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Crossing Number of Simple 3-Plane Drawings

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Abstract. We study 3-plane drawings, that is, drawings of graphs in which every edge has at most three crossings. We show how the recently developed Density Formula for topological drawings of graphs [10] can be used to count the crossings in terms of the number n of vertices. As a main result, we show that every 3-plane drawing has at most $5.5(n-2)$ crossings, which is tight. In particular, it follows that every 3-planar graph on n vertices has crossing number at most $5.5n$, which improves upon a recent bound [3, 4] of $6.6n$. To apply the Density Formula, we carefully analyze the interplay between certain configurations of cells in a 3-plane drawing. As a by-product, we also obtain an alternative proof for the known statement that every 3-planar graph has at most $5.5(n-2)$ edges.

1 Introduction

One of the most basic combinatorial questions one can ask for a class of graphs is: How many edges can a graph from this class have as a function of the number n of vertices? Prominent examples include upper bounds of $\binom{n}{2}$ for the class of all graphs and $\frac{n^2}{4}$ for bipartite graphs. These bounds

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are immediate consequences of the definition of these graph classes, and they are tight, that is, there exist graphs in the class with exactly this many edges. But for several other graph classes good upper bounds on the number of edges are much more challenging to obtain. Notably, this holds for classes that relate to the existence of certain geometric representations. One of the most fundamental questions one can ask about a class of geometrically represented graphs is: What is the minimum number of edge crossings required in such a representation, as a function of the number n of vertices? We study both of these fundamental questions in combination, for the class of 3-planar graphs. A graph is *k-planar* if it can be drawn in the plane such that every edge has at most k crossings. The study of k-planar graphs goes back to Ringel [17] and has been a major focus in graph drawing over the past two decades [9], as a natural generalization of planar graphs ($k = 0$).

The maximum number of edges in a simple k-planar graph on n vertices is known to be at most $c_k(n-2)$, where $c_0 = 3$, $c_1 = 4$ [6], $c_2 = 5$ [15, 16], $c_3 = 5.5$ [11, 12], $c_4 = 6$ [1], and $c_k \leq 3.72\sqrt{k}$, for general $k \geq 5$ [7, Corollary 8]. The bounds for $k \leq 2$ are tight and those for $k \leq 4$ are tight up to an additive constant [1, 5]. The bounds for $k \leq 4$ also generalize to *non-homotopic* drawings of multigraphs [13, 14], that is, drawings where every continuous transformation that transforms one copy of an edge to another passes over a vertex. Interestingly, the upper bound for 3-planar graphs is tight in this more general setting only [5, 7].

The *crossing number* of a drawing Γ is the number of edge crossings in Γ. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum crossing number over all drawings of G. By definition every k-planar graph G admits a k-plane drawing and thus

$$\mathrm{cr}(G) \leq \frac{km}{2}, \tag{S}$$

where m denotes the number of edges in G. For a k-planar graph, this simple inequality connects upper bounds on the number of edges with lower bounds on the crossing number. Both of these come together in the well-known Crossing Lemma [2, Chapter 45], as the best constants in the Crossing Lemma are obtained by analyzing k-plane drawings [1, 7, 11, 12]. Conversely, combining the lower bound on $\mathrm{cr}(G)$ from the Crossing Lemma with an upper bound on $\mathrm{cr}(G)$ we obtain an upper bound on the number of edges in G. While (S) would work here, it is probably not an ideal choice because the graphs for which (S) is tight might be very different from those graphs that have a maximum number of edges, for any fixed n. For instance, for a 1-planar graph G we have $\mathrm{cr}(G) \leq n - 2$ [18, Proposition 4.4], which beats the bound we get by plugging $m \leq 4n - 8$ into (S) by a factor of two. Can we obtain similar improvements by bounding $\mathrm{cr}(G)$ in terms of n, rather than m, for $k \geq 2$?

Indeed, very recently it has been shown that $\mathrm{cr}(G) \leq \mathrm{cr}_2(G) \leq 3.\overline{3}n$ if G is 2-planar [3, 4, Theorem 3] and $\mathrm{cr}(G) \leq \mathrm{cr}_3(G) \leq 6.6n$ if G is 3-planar [3, 4, Theorem 4]. (The *k-planar crossing number* $\mathrm{cr}_k(G)$ is similar to the crossing number, except that the minimum is taken over all k-plane drawings of G.) There is some indication that the bound for 2-planar graphs could be tight up to an additive constant, as it is achieved by the standard drawings of optimal 2-planar graphs (Figure 1). But the crossing number of these graphs is not known.

In contrast, there exists a family of simple 3-planar graphs with $5.5n - 15$ edges whose standard drawings have $5.5n - 21$ crossings (Figure 2). Thus, there is a gap of $1.1n$ between the lower and the upper bound for the crossing number of 3-plane drawings.

Results. We close the gap and present an upper bound on the crossing number of 3-plane drawings that is tight up to an additive constant. Using the same approach we also obtain an alternative proof to show that a 3-planar n-vertex graph has at most $5.5(n-2)$ edges.

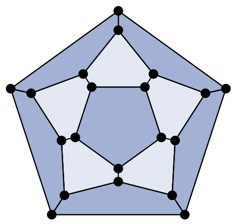


Figure 1: Construction of a 2-plane drawing of a simple n-vertex graph with $3.\overline{3}(n-2)$ crossings and $5(n-2)$ edges by Pach and Tóth [16, Figure 3]. Left: A plane drawing with pentagonal faces. Right: To each pentagonal face all diagonals are added.

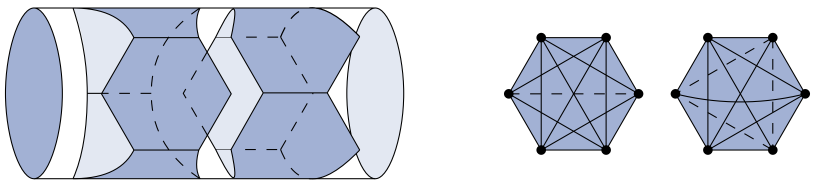


Figure 2: Construction of a 3-plane drawing of a simple n-vertex graph with $5.5n-21$ crossings and $5.5n-15$ edges by Pach and Tóth [12, Figure 8]. Left: A cylinder with two layers, each consisting of three hexagonal faces. Right: To each face of a layer all but one diagonal is added. To the top and bottom face six diagonals can be added without creating parallel edges. Missing diagonals are represented by dashed lines.

Theorem 1. *Every non-homotopic* 3*-plane drawing of a graph on* n *vertices,* $n \geq 3$*, contains at most* $5.5(n-2)$ *edges and at most* $5.5(n-2)$ *crossings.*

Our proof relies on the recently developed Density Formula (cf. Theorem 4 below) for topological drawings of graphs [10]. It relates the number of vertices, edges, and cells of various sizes in a drawing, in a way similar to the Euler Formula in the case of plane graphs. Previously, the Density Formula has been used to derive upper bounds on the number of edges in k-plane drawings, for $k \leq 2$ [10]. In order to apply it to 3-plane drawings, to bound the number of crossings, and to obtain tight bounds, we study cells not only in isolation but also as part of what we call *configurations*, which consist of several connected cells. We then develop a number of new constraints that relate the number of cells and/or configurations of a certain type in any 3-plane drawing. The combination of all these constraints with the Density Formula yields a linear program that we can solve in two different ways—maximizing either the number of edges or the number of crossings—to prove Theorem 1.

As with other proofs involving the Density Formula [10], our technique can be seen as a variant of the discharging method. But we streamline the arguments so that we prove each constraint ad hoc by rather simple means, usually requiring only an easy double counting.

Using Theorem 1 we can derive better upper bounds on the number of edges in k-planar graphs without short cycles. Plugging our bound of at most $5.5n$ crossings into the proofs from [3, 4] we obtain the following.

Theorem 2. *For every integer* n*, each of the following holds.*

- *Every C_3-free 3-planar graph on n vertices has at most* $\sqrt[3]{891/8}\,n < 4.812n$ *edges (down from* $\approx 5.113n$ *[3, 4, Theorem 18]).*
- *Every C_4-free 3-planar graph on n vertices has at most* $\sqrt[3]{1\,254\,825/12\,544}\,n < 4.643n$ *edges (down from* $\approx 4.933n$ *[3, 4, Theorem 20]).*
- *Every 3-planar graph of girth 5 on n vertices has at most* $\sqrt[3]{122\,793/1600}\,n < 4.25n$ *edges (down from* $\approx 4.516n$ *[3, 4, Theorem 21]).*

2 Preliminaries

We consider *drawings* of graphs on the sphere with vertices as points, edges as Jordan arcs, and the usual assumption that any two edges share only finitely many points, each being a common endpoint or a proper crossing, and that no three edges cross in the same point. As is customary, we do not distinguish between the points and curves in Γ and the vertices and edges of G they represent, respectively. We also assume the drawings to be *simple*[1], that is, no edge crosses itself, no two adjacent edges cross, and any two edges cross at most once. We explicitly allow our graphs to contain parallel edges, but no loops. Hence, our graphs are not necessarily simple, while our drawings are simple. In order to avoid an arbitrary number of parallel edges within a small corridor, a drawing Γ is called *non-homotopic* if every region that is bounded by two parallel edges, called a *lens*, contains a crossing or a vertex in its interior; see Figure 3.

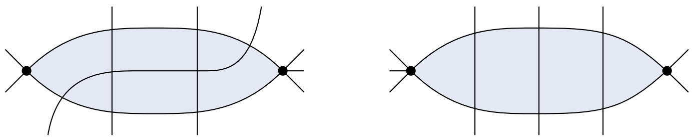


Figure 3: Left: A lens (blue) with two crossings in its interior and two vertices on its boundary. Such a lens can be part of a non-homotopic drawing. Right: An *empty* lens (blue), i.e., a lens that contains neither a vertex nor a crossing. Such a lens cannot be part of a non-homotopic drawing.

Let Γ be a drawing of a graph $G = (V, E)$. If every edge is crossed at most three times, we say that Γ is 3-*plane*. We denote the set of crossings by X. For $i \in \{0, 1, 2, 3\}$, let $E_i \subseteq E$ be the set of all edges with exactly i crossings, and let $E_\times = E_1 \cup E_2 \cup E_3$.

We easily observe the following.

Observation 3. *If Γ is a 3-plane drawing, we have*

$$|E| = |E_\times| + |E_0| \tag{9.A}$$

$$|E_\times| = |E_1| + |E_2| + |E_3| \tag{8.A}$$

$$2|X| = |E_1| + 2|E_2| + 3|E_3| \tag{8.B}$$

Edge-Segments and Cells. An edge with i crossings is split into $i + 1$ parts, called *edge-segments*. An edge-segment is *inner* if both its endpoints are crossings, *uncrossed* if both its endpoints are vertices and *outer*, otherwise. The *planarization* of Γ is the graph obtained by

[1] Our definition is slightly more general than what is usually meant in the literature, namely that "any two edges have at most one point in common". Our simple drawings allow non-crossing parallel edges.

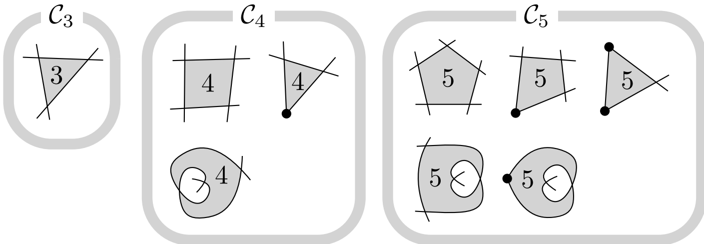


Figure 4: Taken from [10, Figure 2]. All types of cells c of size $\|c\| \leq 5$ in a non-homotopic connected drawing on at least three vertices. The bottom row shows the degenerate cells. Since we consider only simple drawings in this paper, degenerate cells do not appear.

replacing every crossing x with a vertex of degree 4 that is incident to the four edge-segments of x. We say that the drawing Γ is *connected*, if its planarization is a connected graph, and shall henceforth only consider connected drawings. Removing all edges and vertices of Γ splits the sphere into several components, called *cells*. We denote the set of all cells by $\mathcal{C}$. Since Γ is connected, the *boundary* ∂c of a cell c corresponds to a cyclic sequence alternating between edge-segments and elements in $V \cup X$ (i.e., vertices and crossings). If a crossing or a vertex appears multiple times on the boundary of the same cell c, then c is *degenerate*. The *size* of a cell c, denoted by $\|c\|$, is the number of vertex incidences plus the number of edge-segment incidences of c. Note that incidences with crossings are not taken into account. Figure 4 shows all types of cells of size at most 5. Note that, as we forbid two adjacent crossing edges and two edges crossing more than once, all cells of size at most 5 in our drawings are non-degenerate. For $a \in \mathbb{N}$, we denote by $\mathcal{C}_a = \{c \in \mathcal{C} \colon \|c\| = a\}$ the set of all cells of size a.

Theorem 4 (Density Formula [10]). *If Γ is a connected drawing with at least one edge and t is a real number, then*

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} \|c\| - t \right) - |X|.$$

To apply the Density Formula, we count the cells of different sizes. We distinguish several types of cells based on their size and boundary and denote these by small pictograms, such as or . We call a cell *large* if it has size at least 6 and write for this type of cells. By abuse of notation, we denote the number of cells of a certain type by their pictogram.

Configurations and Trails. Configurations are arrangements of several cells and edge-segments in a drawing Γ of a graph G. Formally, a *configuration* is a connected labeled embedded subgraph H of the planarization Λ of Γ. The label of a vertex $v \in V(H)$ indicates whether v arises from a vertex of G or a crossing. Two configurations are of the same *type* if they are isomorphic as labeled embedded graphs (up to reflection). We also denote the types by small pictograms such as and , see Figure 5 (left) for an example.

A cell c is *interior* to a configuration C if all of its boundary ∂c is part of C. We color interior cells gray in the pictograms. An edge-segment is *interior* to C if both its incident cells are interior to C. For example, a -configuration C has an interior -cell and an interior -cell, whose shared outer edge-segment is the only interior edge-segment of C.

Now, trails are specific configurations. Consider a sequence $(A_1, \ldots, A_\ell)$ of $\ell \geq 2$ cells. The configuration T that is the union of $A_1, \ldots, A_\ell$ is a *trail* if

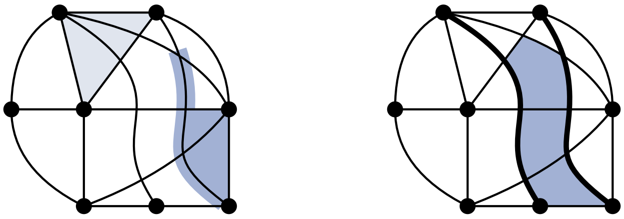


Figure 5: Left: A -configuration (light blue) and a -configuration (dark blue). Right: A - - trail (dark blue) and its bounding edges (thick).

- neither A_1 nor A_ℓ is a -cell,
- for each $i \in [\ell-1]$ cells A_i and A_{i+1} share exactly one inner edge-segment (This edge-segment is then interior to T.), and
- if $\ell \geq 3$, then each of $A_2, \ldots, A_{\ell-1}$ is a -cell c whose two interior edge-segments in T are opposite on the boundary ∂c of c.

See Figure 5 (right) for an example. Note that only two of the four inner edge-segments of each -cell of T are interior to T (as no edge self-intersects). Every trail T has two *bounding edges* e_1 and e_2, which are those crossing all interior edge-segments of T. An A-B-trail is a trail whose endpoints are cells of type A and B respectively. We denote by $(A \leftrightarrow B)$ the number of A-B-trails in Γ. Note in particular that $(A \leftrightarrow B) = (B \leftrightarrow A)$.

Observation 5. *Every inner edge-segment of a drawing is interior to exactly one trail.*

3-Saturated Drawings. Considering a 3-plane drawing, it is convenient to assume that any two vertices are connected by an uncrossed edge whenever this is obviously possible, i.e., whenever both vertices appear on a common cell. A drawing is *filled* if for every cell c and every pair of vertices $u \neq v$ on the boundary ∂c of c, there exists an uncrossed edge uv in ∂c. In other words, a drawing Γ is filled if adding any uncrossed edge to Γ results in a lens bounded by two parallel uncrossed edges. For brevity, we define a 3-*saturated* drawing to be a 3-plane, non-homotopic, connected, filled drawing of a graph on at least three vertices. Note that a 3-saturated drawing may contain parallel edges, but also recall that all our drawings are simple.

In every 3-saturated drawing, all cells of size at most five are of one of the following types: , , , , , and ; cf. Figure 4.

Lemma 1. *Every* 3*-plane drawing* Γ *of a connected graph* G *on at least three vertices can be completed to a* 3*-saturated drawing by only adding edges.*

Proof: Note that the drawing Γ is connected as G is connected. We therefore only need to enforce the filled property, while preserving all other properties. Suppose there is a cell c with vertices u and v on its boundary, but no edge uv lies on ∂c. We can insert an edge uv within the cell c, without creating new crossings. The edge might be a parallel edge. Yet, as no edge connecting u and v lies on the boundary ∂c, we did not create an empty lens. Therefore, the drawing is still non-homotopic. Neither does this create self-crossings or crossings between adjacent edges. Inductively, the claim follows. □

By Lemma 1, it suffices to consider only 3-saturated drawings in order to obtain upper bounds on the number of crossings and edges in any 3-plane drawing.

3 Bounds on Edge-Density and Crossing-Number of 1-, 2- and 3-Plane Drawings

To obtain our upper bounds we prove a number of (in)equalities, each relating the number of certain cells, configurations, edges and crossings. The Density Formula is one such equality. In total, we obtain a system of linear inequalities where each quantity (such as $|E|$, $(|V|-2)$, $|X|$, $|\mathcal{C}_2|$, $|E_1|$, , , etc.) can be considered as a non-negative variable. Setting the "variable" $(|V|-2)$ to 1, we can maximize the value of $|X|$ by solving the obtained linear program (LP). The resulting maximum represents the number of crossings per vertex; more precisely, per $(|V|-2)$.

Suppose we want to prove that the number of crossings in any drawing of a certain type (here we consider 1-, 2-, and 3-plane drawings) on n vertices does not exceed $k(n-2)$ for some fixed k. For this, it suffices to show under the constraints of the LP (where we have normalized $(|V|-2) = (n-2)$ to 1) that the maximum value of the "variable" $|X|$ is at most k. In fact, from what is known as the *dual* of the LP, we obtain a coefficient for each of the LP's inequalities such that multiplying each inequality with the corresponding coefficient and summing the inequalities up yields the desired bound $|X| \leq k(|V|-2)$.

Since we believe that this technique, which may be applicable to other types of drawings, is of independent interest, we first illustrate it on two known examples, namely bounding the crossing number in 1-plane and in 2-plane drawings, and only then turn to 3-plane drawings.

Crossing Number of 1-Plane Drawings. We first illustrate the method on the following known bound for 1-plane drawings.

Theorem 6 ([18, Proposition 4.4])**.** *Every 1-plane drawing Γ of a graph G on n vertices with $n \geq 3$ contains at most $n-2$ crossings.*

In order to prove Theorem 6, we first derive two inequalities relating the number of crossings, edges and vertices in 1-plane drawings. Let Γ be a 1-plane drawing on at least three vertices. Removing uncrossed edges and considering the resulting components separately, we may assume that Γ is connected and only contains crossed edges. We may further assume that Γ contains at least three vertices as this holds as soon as we have one crossing. The characterization of cells of small size [10] (cf. Figure 4) implies that Γ contains only cells of size at least 6. Thus, the Density Formula for $t = 3$, (cf. Theorem 4) simplifies to

$$|X| = 3(|V|-2) - \sum_{c \in \mathcal{C}, \|c\| \geq 6} \left(\frac{1}{2}\|c\| - 3\right) - |E| \leq 3(|V|-2) - |E|. \tag{0.A}$$

Moreover, using the fact that every edge in Γ has at most one crossing (in fact, exactly one crossing), a simple double counting argument yields

$$2|X| \leq |E|. \tag{0.B}$$

Observe that (0.A) and (0.B) can be considered as homogeneous linear inequalities in the variables $|X|, |E|, |V|-2$ and thus any set of values for these variables that satisfy the inequalities can be scaled arbitrarily. Hence, by normalizing $|V|-2$ to 1, we transform the set of inequalities (left) into an LP with non-negative variables represented by the matrix M_1 (right).

$$\begin{cases} |X| + |E| \leq 3(|V|-2), & \text{(0.A)} \\ 2|X| - |E| \leq 0, & \text{(0.B)} \end{cases} \quad \longleftrightarrow \quad M_1 = \begin{pmatrix} 1 & 1 & \bigg| & 3 \\ 2 & -1 & \bigg| & 0 \end{pmatrix} \begin{matrix} \text{(0.A)} \\ \text{(0.B)} \end{matrix}$$

(columns of M_1: $|X|$, $|E|$, $|V|-2$)

We want to determine an upper bound on $|X|$ in terms of $|V|-2$ solely by multiplying the inequalities (0.A) and (0.B) with constant coefficients and summing up. This corresponds to maximizing $|X|$ in the LP. It turns out that in the LP represented by M_1, the maximum value of the "variable" $|X|$ is 1. Computing an optimal solution of the dual LP, we obtain a vector c_1 such that all entries of the vector $s_1 = c_1 \cdot M_1$ are non-negative and the entry corresponding to the "variable" $|X|$ is 1, i.e., the maximum possible value of $|X|$ under the given constraints. Here, we obtain for $c_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ the following.

$$s_1 = c_1 \cdot M_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \overset{\begin{matrix} |X| & |E| & |V|-2 \end{matrix}}{\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array}\right)} = \overset{\begin{matrix} |X| & |E| & |V|-2 \end{matrix}}{\left(\begin{array}{cc|c} 1 & 0 & 1 \end{array}\right)}.$$

The resulting vector $s_1 = \left(\begin{array}{cc|c} 1 & 0 & 1 \end{array}\right)$ can now be reinterpreted as the inequality $|X| \leq |V|-2$. In fact, the entries of the vector c_1 correspond to the coefficients with which we need to multiply the inequalities (0.A) and (0.B) such that summing up yields $|X| \leq |V|-2$. This concludes the proof of Theorem 6.

Crossing Number of 2-Plane Drawings. In [10], the Density Formula is used to show that every 2-plane drawing of a graph on n vertices contains at most $5n-10$ edges. The statement follows from a lower bound on the number of crossings (cf. Lemma 2 (1.A)). From the upper bound on the number of edges, the authors of [3, 4], derive an upper bound on the number of crossings in 2-plane drawings.

Theorem 7 ([3, 4, Theorem 3])**.** *Every 2-plane drawing Γ of a graph G on n vertices with $n \geq 3$ contains at most $3.\overline{3}(n-2)$ crossings.*

They prove that at most two thirds of the edges of a filled 2-plane drawing are crossed (cf. Lemma 2 (1.B)). We reformulate the proof based on our method. To this end, we first collect several observations about local interactions of cells, crossings, and edges.

Lemma 2 ([10, Lemma 8.2, Lemma 3.1], [3, 4, Theorem 3])**.** *For every connected, non-homotopic 2-plane drawing Γ on $n \geq 3$ vertices, we have*

$$3\,\triangledown_3 + 2\,\boxplus_4 + \triangle \leq |X|, \tag{1.A}$$

$$|E_\times| \leq 0.\overline{6}|E|, \tag{1.B}$$

$$|E| \leq 5(|V|-2) + 2\,\triangledown_3 + \boxplus_4 + \triangle - |X|, \tag{1.C}$$

$$2|X| \leq 2|E_\times|. \tag{1.D}$$

Proof: For a proof of (1.A), we refer to [10, Lemma 8.2], a proof of (1.B) is given in [3, 4, Theorem 3]. The constraint (1.C) follows from the Density Formula for $t=5$ (cf. Theorem 4). To prove (1.D), observe that every edge is crossed at most twice and every crossing is incident to two distinct edges in any 2-plane drawing. A simple double-counting argument now yields (1.D). □

Normalizing the variable corresponding to $|V|-2$ in the inequalities (1.A),(1.B),(1.C) and (1.D) yields an LP with non-negative variables that can be represented by the matrix M_2 with

$$M_2 = \overset{\begin{matrix} |X| & |E| & |E_\times| & \triangledown_3 & \boxplus_4 & \triangle & |V|-2 \end{matrix}}{\left(\begin{array}{cccccc|c} -1 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & -1 & -1 & 5 \\ 2 & 0 & -2 & 0 & 0 & 0 & 0 \end{array}\right)} \begin{matrix} \text{(1.A)} \\ \text{(1.B)} \\ \text{(1.C)} \\ \text{(1.D).} \end{matrix}$$

Under the given constraints, the maximum value of the "variable" $|X|$ is $\frac{10}{3}$. In fact, solving the dual LP, we obtain the vector $c_2 = \begin{pmatrix}\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{2}\end{pmatrix}$ that satisfies both $c_2 \geq 0$ and $c_2 \cdot M_2 = s_2$ for $s_2 = \begin{pmatrix}1 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 \mid \frac{10}{3}\end{pmatrix}^{\mathrm{T}}$. The vector s_2 can now be reinterpreted as the inequality

$$|X| + \frac{2}{3}\triangledown_3 + \frac{2}{3}\square_4 \leq \frac{10}{3}(|V| - 2). \tag{1.E}$$

As the quantities $\triangledown_3$ and $\square_4$ are non-negative, we obtain $|X| \leq 3.\overline{3}(|V| - 2)$, which concludes the proof of Theorem 7. Again, note that summing up (1.A)–(1.D) with the coefficients from c_2 yields precisely (1.E). Interpreting a set of inequalities as an LP enables us to easily compute a best-possible bound and the dual coefficients to witness it.

Crossing Number of 3-Plane Drawings. We derive the inequalities for 3-plane drawings in Section 4. Our LP comprises 21 constraints, which are summarized in Figure 13 in the appendix, and whose validity is proven in Section 4. Summing up all constraints using the coefficients shown in Figure 13, we obtain $|X| \leq 5.5(|V| - 2)$.

If we maximize $|E|$ instead of $|X|$ in the LP, we obtain $|E| \leq 5.5(|V| - 2)$ from the same constraints (witnessed by different coefficients; also in Figure 13). Hence, by verifying that all 21 constraints hold for every 3-saturated drawing, we obtain our result by Lemma 1.

As in the case of 1-plane drawings, the LP for 3-plane drawings can be modeled as a matrix M where each column corresponds to a non-negative variable v_i and each row to an inequality of the form $\sum_i a_i v_i \leq b(|V| - 2)$ with constant, integer factors a_i and b. Solving the dual LP, we obtain a vector $c_\times$ such that $s_\times = c_\times \cdot M$ is a vector where each entry is non-negative and the entry corresponding to the number of crossings $|X|$ is 1, the one corresponding to $|V| - 2$ is 5.5. This witnesses $|X| \leq 5.5(|V| - 2)$. Similarly, there exists a vector c_E such that $c_E \cdot M$ yields $|E| \leq 5.5(|V| - 2)$. The explicit definition of M, $c_\times$ and c_E is given in Section A.

4 Relating Crossing, Edge, Cell, Trail, and Configuration Counts

In this section, we present a number of (in)equalities, each relating the number of certain cells, configurations, edges, or crossings. Our proof relies on the Density Formula for $t = 5$. For this value of t, $\bigcirc$-cells contribute negatively in the formula. Intuitively, large cells account for many crossings: If many trails end in large cells, we obtain a lower bound on the sum $\sum_{a \geq 6} a|\mathcal{C}_a|$ of sizes of large cells. This yields a lower bound on the sum $\sum_{c \in \mathcal{C}_{\geq 6}}(\|c\| - 5)$ in the Density Formula, where $\mathcal{C}_{\geq 6}$ denotes the set of large cells. If there are few such trails, we obtain configurations that contain many crossed edges.

4.1 Lower Bounds on the Number of Cells and Configurations

Lemma 3. *If* Γ *is a* 3*-saturated drawing, then*

$$\triangle_4 = (\triangle_4 \leftrightarrow \square_5) + (\triangle_4 \leftrightarrow \pentagon_5) + (\triangle_4 \leftrightarrow \bigcirc) \tag{2.A}$$

$$2\,\square_5 = (\triangle_4 \leftrightarrow \square_5) + 2\,(\square_5 \leftrightarrow \square_5) + (\square_5 \leftrightarrow \triangledown_3) + (\square_5 \leftrightarrow \pentagon_5) + (\square_5 \leftrightarrow \bigcirc) \tag{2.B}$$

$$3\,\triangledown_3 = (\triangledown_3 \leftrightarrow \square_5) + (\triangledown_3 \leftrightarrow \pentagon_5) + (\triangledown_3 \leftrightarrow \bigcirc) \tag{2.C}$$

$$5\,\pentagon_5 = (\triangledown_3 \leftrightarrow \pentagon_5) + (\pentagon_5 \leftrightarrow \triangle_4) + (\pentagon_5 \leftrightarrow \square_5) + 2\,(\pentagon_5 \leftrightarrow \pentagon_5) + (\pentagon_5 \leftrightarrow \bigcirc) \tag{2.D}$$

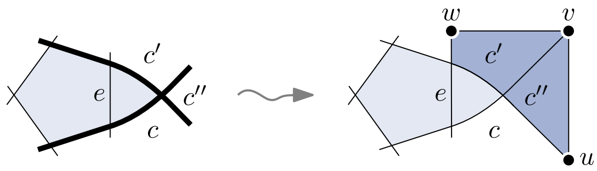


Figure 6: A ▽3 - ⬠5 - trail (light blue) and its boundary (thick) is represented on the left. Such a trail is always adjacent to a •▽⊞• (dark blue). The adjacency is represented on the right.

Proof: To prove each of the equalities above, we double-count the number of pairs (c, T) of cells c which correspond to an end of a trail T. Recall that each inner segment of a cell is interior to exactly one trail of the drawing Γ (cf. Observation 5). That is, a △4-cell is an endpoint of exactly one trail, a ⊞5•-cell of two trails, a ▽3-cell of three trails, and a ⬠5-cell of five trails, respectively. It thus suffices to observe for each trail T how many cells of a given type correspond to one of the two ends of T.

There are only six types of cells of size at most five, i.e., cell-types which are not large: ▽3-, ⊞4-, △4-, ⬠5-, ⊞5•- and △5•-cells (cf. Figure 4). Only four of these are endpoints of trails. Indeed, no end of a trail corresponds to a ⊞4- or a △5•-cells, as the latter do not contain inner segments.

If T is a trail where one end corresponds to a ▽3- or a △4-cell, then the other end can neither be a ▽3- nor a △4-cell. Such a trail T would form an empty lens, a contradiction to Γ being non-homotopic. Thus, ▽3- and △4-cell are only contained in three types of trails: (△4 ↔ ⊞5•), (△4 ↔ ⬠5) and (△4 ↔ ◯). As a △4-cell is the end of one trail and a ▽3-cell the end of three trails, (2.A) and (2.C) follow.

If one end c of a trail T corresponds to a ⊞5•- or a ⬠5-cell, there are no further restrictions on the type of the other end c' of T. That is, c' can be of type ▽3, △4, ⬠5, ⊞5• or ◯. Note that if c and c' are of the same type A, then there are two pairs (c, T), (c', T) which contribute to the number of A-A-trails. Double-counting yields (2.B) and (2.D). □

Lemma 4. *If* Γ *is a* 3*-saturated drawing, then*

$$(▽3 \leftrightarrow ⬠5) \leq •▽⊞•. \tag{3.A}$$

Proof: Consider a ▽3 - ⬠5 - trail T, see Figure 6. We use the same notation as in Figure 6. Note that T contains no ⊞4-cell as each of its bounding edges $u'u$ and $v'v$ is already crossed three times. For the same reason, one of the endpoints of uu' and vv' lies on the boundary of c and c', respectively, i.e., $u \in \partial c$ and $v \in \partial c'$. As the edge e is crossed at least twice, an endpoint w of e is incident to c or c'. We may assume without loss of generality that w lies on $\partial c'$. Observe that w and v are both incident to c'. Since the drawing is 3-saturated, $vw \in E(G)$ and c' is a ⊞6•-cell. Similarly, we see that $uv \in E(G)$ and the cell c'' is a △5•-cell. Thus, the cells c' and c'' form a •▽⊞•-configuration C.

We now observe that no other ▽3 - ⬠5 - trail is adjacent to C. The two outer segments of the △5•-cell in C are part of the two bounding edges of the ▽3 - ⬠5 - trail T, see Figure 6 (right). As no two ▽3 - ⬠5 - trails have the same bounding edges, each is adjacent to a different •▽⊞•-configuration and the inequality above follows. □

Lemma 5. *If* Γ *is a* 3*-saturated drawing, then*

$$(⬠5 \leftrightarrow ⊞5•) \leq ◁⊞▽•. \tag{3.B}$$

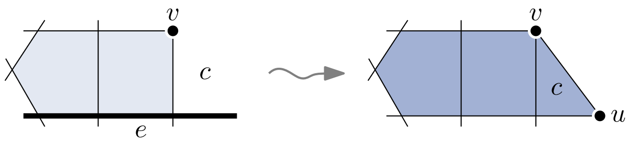


Figure 7: A - - trail (light blue). The bounding edge e (thick) is crossed three times. Such a trail forms a with an adjacent cell. The (dark blue) is represented on the right.

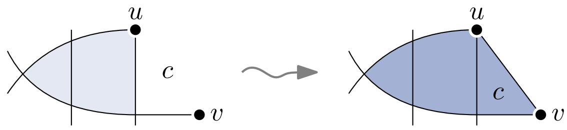


Figure 8: A - - trail (light blue). Such a trail forms a -configuration (dark blue) with an adjacent cell.

Proof: Consider a - - trail. As the drawing is 3-plane, the trail contains no -cell, and we are in the situation represented in Figure 7. The edge e (represented by a thick line) is crossed three times. Thus, one of its endpoints lies on the boundary of the cell c. Let u denote this vertex. Note that the vertex v, which is the only vertex on the -cell, is also incident to c. As the drawing is 3-saturated, uv is an edge and c is a -cell. The - - trail together with c forms a . For every - - trail, we obtain a distinct (which may share a -cell with another such configuration) and the inequality follows. □

Lemma 6. *If* Γ *is a* 3*-saturated drawing, then*

$$(\leftrightarrow) \leq . \tag{3.C}$$

Proof: Consider a - - trail. As every edge is crossed at most three times, the trail contains no -cell and we are in the situation represented in Figure 8. The vertices u and v lie on the boundary of a cell c. As the drawing is 3-saturated, the edge uv is contained in G and the cell c is a -cell. The trail together with c forms a -configuration. As every - - trail is only part of one such configuration, the lower bound on the number of -configurations follows. □

Lemma 7. *If* Γ *is a* 3*-saturated drawing, then*

$$ \leq + . \tag{3.D}$$

Proof: We show that each -cell is contained in a - or a -configuration. Consider the inner edge-segment of a -cell c and let e be the corresponding edge. We denote the two cells sharing an outer edge-segment with c by c' and c''. As the edge e is crossed at most three times, an endpoint v of e lies on the boundary of c' or c''. Without loss of generality, we are therefore in the situation represented in Figure 9. The two vertices u and v lie on the boundary of the cell c''. As Γ is 3-saturated, uv is an edge of G and c'' is a -cell.

If e is crossed three times, the three cells c', c and c'' are in a -configuration. Otherwise, e is crossed twice and the same argument as above shows that c' is also a -cell. In particular, the cells c', c and c'' are in a -configuration.

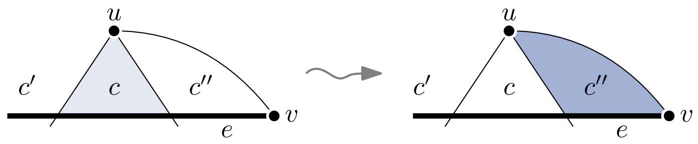


Figure 9: A -cell c (light blue). At least one of the two cells sharing an outer edge-segment with c is a -cell (dark blue).

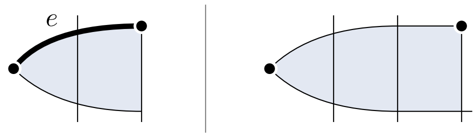


Figure 10: A - - trail with no -cell is depicted on the left, and with one -cell on the right.

As every -cell is only contained in one such configuration, the upper bound on the number of -cells follows. □

Recall that E_1 denotes the set of edges that have exactly one crossing in the drawing.

Lemma 8. *If* Γ *is a* 3*-saturated drawing, then*

$$(\leftrightarrow) \leq \frac{1}{2}|E_1| + . \tag{3.E}$$

Proof: Note that it suffices to show that every - - trail forms a -configuration or has a bounding edge that is crossed at most once. The claim then follows as every edge is a bounding edge of at most two trails.

Consider a - - trail. As every edge is crossed at most three times, the trail contains at most one -cell.

If the trail contains no such cell, we are in the situation depicted on the left of Figure 10. Note that the bounding edge e of the trail is crossed only once.

If the trail contains a -cell, we are in the situation represented on the right of Figure 10. The trail forms a -configuration.

Thus, the number of - - trails yields a lower bound on the sum of half the number of edges that are crossed only once, and the number of -configurations. □

Lemma 9. *If* Γ *is a* 3*-saturated drawing, then*

$$2\,(\leftrightarrow) + (\leftrightarrow) + (\leftrightarrow) - 4 \leq . \tag{4.A}$$

Proof: Let T be a trail ending in a -cell and let e_1 and e_2 be the two edges on its boundary. If the other end of T corresponds to a -, - or -cell, the edges e_1 and e_2 have the same number of crossings within the trail. We therefore call these cells *crossing-even* and use the same term for trails where one endpoint corresponds to a -cell, the other to a crossing-even cell. A -cell where every incident trail ends in a crossing-even cell is called *full*. We denote the number of full -cells by ℓ.

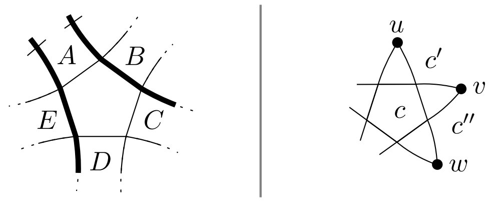


Figure 11: Left: A full -cell with its incident trails is represented. One of the trails is crossed. The corresponding boundary edges are represented with thick lines. Right: A full -cell with three consecutive uncrossed trails.

An upper bound on the number ℓ of full -cells. We first show that every full -cell is part of at least one -configuration. As no such configuration contains two -cells, we then obtain

$$\ell \leq . \qquad (\star)$$

Consider a full -cell c. We say that a trail incident to c is *uncrossed* if its boundary contains no crossing that is not incident to c. Otherwise, we say that it is *crossed*. In fact, all such uncrossed trails end in a -cell and contain no -cell. Every trail T incident to c contains an inner edge-segment $s(T)$ on the boundary of c in its interior. Two such trails T and T' are called *consecutive* if the edge-segments $s(C)$ and $s(C')$ share a crossing.

We first show that there are at least three consecutive uncrossed trails incident to the full cell c. If none of the incident trails of c is crossed, the claim clearly holds. Otherwise, some trail A is crossed, see Figure 11 (left). Recall that A is crossing-even. Thus, the two edges on its boundary are crossed three times. Therefore, the trails C and D are uncrossed. The same argument shows that only one of the two remaining trails B and E is crossed. We therefore obtain three consecutive uncrossed trails.

As every uncrossed trail ends in a -cell and contains no -cell, we are in the situation depicted in Figure 11 (right). Note that the vertices u and v lie on the boundary of a cell c'. Thus, $uv \in E(G)$ and c' is a -cell as Γ is 3-saturated. Similarly, we see that $vw \in E(G)$ and c'' is a -cell. The cells c, c' and c'' form a -configuration. It follows that every full -cell is part of a -configuration.

A lower bound on the number ℓ of full -cells. We define

$$k := 2\,(\leftrightarrow) + (\leftrightarrow) + (\leftrightarrow),$$

i.e., k corresponds to the number of incidences between crossing-even trails and -cells. It remains to show that

$$k - 4 \leq \ell. \qquad (\star\star)$$

The claim then follows by combining $(\star)$ and $(\star\star)$.

Every -cell that is not full is incident to at most four crossing-even trails while every full -cell is incident to five such trails. As there are k incidences between crossing-even trails and -cells, we obtain $k \leq 4(- \ell) + 5\ell = 4 + \ell$. Thus, $(\star\star)$ holds. □

Lemma 10. *If Γ is a 3-saturated drawing, then*

$$\geq . \tag{4.B}$$

Proof: Each -configuration contains a . As no two -configurations share a , the claim follows. □

Lemma 11. *For every drawing Γ, we have*

$$2 \geq + + + + 2 . \tag{9.C}$$

Proof: We give a lower bound on the number of -cells by double-counting the number of outer edge-segments incident to a -cell.

Every -cell is incident to two outer edge-segments.

Recall that an edge-segment is interior to a configuration C if both its incident cells are interior to C. In each of the configurations , , , and , exactly one outer edge-segment s_c of each interior -cell c is interior to the configuration. Note in particular that we did not count any such edge-segment twice due to the choice of the configurations we consider. Indeed, any two configurations may share a -cell c, but the corresponding outer edge-segments s_c do not coincide. As each , , and contains one outer edge-segment that is incident to a -cell in its interior, and contains two such segments, the lower bound on the number of -cells follows. □

4.2 Lower Bounds on the Number of Edges

Recall that $E_\times$ denotes the set of edges with at least one crossing.

Lemma 12. *For every drawing Γ, we have*

$$4|E_\times| \geq 2 + 2 + 2 . \tag{6}$$

Proof: We double count the number ℓ of incidences between outer edge-segments and cells.

Every crossed edge has exactly two outer edge-segments. Each of them has two incidences to cells. Thus, $4|E_\times| = \ell$.

Every -cell, every -cell and every -cell is incident to exactly two outer edge-segments. We therefore have, $\ell \geq 2 + 2 + 2$ which yields the claim. □

Recall that for an integer $i \geq 0$, we denote by E_i the set of edges with exactly i crossings.

Lemma 13. *For every drawing Γ, we have*

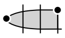
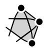
$$2|E_2| + 4|E_3| \geq 3 + + 4 + 2 + 5 \\ + (\leftrightarrow \bigcirc) + (\leftrightarrow \bigcirc) + (\leftrightarrow \bigcirc) + (\leftrightarrow \bigcirc) . \tag{7}$$

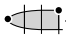
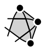
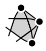
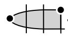
Proof: We double count the number ℓ of incidences between inner edge-segments and cells.

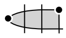
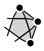
An edge that is crossed k times is split into $k+1$ edge-segments, $k-1$ of which are inner edge-segments. Each inner edge-segment is incident to two cells. We therefore have $2|E_2| + 4|E_3| \geq \ell$.

On the other hand, every -cell, every -cell, every -cell, every -cell and every -cell is incident to 3, 1, 4, 2 and 5 inner edge-segments respectively. In order to obtain a lower bound for the number of inner edge-segment incidences with large cells, it suffices to count the number of trails ending in large cells. Indeed, each such trail enters a large cell through an inner edge-segment. This yields the right hand side of the inequality above. □

Lemma 14. *For every drawing* Γ, *we have*

$$|E_2| \geq \frac{1}{2}$$  $+$  (8.C)

Proof: Every -configuration contains an edge that is crossed exactly twice. This edge may be contained in at most two such configurations. Similarly, every -configuration contains an edge that is crossed twice. Note that this edge cannot be contained in any other - or -configuration.

Thus, the sum of $\frac{1}{2}$  and  provides a lower bound on the number $|E_2|$ of edges that are crossed exactly twice. □

Double-counting the number of incidences of uncrossed edges with cells yields the following.

Lemma 15. *For every drawing* Γ, *we have*

$$2|E_0| \geq$$ . (9.B)

4.3 Relating the Number of Cells, Crossings and Edges

Let us start with a convenient consequence of the Density Formula with parameter $t = 5$. We write $\mathcal{C}$ for the set of all cells. For $a \in \mathbb{N}$, we denote by $\mathcal{C}_a$ the set of all cells of size a.

Theorem 8. *If* Γ *is a* 3*-saturated drawing on at least three vertices, then*

$$|X| \leq 5|V| + 2$$  $+$  $+$  $$- \frac{1}{6}\sum_{a\geq 6} a|\mathcal{C}_a| - |E|.$$ (5.B)

Proof: Recall that every 3-saturated drawing is in particular connected. As the graph contains at least three vertices, every cell has size at least 3. Considering $t = 5$ in the Density Formula (Theorem 4) yields

$$|E| \leq 5|V| - \sum_{c\in\mathcal{C}} (\|c\| - 5) - |X|. \qquad (\star)$$

Thus, only cells of size at most 4 have a positive contribution to the right side. Recall that there is only one cell-type of size 3, namely , and only two types of size 4:  and . Cells of size 5 have no contribution. For $a \geq 6$, we have $a - 5 \geq \frac{1}{6}a$. We therefore obtain

$$\sum_{c\in C, \|c\|\geq 6} (\|c\| - 5) = \sum_{a\geq 6} |\mathcal{C}_a|(a-5) \geq \frac{1}{6}\sum_{a\geq 6} a|\mathcal{C}_a|.$$

Together with ($\star$), we get

$$|E| \leq 5|V| + 2$$  $+$  $+$  $$- \frac{1}{6}\sum_{a\geq 6} a|\mathcal{C}_a| - |X|.$$

□

Lemma 16. *If* Γ *is a* 3*-saturated drawing, then*

$$\sum_{a\geq 6} a|\mathcal{C}_a| \geq$$ ( $\leftrightarrow$ ) $+$ ( $\leftrightarrow$ ) $+$ ( $\leftrightarrow$ ) $+$ ( $\leftrightarrow$ ) . (5.A)

Proof: As we want to obtain a lower bound on the sum $\sum_{a\geq 6} a|\mathcal{C}_a|$, it suffices to count the number of vertex and edge-segment incidences of large cells. Each trail that ends in a large cell enters this cell via an inner edge-segment. As no two trails share such an inner edge-segment, we obtain one edge-segment incidence for each such trail. □

5 Discussion

Our main result (Theorem 1) states that every non-homotopic 3-plane drawing with n vertices contains at most $5.5n - 11$ crossings. This bound is best-possible. In fact, in the construction in Figure 2 we can insert in *every* hexagonal face (i.e., also in the top and bottom face) of the cylinder 8 diagonals giving 11 crossings. The resulting 3-plane drawing is non-homotopic (but no longer simple) and has exactly $5.5n - 11$ crossings. In fact, the resulting 3-planar graph also has the maximum number of edges among all 3-planar n-vertex graphs.

In general, *optimal k-planar* graphs are k-planar graphs with the maximum number of edges (for the given number of vertices). For $k = 2, 3$, they turn out to admit k-plane drawings with the maximum number of crossings among all k-plane drawings of n-vertex graphs. Intuitively it makes sense to use graphs with many edges to obtain many crossings. But the situation is not so clear for $k \geq 4$. For example, for $k = 4$, there exists a family of simple 4-plane drawings (again based on a cylinder with hexagonal faces) with $6n - 18$ edges and $7.5n - 33$ crossings (Figure 12). It is known that the number of edges $6n - 18$ is indeed best-possible for n-vertex 4-planar graphs up to the additive constant [1, Theorem 4]. But it remains open whether there are 4-plane n-vertex drawings with more than $7.5n$ crossings.

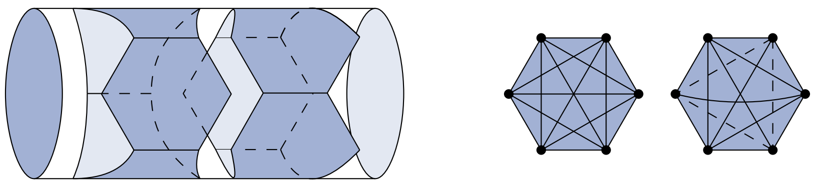


Figure 12: Construction of a 4-plane drawing of a simple n-vertex graph with $7.5n - 33$ crossings and $6n - 18$ edges from [1, Figure 35]. Left: A cylinder with two layers, each consisting of three hexagonal faces. Right: To each face of a layer all diagonals are added. To the top and bottom face six diagonals can be added without creating parallel edges. Missing diagonals are represented by dashed lines.

Question 1. *Does every simple 4-plane n-vertex drawing contain at most $7.5n$ crossings?*

As already mentioned in the introduction, the *k-planar crossing number* $\text{cr}_k(G)$ is similar to the crossing number, except that the minimum is taken over all k-plane drawings of G. Clearly, $\text{cr}(G) \leq \text{cr}_k(G)$ for all k and G. But there are k-planar n-vertex graphs G with $\text{cr}(G) \in \mathcal{O}(k)$ and $\text{cr}_k(G) \in \Omega(kn)$ [8, Theorem 2]. By Theorem 1, every 3-plane drawing of an n-vertex graph G has at most $5.5(n-2)$ crossings, and hence $\text{cr}(G) \leq \text{cr}_3(G) \leq 5.5(n-2)$. Although Theorem 1 is tight, we could have $\text{cr}(G) < 5.5(n-2)$, and a similar question arises for 2-planar graphs.

Question 2.

(i) Are there 3-planar n-vertex graphs G with $\text{cr}(G) = \text{cr}_3(G) = 5.5(n-2)$?

(ii) Are there 2-planar n-vertex graphs G with $\mathrm{cr}(G) = \mathrm{cr}_2(G) = 3.\overline{3}(n-2)$*?*

We give a positive answer to Question 2 (ii) (cf. Corollary 10) by using a recent result of Büngenger and Kaufmann that improves upon the Crossing Lemma.

Theorem 9 (Büngener, Kaufmann [7, Theorem 6])**.** *For every graph G on $n \geq 2$ vertices and m edges, we have* $\mathrm{cr}(G) \geq \frac{73}{18}m - \frac{305}{18}(n-2)$.

Optimal 2-planar graphs G on $n \geq 5$ vertices contain $5(n-2)$ edges [15, 16]. Plugging this edge count into Theorem 9, it follows that every (not necessarily 2-plane) drawing of an optimal 2-planar graph G contains at least $3.\overline{3}(n-2)$ crossings. And in fact, this number of crossings is attained in every 2-plane drawing of G by Theorem 7.

Theorem 10. *For every optimal 2-planar graph G on $n \geq 5$ vertices, we have*

$$\mathrm{cr}(G) = \mathrm{cr}_2(G) = 3.\overline{3}(n-2).$$

Yet, Theorem 9 does not immediately give a positive answer to Question 2 (i). We have that optimal 3-planar (multi)-graphs on $n \geq 6$ vertices contain $5.5(n-2)$ edges [11, 12]. Plugging this edge count into Theorem 9, only yields a lower bound of $5.36\overline{1}(n-2)$ on the crossing number of optimal n-vertex 3-planar graphs. However, using another result by Büngener and Kaufmann [7, Cor. 4], we can give at least a partial answer to Question 2 (i).

Proposition 11. *For every $n \geq 6$, every optimal 3-planar n-vertex (multi-)graph G satisfies* $5.36\overline{1}(n-2) \leq \mathrm{cr}(G) \leq \mathrm{cr}_3(G) = 5.5(n-2)$.

Proof: The lower bound immediately follows from Theorem 9.

To obtain the upper bound on the 3-crossing number, let G be an optimal 3-planar graph and let Γ be a 3-plane drawing of G with minimum number of crossings. It suffices to show that $|X| = 5.5(n-2)$.

First note that Γ is filled (i.e., any two vertices $u \neq v$ on the boundary ∂c of a cell c are joined by an uncrossed edge along ∂c) since G is optimal. Thus, in Γ we have  = .

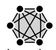
Now, recall that optimal 3-planar n-vertex (multi)-graphs contain exactly $5.5(n-2)$ edges. A result by Büngener and Kaufmann [7, Cor. 4] then implies that

 =  $\geq \frac{1}{2}(n-2)$. (⋆)

Double-counting the edge-incidences of uncrossed edges with -configurations, we obtain $2|E_0| \geq$ 6 .

Observe that no two -configurations share a crossed edge, and that every -configuration contains eight crossed edges. Together with (⋆), we can conclude that

$5.5(n-2) = |E| = |E_0| + |E_\times| \geq 3$  $+ 8$  $= 11$  $\geq 5.5(n-2).$

It follows that the above inequality, and hence also (⋆), holds with equality. In particular, we have  $= \frac{1}{2}(n-2)$. As each  accounts for eleven crossings, we obtain the desired count for the total number of crossings: $|X| = 5.5(n-2)$. □

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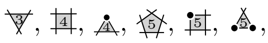
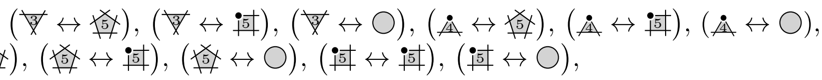
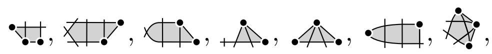
A Certificates

The LP for 3-plane drawings (see Figure 13 for an overview) corresponds to a coefficient matrix M. Each column of M corresponds to a variable v_i and each row to an inequality of the form $\sum_i a_i v_i \leq b(|V| - 2)$ with constant, integer coefficients a_i and b.

The columns of M correspond (in that order) to

- $|X|$, $|E|$, $|E_0|$, $|E_1|$, $|E_2|$, $|E_3|$, $|E_\times|$,
- the cells 
- the trails 
- the configurations 
- and the terms $\sum_{a \geq 6} a|\mathcal{C}_a|$, $|V| - 2$.

The rows of M correspond (in that order) to the inequalities (2.A), (2.B), (2.C), (2.D), (3.A), (3.B), (3.C), (3.D), (3.E), (4.A), (4.B), (5.A), (5.B), (6), (7), (8.A), (8.B), (8.C), (9.A), (9.B), (9.C). There is a vector $c_\times$ such that $s_\times = c_\times \cdot M$ is a vector where each entry is non-negative, the entry corresponding to the number of crossings $|X|$ is 1, and the entry corresponding to the number of vertices $|V| - 2$ is 5.5. That is, $|X| \leq 5.5(|V| - 2)$. Similarly, there exists a vector c_E such that $c_E \cdot M$ yields $|E| \leq 5.5(|V| - 2)$. The computation of $c_\times \cdot M$ and $c_E \cdot M$ can be verified with the code given in Listing 1.

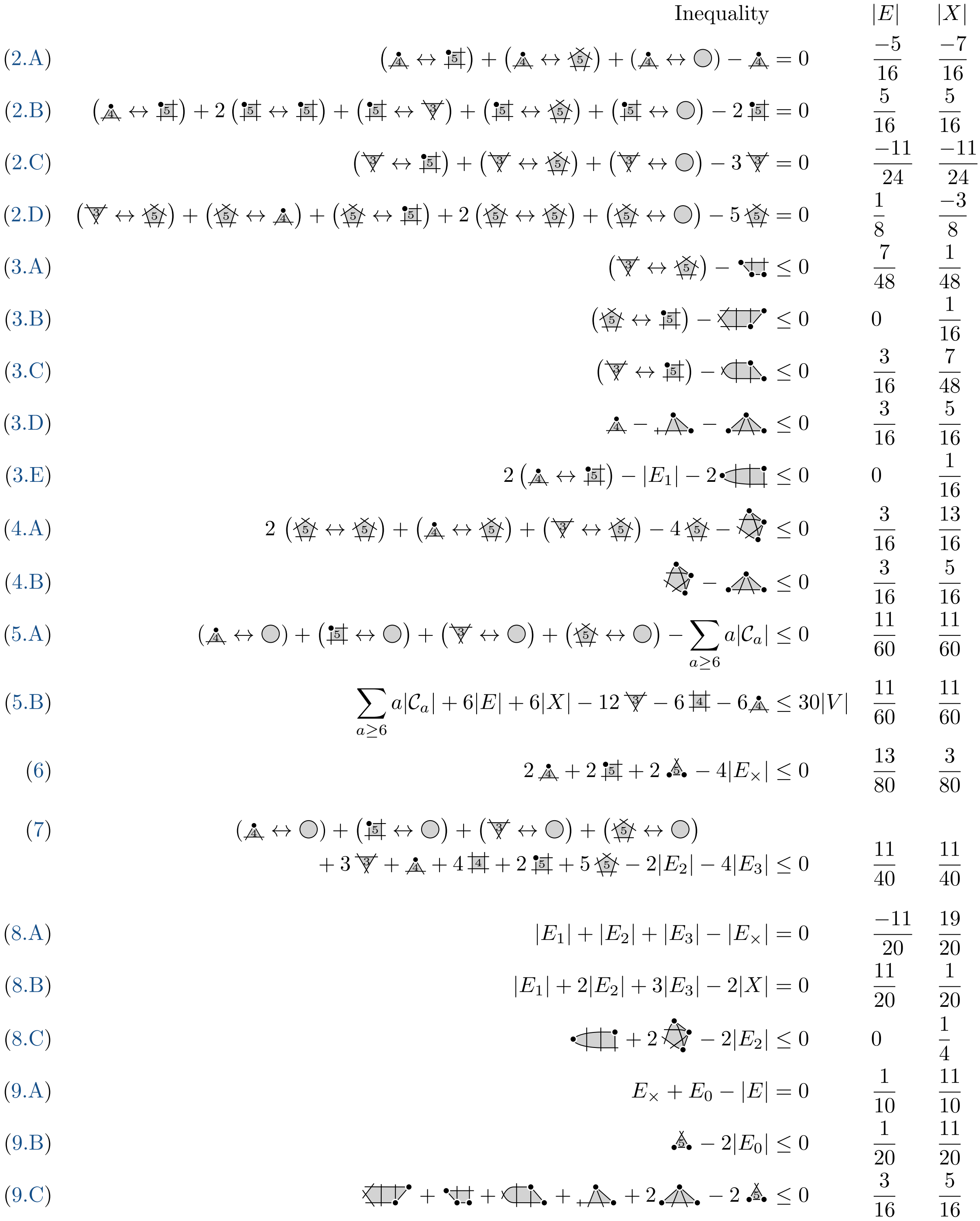


Figure 13: Certificates for the upper bound on the number of edges and crossings in 3-saturated drawings in terms of the number of vertices. Each row corresponds to one inequality. In order to obtain the upper bound on the number of edges, we multiply each inequality with the third entry in the corresponding row and sum up all the inequalities. To obtain the upper bound on the number of crossings we proceed likewise using the fourth entry of each row as a coefficient.

Listing 1: Python code for verifying the matrix multiplication of $c_{\times} \cdot M$ and $c_e \cdot M$.

```
import numpy as np
from fractions import Fraction as fr

ce = [fr(-5,16),fr(5,16),fr(-11,24),fr(1,8),fr(7,48),0,fr(3,16),
fr(3,16),0,fr(3,16),fr(3,16),fr(11,60),fr(11,60),fr(13,80),
fr(11,40),fr(-11,20),fr(11,20),0,fr(1,10),fr(1,20),fr(3,16)]


cx = [fr(-7,16),fr(5,16),fr(-11,24),fr(-3,8),fr(1,48),fr(1,16),fr(7,48),
fr(5,16),fr(1,16),fr(13,16),fr(5,16),fr(11,60),fr(11,60),fr(3,80),
fr(11,40),fr(19,20),fr(1,20),fr(1,4),fr(11,10),fr(11,20),fr(5,16)]

M = [[0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,-2,0,0,1,0,0,1,0,0,1,0,2,1,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,-3,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,-5,0,0,1,0,0,1,0,0,2,1,1,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,-1,0,0,0,0],
[0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0],
[0,0,0,0,0,0,0,0,0,0,-4,0,0,1,0,0,1,0,0,2,0,0,0,0,0,0,0,0,0,0,-1,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,0,1,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,-1,0],
[6,6,0,0,0,0,0,-12,-6,-6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,30],
[0,0,0,0,0,0,-4,0,0,2,0,2,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,-2,-4,0,3,4,1,5,2,0,0,0,1,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,0,0],
[0,0,0,1,1,1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[-2,0,0,1,2,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,-2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,2,0,0],
[0,-1,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,-2,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],
[0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,2,0,0,0,0]]


se = np.matmul(ce,M)
sx = np.matmul(cx,M)

print()
print(f'cx = [',' '.join([format(item,'') for item in cx]),']')
print(f'ce = [',' '.join([format(item,'') for item in ce]),']')
print(f'M = ')
print('\n'.join([''.join(['{:4}'.format(item) for item in row])
      for row in M]))
print(f'cx * M = [',' '.join([format(item,'') for item in sx]),']')
print(f'ce * M = [',' '.join([format(item,'') for item in se]),']')
```