the first entry in each cell being a lower bound on the grid-size, and the second entry being a lower bound on the number of bends:*

| Plane drawings of planar graphs | | | | |
|---|---|---|---|---|
| maximum degree $\Delta$ | $c$ | Degree of simplicity $\alpha$ | | |
| | | Simple graph | Multigraph | Graph with loops |
| 4-graph | 4 | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$<br>$n+4$ bends | –<br>– | –<br>– |
| | 3 | $(\frac{2}{3}n+1)\times(\frac{2}{3}n+1)$<br>$\frac{4}{3}n+4$ bends | $(\frac{2}{3}n+\frac{7}{3})\times(\frac{2}{3}n+\frac{7}{3})$<br>$\frac{10}{7}n$ bends (*) | –<br>– |
| | 2 | $(n-1)\times(n-1)$<br>$2n-2$ bends [21] | $(n+1)\times(n+1)$<br>$2n+4$ bends [21] | $(n+3)\times(n+3)$<br>$3n$ bends (*) |
| | 1 | $(\frac{6}{5}n-\frac{11}{5})\times(\frac{6}{5}n-\frac{11}{5})$<br>$\frac{12}{5}n-\frac{22}{5}$ bends | $(2n-3)\times(2n-3)$<br>$4n-4$ bends | $(2n+1)\times(2n+1)$<br>$4n+4$ bends |
| 3-graph | 3 | $(\frac{n}{3}+1)\times(\frac{n}{3}+1)$<br>$\frac{n}{3}$ bends (*) | –<br>– | –<br>– |
| | 2 | $\frac{n}{2}\times\frac{n}{2}$<br>$\frac{n}{2}+1$ bends | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$<br>$n$ bends (*) | – |
| | 1 | $(\frac{2}{3}n-1)\times(\frac{2}{3}n-1)$<br>$\frac{5}{6}n-1$ bends | $(n-1)\times(n-1)$<br>$\frac{3}{2}n-1$ bends | $(n+1)\times(n+1)$<br>$\frac{3}{2}n+4$ bends |

| Graphs | $n$ | $k$ | $2k-1$ | $m$ | $m-2n$ | $b$ |
|---|---|---|---|---|---|---|
| $P[4,4,s]$ | $4k-4$ | $\frac{1}{4}n+1$ | $\frac{1}{2}n+1$ | $2n$ | $0$ | $n+4$ |
| $P[4,3,s]$ | $3k-3$ | $\frac{1}{3}n+1$ | $\frac{2}{3}n+1$ | $2n$ | $0$ | $\frac{4}{3}n+4$ |
| $P[4,3,m]$ | $3k-5$ | $\frac{1}{3}n+\frac{5}{3}$ | $\frac{2}{3}n+\frac{7}{3}$ | (*) | (*) | (*) |
| $P[4,2,s]$ | $2k$ | $\frac{1}{2}n$ | $n-1$ | $2n-2$ | $-2$ | $2n-2$ |
| $P[4,2,m]$ | $2k-2$ | $\frac{1}{2}n+1$ | $n+1$ | $2n$ | $0$ | $2n+4$ |
| $P[4,2,l]$ | $2k-4$ | $\frac{1}{2}n+2$ | $n+3$ | (*) | (*) | (*) |
| $P[4,1,s]$ | $\frac{5}{3}k+1$ | $\frac{3}{5}n-\frac{3}{5}$ | $\frac{6}{5}n-\frac{11}{5}$ | $2n-2$ | $-2$ | $\frac{12}{5}n-\frac{22}{5}$ |
| $P[4,1,m]$ | $k+1$ | $n-1$ | $2n-3$ | $2n$ | $0$ | $4n-4$ |
| $P[4,1,l]$ | $k-1$ | $n+1$ | $2n+1$ | $2n$ | $0$ | $4n+4$ |
| $P[3,3,s]$ | $6k-6$ | $\frac{1}{6}n+1$ | $\frac{n}{3}+1$ | (*) | (*) | (*) |
| $P[3,2,s]$ | $4k-2$ | $\frac{1}{4}n+\frac{1}{2}$ | $\frac{1}{2}n$ | $\frac{3}{2}n-1$ | $-\frac{1}{2}n-1$ | $\frac{n}{2}+1$ |
| $P[3,2,m]$ | $4k-4$ | $\frac{1}{4}n+1$ | $\frac{1}{2}n+1$ | (*) | (*) | (*) |
| $P[3,1,s]$ | $3k$ | $\frac{1}{3}n$ | $\frac{2}{3}n-1$ | $\frac{3}{2}n-1$ | $-\frac{1}{2}n-1$ | $\frac{5}{6}n-1$ |
| $P[3,1,m]$ | $2k$ | $\frac{1}{2}n$ | $n-1$ | $\frac{3}{2}n-1$ | $-\frac{1}{2}n-1$ | $\frac{3}{2}n-1$ |
| $P[3,1,l]$ | $2k-2$ | $\frac{1}{2}n+1$ | $n+1$ | $\frac{3}{2}n$ | $-\frac{1}{2}n$ | $\frac{3}{2}n+4$ |

Table 3: For the proof of Theorem 4.

**Proof:** The results marked (*) are identical to the claims of Theorem 3 and have been proved in Theorem 2. For all other cases, the lower bounds follow from reformulating Corollary 4 in terms of the number of vertices. This is done in Table 3.

We list for all defined graph classes the number of vertices $n$ and reformulate $k$ in terms of $n$. This yields the lower bound of $2k-1$ on the width and height by Corollary 4. Then, if needed, we list $m$ relative to $n$, and $m-2n$, which allows us to compute the lower bound of the number of bends $b = 4k+m-2n$. □

Remark: One might think that the lower bound for simple 3-connected 4-graphs could be improved from $\frac{4}{3}n+4$ bends to $\frac{10}{7}n$ by using class $N[4,3,s]$. However, this is not true because $N[4,3,s](k)$ is not planar; we could only use graph $N_{pl}[4,3,s](k)$, but the lower bound of $\frac{6}{5}n$ bends for this graph is not better than the lower bound of $\frac{4}{3}n+4$ shown above.

# 6 Planar drawings of planar graphs

To the author's knowledge no research has been done into lower bounds for planar orthogonal drawings when we can choose the combinatorial embedding of the graph. We provide such results here. Our main contribution are lower bounds on the grid-size; we prove lower bounds on the number of bends as well, but these are frequently not better than the ones known for non-planar drawings of planar graphs (Theorem 3).

The difficulty in proving lower bounds for planar drawings lies in the fact that possible combinatorial embeddings and outer-faces of the graph have to be tested. We deal with this by using graph classes that have only one possible combinatorial embedding, up to renaming of vertices. For 4-connected and 3-connected planar graphs, the combinatorial embedding is unique. For the 2-connected planar graphs without loops defined in Section 5.1, one can show that the combinatorial embedding is also unique up to renaming of vertices. Unfortunately, the combinatorial embedding is not unique for our graph classes that have loops or are not 2-connected. For this reason, we will use 2-connected graphs without loops to obtain lower bounds for 1-connected graphs and graphs with loops.

In fact, a weaker property than uniqueness of the combinatorial embedding will suffice for our lower bound argument. This property is detailed in the following lemma.

**Lemma 5** *If $\Delta, c, \alpha, k$ is a combination for which $P[\Delta, c, \alpha](k)$ is defined, and if $k \geq 3$, $c \geq 2$ and $\alpha \neq l$, then for any planar drawing of $P[\Delta, c, \alpha](k)$ there exists some $i$, $1 \leq i < k$, such that the cycles $C_1, \ldots, C_i$ are stacked, and the cycles $C_k, C_{k-1}, \ldots, C_{i+1}$ are stacked.*

**Proof:** For $c = 2$ this will be proved in the appendix. For $c \geq 3$ the combinatorial embedding is unique, thus in any planar drawing the outer-face $F$ must be a face in the drawing of Section 5.1. One verifies that there are only three possibilities for $F$: It can be $C_1$, it can be $C_k$, or it can be composed of edges of two cycles $C_i$ and $C_{i+1}$, $i \in \{1, \ldots, k-1\}$ and (for $\Delta = 3$) edges that were added to connect $C_i$ and $C_{i+1}$. See also Figure 4. In the first case, set $i = 1$; in the second case, set $i = k - 1$, and in the third case, use $C_i$ as defined. One verifies the claim. $\square$

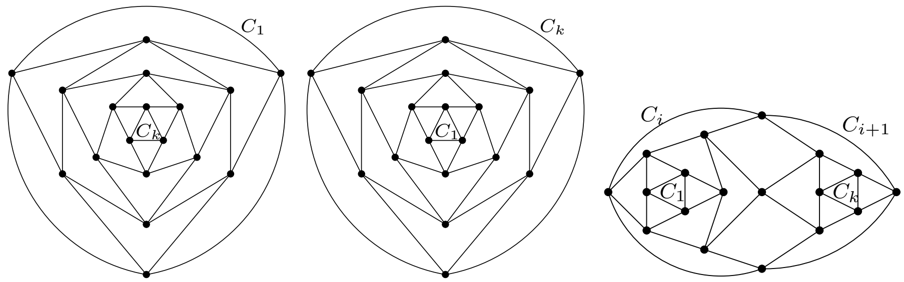



Figure 4: Three possibilities for the outer-face: it can be $C_1$, it can be $C_k$, or it can be incident to two cycles $C_i$ and $C_{i+1}$. Shown here is $P[4, 3, s](7)$.

To obtain a slightly stronger lower bound, we will use only those graphs $P[\Delta, c, \alpha](k)$ for which $k$ is odd.

**Lemma 6** *If $\Delta, c, \alpha, k$ is a combination for which $P[\Delta, c, \alpha](k)$ is defined, and if $k \geq 3$ is odd, $c \geq 2$ and $\alpha \neq l$, then $P[\Delta, c, \alpha](k)$ needs a $k \times k$-grid and at least $4k + m - 2n - (\Delta - 2)c$ bends in any planar orthogonal drawing.*

**Proof:** Let $1 \leq i < k$ be such that the cycles $C_1, \ldots, C_i$ and $C_k, \ldots, C_{i+1}$ are stacked (Lemma 5). Let $G_1$ be the graph induced by the vertices in $C_1, \ldots, C_i$ and let $G_2$ be the graph induced by the vertices in $C_{i+1}, \ldots, C_k$.

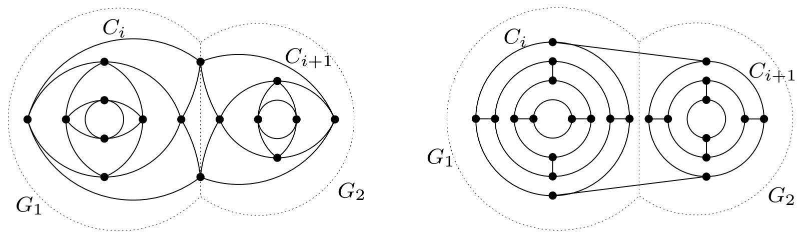



Figure 5: Graph $G_1$ consists of stacked cycles $C_1, \ldots, C_i$ and $G_2$ consists of stacked cycles $C_k, \ldots, C_{i+1}$. Shown here is $P[4, 2, m](9)$ with $i = 5$ and $P[3, 2, m](7)$ with $i = 4$.

Let $n_1, m_1, n_2, m_2$ be the number of vertices and edges of $G_1$ and $G_2$, respectively. Graph $G_1$ has $i$ stacked cycles and hence needs $2i$ rows, $2i$ columns and $4i + m_1 - 2n_1$ bends by Lemma 3. Graph $G_2$ has $k - i$ stacked cycles and hence needs $2(k - i)$ rows, $2(k - i)$ columns and $4(k - i) + m_2 - 2n_2$ bends by Lemma 3.

For the claim on the grid-size, observe that $\max\{i, k - i\} \geq \lceil \frac{k}{2} \rceil = \frac{k+1}{2}$ since $k$ is odd, so at least one of $G_1$ and $G_2$ needs $k + 1$ rows and $k + 1$ columns, and thus width and height $k$.

For the claim on the number of bends, we distinguish by the maximum degree $\Delta$. If $\Delta = 4$, then $C_i$ and $C_{i+1}$ were connected by identifying at most $c$ vertices, so $n_1 + n_2 \leq n + c$, $m_1 + m_2 = m$, and the number of bends is at least

$$4i + m_1 - 2n_1 + 4(k - i) + m_2 - 2n_2 \geq 4k + m - 2n - 2c = 4k + m - 2n - (\Delta - 2)c.$$

If $\Delta = 3$, then $C_i$ and $C_{i+1}$ were connected by adding $c$ edges, so $n_1 + n_2 = n$ and $m_1 + m_2 = m - c$, and the number of bends is at least

$$4i + m_1 - 2n_1 + 4(k - i) + m_2 - 2n_2 = 4k + m - c - 2n = 4k + m - 2n - (\Delta - 2)c.$$

Either way, the second claim follows. $\square$

**Theorem 5** *There exist lower bounds for plane orthogonal drawings as indicated below, with the first entry in each cell being a lower bound on the grid-size, and the second entry being a lower bound on the number of bends:*

| Planar drawings of planar graphs | | | | |
|---|---|---|---|---|
| Maximum degree $\Delta$ | $c$ | Degree of simplicity $\alpha$ | | |
| | | Simple graph | Multigraph | Graph with loops |
| 4-graph | 4 | $(\frac{n}{4}+1)\times(\frac{n}{4}+1)$ | – | – |
| | | $n-4$ bends | – | – |
| | 3 | $(\frac{n}{3}+1)\times(\frac{n}{3}+1)$ | $(\frac{n}{3}+\frac{5}{3})\times(\frac{n}{3}+\frac{5}{3})$ | – |
| | | $\frac{4}{3}n-2$ bends | $\frac{10}{7}n$ bends (*) | – |
| | 2 | $\frac{n}{2}\times\frac{n}{2}$ | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$ | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$ |
| | | $2n-6$ bends | $2n$ bends | $3n$ bends (*) |
| | 1 | $\frac{n}{2}\times\frac{n}{2}$ | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$ | $(\frac{n}{2}+1)\times(\frac{n}{2}+1)$ |
| | | $2n-6$ bends | $\frac{7}{3}n$ bends (*) | $3n$ bends (*) |
| 3-graph | 3 | $(\frac{n}{6}+1)\times(\frac{n}{6}+1)$ | – | – |
| | | $\frac{n}{3}$ bends (*) | – | – |
| | 2 | $(\frac{n}{4}+\frac{1}{2})\times(\frac{n}{4}+\frac{1}{2})$ | $(\frac{n}{4}+1)\times(\frac{n}{4}+1)$ | – |
| | | $\frac{n}{2}$ bends (*) | $n+1$ bends (*) | – |
| | 1 | $(\frac{n}{4}+\frac{1}{2})\times(\frac{n}{4}+\frac{1}{2})$ | $(\frac{n}{4}+1)\times(\frac{n}{4}+1)$ | $(\frac{n}{4}+1)\times(\frac{n}{4}+1)$ |
| | | $\frac{n}{2}+1$ bends (*) | $n$ bends (*) | $\frac{3}{2}n+3$ bends (*) |

**Proof:** The results marked (*) are identical to the claims of Theorem 3 and have been proved in Theorem 2. For all other cases, the lower bounds follow from reformulating Lemma 6 for an appropriate graph class in terms of the number of vertices. For each case of maximum degree $\Delta$, connectivity $c$ and degree of simplicity $\alpha$, we list in Table 4 the used graph class $P[\Delta', c', \alpha'](k)$.

For each graph class $P[\Delta', c', \alpha'](k)$, we list $k$, which is the lower bound on the width and height by Lemma 6, and which we computed relative to $n$ already in Table 3. Where needed, we then compute $4k$ and $m-2n$; the latter is again taken from the proof of Theorem 4. Finally we compute $(\Delta'-2)c'$; note that here we have to take the parameters of the graph class used for the lower bound, not the parameters of the case under consideration. Combining these values, we get $b = 4k + m - 2n - (\Delta'-2)c'$, which is the lower bound on the number of bends by Lemma 6. $\square$

# 7 Remarks and open problems

In this paper we studied lower bounds on the grid-size and the number of bends for orthogonal drawings. We provided lower bounds for various graph classes, depending on degree of planarity, maximum degree, connectivity, and degree of simplicity. In most cases, we gave lower bounds for the first time or considerably improved previous ones.

Not for all graph classes do there exist specialized algorithms, thus not all lower bounds can be compared to upper bounds. As far as algorithms do exist, the upper bounds and lower bounds are generally very close for plane drawings

| Case | Graph | Grid-size | Bends | | | |
|---|---|---|---|---|---|---|
| $(\Delta, c, \alpha)$ | class | $k$ | $4k$ | $m - 2n$ | $(\Delta'-2)c'$ | $b$ |
| $(4,4,s)$ | $P[4,4,s]$ | $\frac{1}{4}n+1$ | $n+4$ | 0 | 8 | $n-4$ |
| $(4,3,s)$ | $P[4,3,s]$ | $\frac{1}{3}n+1$ | $\frac{4}{3}n+4$ | 0 | 6 | $\frac{4}{3}n-2$ |
| $(4,3,m)$ | $P[4,3,m]$ | $\frac{1}{3}n+\frac{5}{3}$ | (*) | (*) | (*) | (*) |
| $(4,2,s)$ | $P[4,2,s]$ | $\frac{1}{2}n$ | $2n$ | $-2$ | 4 | $2n-6$ |
| $(4,2,m)$ | $P[4,2,m]$ | $\frac{1}{2}n+1$ | $2n+4$ | 0 | 4 | $2n$ |
| $(4,2,l)$ | $P[4,2,m]$ | $\frac{1}{2}n+1$ | (*) | (*) | (*) | (*) |
| $(4,1,s)$ | $P[4,2,s]$ | $\frac{1}{2}n$ | $2n$ | $-2$ | 4 | $2n-6$ |
| $(4,1,m)$ | $P[4,2,m]$ | $\frac{1}{2}n+1$ | (*) | (*) | (*) | (*) |
| $(4,1,l)$ | $P[4,2,m]$ | $\frac{1}{2}n+1$ | (*) | (*) | (*) | (*) |
| $(3,3,s)$ | $P[3,3,s]$ | $\frac{1}{6}n+1$ | (*) | (*) | (*) | (*) |
| $(3,2,s)$ | $P[3,2,s]$ | $\frac{1}{4}n+\frac{1}{2}$ | (*) | (*) | (*) | (*) |
| $(3,2,m)$ | $P[3,2,m]$ | $\frac{1}{4}n+1$ | (*) | (*) | (*) | (*) |
| $(3,1,s)$ | $P[3,2,s]$ | $\frac{1}{4}n+\frac{1}{2}$ | (*) | (*) | (*) | (*) |
| $(3,1,m)$ | $P[3,2,m]$ | $\frac{1}{4}n+1$ | (*) | (*) | (*) | (*) |
| $(3,1,l)$ | $P[3,2,m]$ | $\frac{1}{4}n+1$ | (*) | (*) | (*) | (*) |

Table 4: For the proof of Theorem 5.

(with the exception mentioned below). For planar drawings, the upper and lower bounds are close with respect to the number of bends, but do not match with respect to the grid-size. Our conjecture is here that the upper bounds should be improved, as most algorithms do not change the embedding of the planar graphs, or not by much. Finally, much work remains to be done for non-planar drawings, in particular with respect to improving the lower bound on the area.

Some remaining open problems are the following:

- For which classes can the lower bounds be improved? In particular, are there better lower bounds for planar drawings of 1-connected graphs? Are there better lower bounds on the area of non-planar drawings?
- We verified that the presented graph classes indeed have an orthogonal drawing which matches the lower bounds, up to a small additive constant, with two exceptions:
  - We did not find a drawing of $N[4,4,s](k)$ with less than $\frac{3}{2}n$ bends, or a drawing of $N_{pl}[4,4,s](k)$ with less than $\frac{6}{5}n$ bends, and conjecture that these numbers are the correct lower bound on the number of bends. How can this be shown?

  - The lower bound on the grid-size for planar orthogonal drawings is computed by taking the maximum of the two subgraphs defined by the two sets of stacked cycles. However, this disregards that both subgraphs need grid-space. Intuitively, one would think that if one subgraph needs $2i$ rows and columns, and the other subgraph needs $2k - 2i$ rows and columns, then, to place the drawings next to each other, one needs at least $2k$ rows or columns, thus yielding a lower bound of roughly a $k \times 2k$-grid.

    This agrees with our experience of trying to draw the planar graphs while changing the outer-face. However, to prove this lower bound, one would have to show that each subgraph "almost completely" fills its grid in any drawing. What is the appropriate definition of "almost completely", and how can this be shown?

- No algorithms are known that use the fact that a graph is 4-connected. No algorithms are known for non-planar 3-connected graphs. Certainly, drawings of such graphs can be obtained by applying an algorithm for graphs of lower connectivity, but could the upper bounds be improved with specialized algorithms for graphs of high connectivity?

- Are there better upper bounds for plane drawings of 1-connected 3-graphs? The algorithm by Kant [12] does not work, as it may change the combinatorial embedding. (This can be seen already from the fact that Kant's algorithm achieves $\frac{n}{2} + 1$ bends, while the lower bound for plane drawings of 3-graphs is $\frac{5}{6}n - 1$ bends.) The cited upper bound results from an algorithm for 4-graphs [2], and can thus likely be improved.

- The algorithm by Leiserson [15] creates small non-planar drawings of planar graphs, but has not been analyzed with respect to the constants involved in the area, and with respect to the number of bends. Is there an algorithm that draws every planar graph in $\mathcal{O}(n \log^2 n)$ area (preferably with small constant) and with at most $\frac{11}{7}n$ bends?

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# A Connectivity-proofs

We now prove the claims on the connectivity for $c = 2, 3, 4$. To limit the excessive number of cases, we restrict the attention to the simple graphs, and leave the rest to the reader.

We use the following notations: Two paths $P_1$ and $P_2$ will be called *fully vertex-disjoint* if no vertex belongs to both $P_1$ and $P_2$, and *weakly vertex-disjoint* if $P_1$ and $P_2$ may begin or end at the same vertex, but are otherwise vertex-disjoint.

## A.1 Graphs $N[\Delta, c, \alpha](k)$ and $N_{pl}[\Delta, c, \alpha](k)$

### A.1.1 4-connectivity

**Lemma 7** *$N[4, 4, s](k)$ is 4-connected for all $k \geq 2$.*

**Proof:** Recall that $N[4, 4, s](k)$ was built using $k$ copies $K_4^1, \ldots, K_4^k$ of $K_4$. Let the four vertices of $K_4^i$ be $v_1^i, v_2^i, v_3^i$ and $v_4^i$, named such that the added edges are $(v_2^i, v_1^{i+1})$ and $(v_4^i, v_3^{i+1})$, $i = 1, \ldots, k$ (addition is modulo $k$). See Figure 6(a).

Let $w_1$ and $w_2$ be two arbitrary vertices of $N[4, 4, s](k)$. We will show that there are four weakly vertex-disjoint paths from $w_1$ to $w_2$; this proves the claim by Menger's theorem. By symmetry we may assume that $w_1$ belongs to $K_4^1$. Let $l$ be such that $w_2$ belongs to $K_4^l$.

If $l > 1$, then define the following four fully vertex-disjoint paths connecting the four vertices of $K_4^1$ with the four vertices of $K_4^l$ (see Figure 6(a)):

- $v_2^1 - v_1^2 - v_2^2 - v_1^3 - \ldots - v_2^{l-1} - v_1^l$,
- $v_4^1 - v_3^2 - v_4^2 - v_3^3 - \ldots - v_4^{l-1} - v_3^l$,
- $v_1^1 - v_2^k - v_1^k - v_2^{k-1} - \ldots - v_1^{l+1} - v_2^l$, and
- $v_3^1 - v_4^k - v_3^k - v_4^{k-1} - \ldots - v_3^{l+1} - v_4^l$,

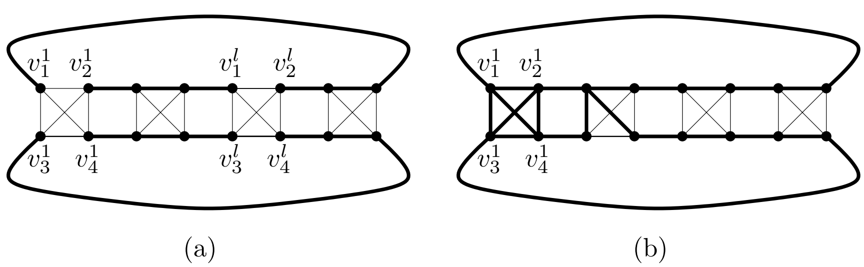


Figure 6: $N[4, 4, s](k)$ is 4-connected: (a) Four fully vertex-disjoint paths can be found between $K_4^1$ and $K_4^l$, and (b) $N[4, 4, s](k)$, $k \geq 2$ contains a subdivision of $K_5$.

For each of these paths, vertex $w_1$ either is its endpoint in $K_4^1$, or is incident to its endpoint in $K_4^1$, and $w_2$ either is its endpoint in $K_4^l$ or incident to the endpoint in $K_4^l$. Thus, the four paths can be completed to four weakly vertex-disjoint paths connecting $w_1$ and $w_2$.

If $l = 1$, then for $k \geq 2$ $N[4,4,s](k)$ contains a subdivision of $K_5$ such that each vertex in $K_1^1$ is mapped to a vertex of $K_5$. See Figure 6(b). Four weakly vertex-disjoint paths exist between any two vertices in a $K_5$, and therefore also between any two vertices in $K_4^1$. □

**Lemma 8** *$N_{pl}[4,4,s](k)$ is 4-connected for $k \geq 2$.*

**Proof:** The proof the same as the proof of the above lemma, except that the 4-wheel $W$ plays the role of $K_4$ and the octahedron $O$ plays the role of $K_5$. Recall that $N_{pl}[4,4,s](k)$ was built using $k$ copies $W^1, \ldots, W^k$ of $W$. Let the five vertices of $W^i$ be $v_1^i, \ldots, v_5^i$, named such that the added edges are $(v_2^i, v_1^{i+1})$ and $(v_4^i, v_3^{i+1})$, $i = 1, \ldots, k$ (addition is modulo $k$). See Figure 7(a).

Let $w_1$ and $w_2$ be two arbitrary vertices of $N_{pl}[4,4,s](k)$. We will show that there are four weakly vertex-disjoint paths from $w_1$ to $w_2$; this proves the claim by Menger's theorem. By symmetry we may assume that $w_1$ belongs to $W^1$. Let $l$ be such that $w_2$ belongs to $W^l$.

If $l > 1$, then define the same four fully vertex-disjoint paths between $W^1$ and $W^l$ as in the proof of Lemma 7; see also Figure 7. Every vertex in $W$ can be connected to the four vertices of degree 3 with four weakly vertex-disjoint paths. Thus, the four paths above can be completed to four weakly vertex-disjoint paths connecting $w_1$ and $w_2$.

If $l = 1$, then for $k \geq 2$ $N_{pl}[4,4,s](k)$ contains a subdivision of the octahedron $O$ such that every vertex of $W^1$ is mapped to a vertex of $O$. See Figure 7(b). Because the octahedron is a triangulated planar graph without a separating triangle, it is 4-connected. Thus four weakly vertex-disjoint paths exist between any two vertices in an octahedron, and therefore also between any two vertices in $W^1$. □

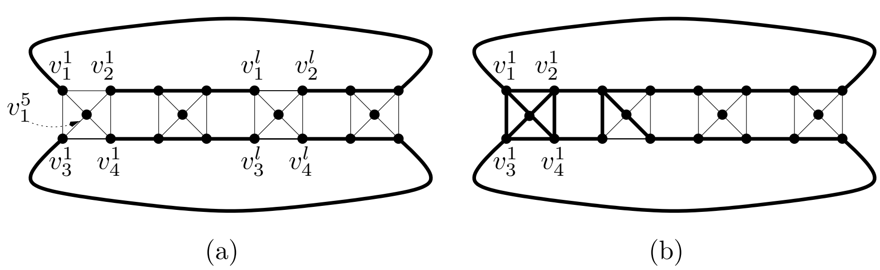


Figure 7: $N_{pl}[4,4,s](k)$ is 4-connected: (a) Four vertex-disjoint paths can be found between $W^1$ and $W^l$, and (b) $N_{pl}[4,4,s](k)$, $k \geq 2$ contains a subdivision of the octahedron.

### A.1.2 3-connectivity

**Lemma 9** *$N[4,3,s](k)$ is 3-connected for $k \geq 3$.*

**Proof:** In Figure 8(a) we show the *$k$-prism* $P_k$ which has $2k$ vertices. This graph is 3-connected for $k \geq 3$, because it is the graph of a convex polyhedron (see also Figure 10). $N[4,3,s](k)$ can be derived from $P_k$ as follows: Subdivide all edges of $P_k$ and replace each original vertex with a quadruple edge with three edges subdivided.

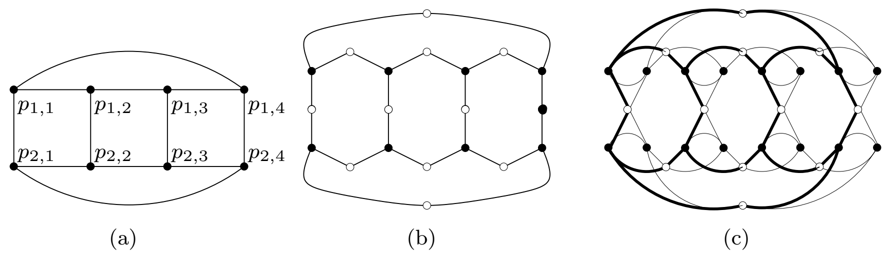


Figure 8: (a) The $k$-prism ($k = 4$ in this example). (b) By subdividing the edges of a $k$-prism, and (c) substituting a subdivided quadruple edge for each vertex of the $k$-prism, we obtain $N[4,3,s](k)$. This graph contains a subdivision of the $k$-prism.

Let $w_1$ and $w_2$ be two arbitrary vertices of $N[4,3,s](k)$. We will show that there are three weakly vertex-disjoint paths from $w_1$ to $w_2$; this proves the claim by Menger's theorem.

Let $p_{i,j}$ be the vertex in the $i$th row and $j$th column of $P_k$ as shown in Figure 8(a). Let $Q_{i,j}$ be the subdivided quadruple edge that replaces $p_{i,j}$ in $N[4,3,s](k)$.

Assume first that $w_1$ and $w_2$ belong to different copies of the subdivided quadruple edge, say $w_1$ belongs to $Q_{i_1,j_1}$ and $w_2$ belongs to $Q_{i_2,j_2}$. There are three weakly vertex-disjoint paths from $p_{i_1,j_1}$ to $p_{i_2,j_2}$ in $P_k$ because $P_k$ is 3-connected. These paths can be transformed to three fully vertex-disjoint paths in $N[4,3,s](k)$ between the three subdivision vertices of $Q_{i_1,j_1}$ and the three subdivision vertices of $Q_{i_2,j_2}$, because $N[4,3,s](k)$ contains a subdivision of the $k$-prism (see Figure 8(c) and Figure 9). From every vertex in $Q_{i,j}$ we can find three weakly vertex-disjoint paths to the three subdivision-vertices of $Q_{i,j}$, therefore we can complete the paths to three weakly vertex-disjoint paths connecting $w_1$ and $w_2$.

Now assume that $w_1$ and $w_2$ belong to the same copy of the subdivided quadruple edge. If one of $w_1$ or $w_2$ is a subdivision vertex, then it also belongs to some other copy of a subdivided quadruple edge and we are done by the above case. So $w_1$ and $w_2$ are the original vertices of the quadruple edge, and hence connected by three (actually, four) weakly vertex-disjoint paths within the quadruple edge. □

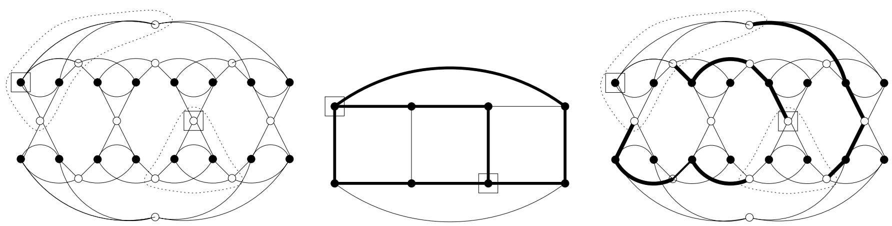

Figure 9: For any two vertices, we can find three weakly vertex-disjoint paths in the $k$-prism, and therefore also three fully vertex-disjoint paths in $N[4,3,s](k)$.

The proof that $N_{pl}[4,3,s](k)$ is 3-connected for $k \geq 2$ is identical to the above proof, except that octahedron replaces the quadruple edge. We leave the details to the reader.

**Lemma 10** *$N[3,3,s](k)$ is 3-connected for $k \geq 3$.*

**Proof:** In Figure 10 we show a polyhedron, the graph of which is the $k$-prism. Cutting off each corner of this polyhedron, we obtain a polyhedron the graph of which is $N[3,3,s](k)$, so this graph is triconnected. □

Figure 10: By cutting off the corners of the $k$-prism, we obtain the polyhedron of $N[3,3,s](k)$.

#### A.1.3 2-connectivity

**Lemma 11** *$N[4,2,s](k)$ , $N_{pl}[4,2,s](k)$, and $N[3,2,s](k)$ are 2-connected for $k \geq 2$.*

**Proof:** As shown in Figure 11, these graphs are Hamiltonian, i.e., they have a simple cycle with $n$ vertices. Any Hamiltonian graph is 2-connected. □

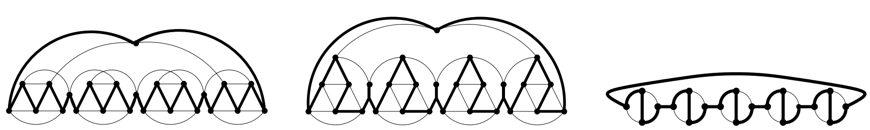

Figure 11: $N[4,2,s](k)$, $N_{pl}[4,2,s](k)$, and $N[3,2,s](k)$ are Hamiltonian.

## A.2 Graphs $P[\Delta, c, \alpha](k)$

**Lemma 12** *$P[4, c, s](k)$, $c = 3, 4$ is $c$-connected for $k \geq 3$.*

**Proof:** Recall that $P[4, c, s](k)$, $c = 3, 4$ consists of $k$ stacked cycles $C_1, \dots, C_k$ and that vertex $v_j^p$ is the $j$th vertex in clockwise order of cycle $C_i$. As a first step, define a path connecting $v_j^1$ with $v_j^k$, $j = 1, \dots, c$ as

$$v_j^1{=}v_{2j-1}^2 - v_{2j}^2{=}v_{2j-1}^3 - v_{2j}^3{=}v_{2j-1}^4 - \dots v_{2j}^{k-2}{=}v_{2j-1}^{k-1} - v_{2j}^{k-1}{=}v_j^k.$$

See also Figure 12. These $c$ fully vertex-disjoint paths will be called *spirals*.

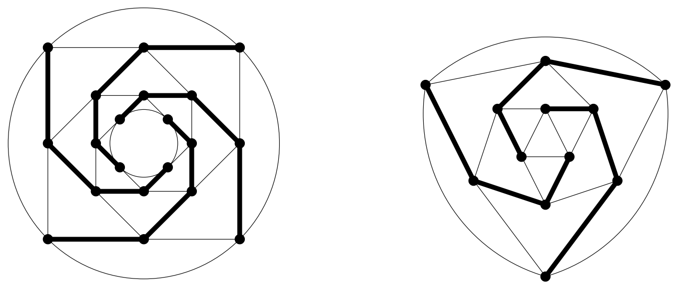

Figure 12: The spirals of $P[4, 4, s](6)$ and $P[4, 3, s](5)$.

Next, we show that for any vertex $w$ and any spiral $S$ that does not contain $w$, there are four weakly vertex-disjoint paths from $w$ to a vertex in $S$. Let $w$ belong to cycle $C_i$; since every vertex belongs to two cycles and by $k \geq 3$ we can choose $i$ such that $1 < i < k$. Two neighbors of $w$ are on $C_{i-1}$, thus we can find two weakly vertex-disjoint paths from $w$ to a vertex in $S \cap C_{i-1}$ using edges of $C_{i-1}$. Two neighbors of $w$ are on $C_{i+1}$, thus we can find two weakly vertex-disjoint paths from $w$ to a vertex in $S \cap C_{i+1}$ using edges of $C_{i+1}$. See Figure 13.

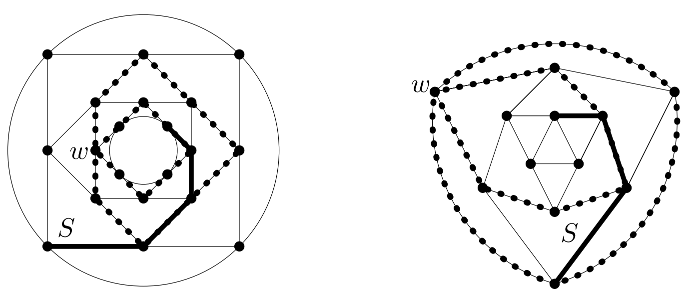


Figure 13: Any vertex $w$ not in $S$ has four weakly vertex-disjoint paths to a vertex in $S$.

Let $w_1, \dots, w_{c-1}$ be $c - 1$ arbitrary vertices of $P[4, c, s](k)$. We will show that the graph that results from removing these vertices is connected, hence $P[4, c, s](k)$ is $c$-connected.

Since $c-1$ vertices are removed, but there are $c$ spirals, at least one spiral, say $S$, remains intact. Let $w$ be an arbitrary vertex $\neq w_1, \ldots, w_{c-1}$. Then either $w \in S$, or there were four weakly vertex-disjoint paths from $w$ to a vertex in $S$ in $P[4,c,s](k)$. By $c \leq 4$, and because no vertex in $S$ is removed, one of these paths remains intact after removing $w_1, \ldots, w_{c-1}$, so $w$ is connected to a vertex in $S$. Thus, all vertices in the remaining graph are either in $S$, or connected to $S$, and $S$ itself is connected, so the remaining graph is connected. □

**Lemma 13** *$P[3,3,s](k)$ is 3-connected for $k \geq 2$.*

**Proof:** The proof is the same as in Lemma 12 with two exceptions: (a) The spirals are defined differently, because $C_i$ and $C_{i+1}$ are now connected by added edges rather than identified vertices. (b) For any vertex $w$ and any spiral $S$ with $w \notin S$, there are only three weakly vertex-disjoint paths from $w$ to vertices in $S$. These paths use the cycle $C_i$ containing $w$, and either $C_{i+1}$ or $C_{i-1}$ (whichever one contains a neighbor of $w$). See Figure 14. □

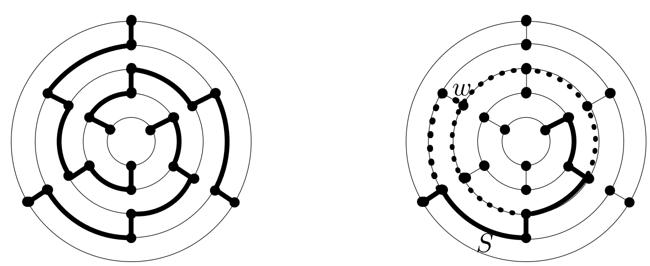


Figure 14: The new definition of spirals, and the three vertex-disjoint paths from $w$ to vertices in spiral $S$.

**Lemma 14** *$P[4,2,s](k)$ and $P[3,2,s](k)$ are 2-connected for $k \geq 2$.*

**Proof:** As shown in Figure 15, these graphs are Hamiltonian. Any Hamiltonian graph is 2-connected. □

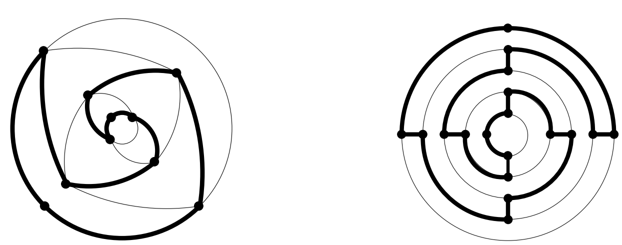

Figure 15: $P[4,2,s](k)$ and $P[3,2,s](k)$ are Hamiltonian.

# B Many stacked cycles

In this section, we prove Lemma 5 for the case $c = 2$, i.e., we prove that in any planar drawing of $P[\Delta, 2, \alpha](k)$, $\alpha \neq l$, $k \geq 3$, there exists an $i$, $1 \leq i < k$ such that the cycles $C_1, \ldots, C_i$ and the cycles $C_k, \ldots, C_{i+1}$ are stacked. We do this in detail for $P[4, 2, m](k)$, $k \geq 3$, and then sketch the proof for the other graph classes.

Recall that $P[4, 2, m](k)$ consists of $k$ cycles $C_1, \ldots, C_k$, where $C_1$ and $C_k$ have length 2 while all other cycles have length 4. In the original planar drawing, these cycles were *stacked*, i.e., all edges of $C_i$ were inside $C_{i+1}$, $i = 1, \ldots, k-1$. Let from now on an arbitrary planar drawing of $P[4, 2, m](k)$ be fixed. We will use the notion of *outside*: An edge is *outside* a cycle $C$ if it is not an edge of $C$ and not inside $C$. A cycle $C'$ is *outside* a cycle $C$ if all edges of $C'$ are outside $C$.

**Claim 1** *For $1 \leq i < k$, $C_i$ is either inside or outside $C_{i+1}$.*

**Proof:** Assume to the contrary that there exists an $i$ such that some edges of $C_i$ are inside $C_{i+1}$ and some are outside $C_{i+1}$. By $k \geq 3$, we have $i > 1$ or $i + 1 < k$. We assume the former, the other case is proved similarly using cycle $C_{i+2}$. So assume $i > 1$, thus $C_i$ is a 4-cycle with vertices $v_1^i, v_2^i, v_3^i, v_4^i$; vertices $v_2^i$ and $v_4^i$ also belong to $C_{i+1}$.

Because we have a planar drawing, and some edges of $C_i$ are inside and some are outside $C_{i+1}$, one vertex of $C_i$ which is not in $C_{i+1}$, say $v_1^i$, must be inside $C_{i+1}$, and the other such vertex $v_3^i$ must be outside $C_{i+1}$.

By $i > 1$, vertices $v_1^i$ and $v_3^i$ both belong to $C_{i-1}$ and are thus connected with a path $P$ using only edges of $C_{i-1}$. This path thus leads from inside $C_{i+1}$ (at $v_1^i$) to outside $C_{i+1}$ (at $v_3^i$). Because $C_{i+1}$ and $C_{i-1}$ are vertex-disjoint, this path must cross $C_{i+1}$ at an edge, a contradiction to planarity. See Figure 16(a). $\square$

**Claim 2** *For $1 < i < k$, either $C_{i+1}$ or of $C_{i-1}$ must be inside $C_i$.*

**Proof:** Let $v_1^i, \ldots, v_4^i$ be the four vertices of $C_i$. Assume that $C_{i+1}$ and $C_{i-1}$ are both not inside $C_i$, so they are both outside $C_i$ by Claim 1. In the induced planar drawing of the subgraph consisting of $C_i$, $C_{i-1}$ and $C_{i+1}$, cycle $C_i$ then is a face. Add an extra vertex $v_5$ inside this face and connect it to the vertices of $C_i$. Thus, using edges of the three circles, we obtain a planar drawing of $K_5$, a contradiction. See Figure 16(b). $\square$

Now we are ready to prove the main claim for $P[4, 2, m](k)$, $k \geq 3$.

**Lemma 15** *In any planar drawing of $P[4, 2, m](k)$, $k \geq 3$, there exists an $i$, $1 \leq i < k$, such that the cycles $C_1, \ldots, C_i$ are stacked, and the cycles $C_k, C_{k-1}, \ldots, C_{i+1}$ are stacked.*

**Proof:** Let $i$ be the smallest integer for which not all edges of $C_i$ are inside $C_{i+1}$; $i = k-1$ if no such integer exists. Thus, the cycles $C_1, \ldots, C_i$ are stacked.

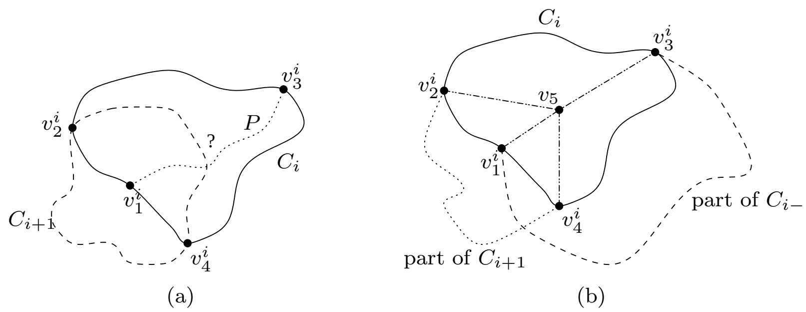


Figure 16: (a) Edges of $C_i$ cannot be both inside $C_{i+1}$ and outside $C_{i+1}$, otherwise cycle $C_{i-1}$ would cause a crossing. (b) One of $C_{i-1}$ and $C_{i+1}$ must be inside $C_i$, otherwise there would be a planar drawing of a $K_5$.

If $i = k-1$ then nothing is left to prove. If $i < k-1$, then not all edges of $C_i$ are inside $C_{i+1}$, so $C_i$ is not inside $C_{i+1}$, so by Claim 1 $C_i$ is outside $C_{i+1}$. By Claim 2, therefore $C_{i+2}$ must be inside $C_{i+1}$, which means that $C_{i+1}$ is outside $C_{i+2}$. Applying induction, one shows that for $j = i+1, \ldots, k-1$, $C_j$ is outside $C_{j+1}$; thus the cycles $C_k, \ldots, C_{i+1}$ are stacked. $\square$

**Lemma 16** *In any planar drawing of $P[3,2,m](k)$, $k \geq 3$, there exists an $i$, $1 \leq i < k$, such that the cycles $C_1, \ldots, C_i$ are stacked, and the cycles $C_k, C_{k-1}, \ldots, C_{i+1}$ are stacked.*

**Proof:** The proof is similar to the one of the previous lemma; we will only sketch an outline here. Claim 1 holds by planarity because $C_i$ and $C_{i+1}$ are vertex-disjoint. Claim 2 holds because otherwise we could again construct a planar drawing of $K_5$. Exactly as in the proof of Lemma 15 we can thus find the integer $i$. $\square$

**Lemma 17** *In any planar drawing of $P[4,2,s](k)$ and $P[3,2,s](k)$, $k \geq 3$, there exists an $i$, $1 \leq i < k$, such that the cycles $C_1, \ldots, C_i$ are stacked, and the cycles $C_k, C_{k-1}, \ldots, C_{i+1}$ are stacked.*

**Proof:** These simple graphs are obtained by subdividing two edges of the corresponding multigraph. For any planar drawing of the simple graph, we can remove the subdivision vertices and obtain a planar drawing of the multigraph; the claim thus holds by the above lemmas. $\square$