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A Systematic Approach to Crossing Numbers of Cartesian Products with Paths

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Abstract. Determining the crossing numbers of Cartesian products of small graphs with arbitrarily large paths has been an ongoing topic of research since the 1970s. Doing so requires the establishment of coincident upper and lower bounds; the former is usually demonstrated by providing a suitable drawing procedure, while the latter often requires substantial theoretical arguments. Many such papers have been published, which typically focus on just one or two small graphs at a time, and use ad hoc arguments specific to those graphs. We propose a general approach which, when successful, establishes the required lower bound. This approach can be applied to the Cartesian product of any graph with arbitrarily large paths, and in each case involves solving a modified version of the crossing number problem on a finite number (typically only two or three) of small graphs. We demonstrate the potency of this approach by applying it to Cartesian products involving all 133 graphs of orders five or six, and show that it is successful in 128 cases. This includes 60 cases which a recent survey listed as either undetermined, or determined only in journals without adequate peer review.

1 Introduction

In this paper, we will refer to a number of common graph families, and so we list them upfront to aid the reader. P_n, C_n, S_n is the path, cycle, star with n edges and thus on $n+1$, n, $n+1$ vertices respectively. The Cartesian product of two graphs G and H is written as $G \square H$. What results is a graph with vertex set $V(G) \times V(H)$, and edges between vertices (u, u') and (v, v') if and only if either $u = v$ and $(u', v') \in E(H)$, or $u' = v'$ and $(u, v) \in E(G)$.

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A *drawing* D of a graph G is a representation of G in the plane, such that each vertex is mapped to a discrete point, and each edge (a, b) is mapped to a closed curve between the points corresponding to vertices a and b, which does not intersect with a point corresponding to any other vertex. If two curves intersect in such a drawing, we say that there is a *crossing* between their corresponding edges, and we denote the number of crossings in the drawing D as $cr_D(G)$. Then the *crossing number problem* (CNP) is to determine $\mathrm{cr}(G) \coloneqq \min_D cr_D(G)$, the minimum number of crossings among all possible drawings of G. We can assume that all intersections are crossings, rather than tangential, and also that three edges never intersect at a common point. Furthermore, a drawing is said to be *simple* if edges incident with a common vertex do not intersect, and no two edges intersect more than once. It is well known that for any graph G, $\mathrm{cr}(G)$ crossings are achieved by a simple drawing.

CNP is known to be NP-hard [18], and is notoriously difficult to solve, even for relatively small graphs. Indeed, $\mathrm{cr}(K_{13})$ and $\mathrm{cr}(K_{9,9})$ are still unknown despite significant effort [1, 15, 33, 42]. Nonetheless, there are some infinite families of graphs for which the crossing numbers are known; these are summarised in the recent survey paper by Clancy et al. [15]. The most common of these are families which result from Cartesian products, the study of which originated with a conjecture by Harary et al. [20] in 1973 that $\mathrm{cr}(C_m \square C_n) = (m-2)n$ for $n \geq m \geq 3$. Despite significant effort [2–4, 6, 17, 19, 32, 35, 36] to date the conjecture has only been resolved for the cases $m \leq 7$ or $n \geq m(m+1)$. While resolving the $m = 4$ case, Beineke and Ringeisen [6] considered $\mathrm{cr}(G \square C_n)$ for all six non-isomorphic connected graphs G of order four, and successfully determined all of them except for the case $G = S_3$. This latter case was subsequently settled by Jendrol' and Ščerbová [22]. Following this, Klešč [24] determined $\mathrm{cr}(G \square P_n)$ and $\mathrm{cr}(G \square S_n)$ for the same six connected graphs G of order four. This was then followed by a decade-long effort by Klešč [23, 25–29] to determine $\mathrm{cr}(G \square P_n)$ for all 21 non-isomorphic connected graphs G of order five, which was finally completed in 2001. In the two decades since, significant efforts have been made by multiple authors (including Klešč) to extend these results to the 112 non-isomorphic connected graphs of order six. Currently, slightly less than half of these have been resolved; the progress is chronicled in [15].

In order to determine $\mathrm{cr}(G \square P_n)$ for a given graph G, one needs to establish lower and upper bounds, and show that they coincide. Establishing a valid upper bound is generally straightforward, and is usually either achieved by providing a drawing procedure which results in the desired number of crossings, or else uses an existing result and the monotonicity property $\mathrm{cr}(G \square P_n) \leq cr(H \square P_n)$ if G is a subgraph of H. Establishing a lower bound is typically much more complicated, usually involving ad hoc arguments specific to the graph in question. Due to this ad hoc nature, it is not uncommon for publications to focus on a single graph G and determine $\mathrm{cr}(G \square P_n)$ for that one case (e.g. see [27, 28, 30, 34]). Even in papers which focus on several graphs, often the complicated arguments are needed for only one or two graphs, and the remaining results follow as corollaries.

In this paper, we propose a new approach to determining, $\mathrm{cr}(G \square P_n)$ which can be applied to any graph G. There are two possible outcomes to this approach; either the required lower bound is established, or else nothing is established. We will demonstrate that the approach is successful in resolving 128 of the 133 non-isomorphic connected graphs of orders five or six, including 60 cases which were hitherto unresolved. Of the five remaining graphs, four have been previously solved in the literature, while the one remaining graph is equivalent to K_5 with a pendant edge attached to one of its vertices. We discuss these five graphs further towards the end of Section 5.

The approach we describe requires a modified version of CNP to be solved for $G \square P_d$, where d is a small number (typically $d = 2$ or $d = 3$ suffices). As such, this approach is tractable for sufficiently small graphs G.

The remainder of this paper is laid out as follows. In Section 2 we introduce the modified version of CNP, which we call *binary-weighted capacity-constrained CNP* (BCCNP). In Section 3 we describe the approach which, when successful, uses this modified version of CNP to obtain the required lower bounds for $\mathrm{cr}(G \square P_n)$. In Section 4 we briefly discuss the manner in which we will establish the required upper bounds and base cases, and also the manner in which we solve instances of BCCNP. In Section 5 we report on calculations which test the efficacy of the approach, and show that we are able to determine $\mathrm{cr}(G \square P_n)$ for 60 new graphs G on six vertices, and also explore its efficacy on graphs G with seven vertices. Finally, we conclude the paper in Section 6.

2 Binary-Weighted Capacity-Constrained CNP (BCCNP)

For a given graph G, denote by $C(G) \subseteq E(G) \times E(G)$ the set of all possible pairs of edges between which a crossing may occur in a simple drawing. That is, $C(G)$ contains all pairs of non-adjacent edges. Then, for a simple drawing D of G, and $c \in C(G)$, define a binary variable x_c^D to be equal to 1 if crossing c occurs in D, and 0 otherwise. CNP is then equivalent to

$$\min_D \sum_{c \in C(G)} x_c^D, \text{ over all simple drawings } D.$$

The above formulation considers every possible crossing that could occur in D and then counts those that do. However, suppose that we additionally define some subset $B \subseteq C(G)$, and are only interested in simple drawings that minimize the crossings in B:

$$\min_D \sum_{c \in B} x_c^D, \text{ over all simple drawings } D.$$

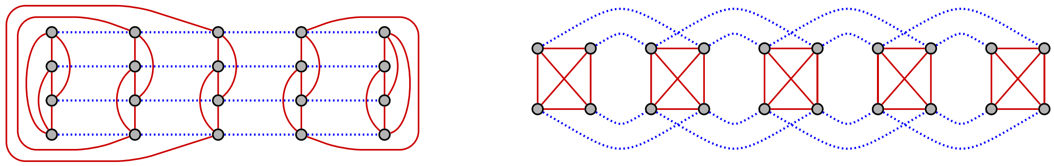
This is equivalent to the *weighted CNP* (cf. Schaefer [37]) when all the weights are binary.

Another variant that could be considered would be to add some additional constraints to the formulation. Suppose that we have a family $\mathcal{F} = \{\mathcal{F}_i\}_i$ which contains subsets $\mathcal{F}_i \subseteq C(G)$, and an $|\mathcal{F}|$-dimensional vector $\mathbf{f} = (f_i)$ of non-negative integer capacities. Then a set of constraints of the form $\sum_{c \in \mathcal{F}_i} x_c^D \leq f_i$, for all $1 \leq i \leq |\mathcal{F}|$, could be imposed: we only allow simple drawings that, for each i, contain no more than f_i crossings from the subset $\mathcal{F}_i$. We say $\mathcal{F}_i$ has a *maximum capacity* of f_i.

The two concepts above can be combined together; we refer to the resulting problem as *binary-weighted capacity-constrained CNP* (BCCNP), formulated as:

$$\begin{aligned} &\min_D \sum_{c \in B} x_c^D \\ &\text{s.t. } D \text{ is a simple drawing, and} \\ &\quad \sum_{c \in \mathcal{F}_i} x_c^D \leq f_i \qquad\qquad \forall i = 1, \dots, |\mathcal{F}| \end{aligned}$$

An instance of BCCNP requires us to specify not only the graph G, but also the subset B, the family $\mathcal{F} \subseteq 2^X$ of subsets (where $X \subseteq C(G)$), and the capacity vector $\mathbf{f}$. Thus, let $\mathrm{cr}(G, B, \mathcal{F}, \mathbf{f})$ denote the solution of a BCCNP instance. Naturally, $\mathrm{cr}(G, C(G), \emptyset, \emptyset)$ yields the standard CNP. BCCNP has potential applications in its own right. However, our reason for defining it is that it will assist us in determining the crossing numbers for various families of graphs.



(a) Optimal drawing with 8 crossings; the three inner K_4-copies are involved in 2 crossings each.

(b) a-restricted drawing for $a = 2$ with 11 crossings.

Figure 1: Two drawings of $K_4 \square P_4$. The K_4-copies are solid red; the path edges are dotted blue.

3 Using BCCNP to Determine Crossing Numbers

Suppose that we have a graph G, and that we have reason to believe there are integers s, a, b (with $s, a > 0$) such that $\mathrm{cr}(G \square P_n) = an - b$ for $n \geq s$. Certainly this is the case for every established result to date [15]. Furthermore, suppose that we already possess a proof that this formula holds for $n = s, s+1, \ldots, t$ for some integer t, and also that we have established the corresponding upper bound $\mathrm{cr}(G \square P_n) \leq an - b$ for $n \geq s$. Then all that remains is to determine the lower bound, $\mathrm{cr}(G \square P_n) \geq an - b$ for $n > t$.

In order to determine the lower bound, we will seek to look at a fixed-size graph $G \square P_d$ for some small integer d, and use it to infer results about $G \square P_n$ for arbitrarily large n. As such, in what follows, we will tend to use P_d when referring to a path of a fixed size, and P_n otherwise.

We begin by giving some definitions and notation that will be useful in the upcoming discussion. Note that the graph $G \square P_n$ contains $n+1$ copies of G, with consecutive copies linked together by some edges. Denote the copies of G as G^0, G^1, ..., G^n, each with edges $E(G^i)$. We call remaining edges *path edges*; those linking G^i to G^{i+1} form the subgraph H^i, for $i = 0, \ldots, n-1$. Note that each H^i corresponds to a matching between G^i and G^{i+1}; see Fig. 1a for an example.

Definition 1. *Consider a positive integer a. A simple drawing of $G \square P_n$ is* a-restricted *(cf. Fig. 1b) if each copy of G has fewer than a crossings on its edges, i.e., the drawing has fewer than a crossings from $E(G^i) \times E(G \square P_n)$ for every $0 \leq i \leq n$.*

For sets of crossings, we use the shorthand $H' \otimes H'' \coloneqq E(H') \times E(H'')$ and define

$$\mathrm{cr}_a(G, d, B) \coloneqq \mathrm{cr}(G \square P_d, B, \left\{G^i \otimes G \square P_d\right\}_{0 \leq i \leq d}, (a-1) \cdot \mathbf{1}_{d+1}),$$

which is the minimum number of crossings on the set of edge pairs B over all a-restricted drawings of $G \square P_d$. Here, $\mathbf{1}_{d+1}$ denotes the $(d+1)$-dimensional unit vector.

Lemma 1. *Suppose that there is an integer $n \geq 2$ such that $\mathrm{cr}(G \square P_{n-1}) \geq a(n-1) - b$, but $\mathrm{cr}(G \square P_n) < an - b$, for some integers a and b. Then every crossing-minimal drawing of $G \square P_n$ is a-restricted.*

Proof: Suppose that there is a crossing-minimal drawing D_n of $G \square P_n$ which is not a-restricted. Recall that $G \square P_n$ has $n+1$ copies of G. Since D_n is not a-restricted, at least one of those copies, say G^i, must have a or more crossings on its edges. By deleting the edges $E(G^i)$ from the drawing, we obtain a drawing of a new graph with fewer than $a(n-1) - b$ crossings. However, since this new drawing is of a graph that is either homeomorphic to $G \square P_{n-1}$, or else, if $i = 0$ or $i = n$, contains $G \square P_{n-1}$ as a subgraph, this violates the initial assumption. □

	G_0	H_0	G_1	H_1	G_2	H_2	G_3	H_3	G_4	H_4	G_5	H_5	G_6	H_6	G_7
G^0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4
H^0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4
G^1	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
H^1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
G^2	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5
H^2	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5
G^3	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5
H^3	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5
G^4	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6
H^4	2	3	3	3	3	4	4	4	4	5	5	5	5	6	6
G^5	3	3	3	3	4	4	4	4	5	5	5	5	6	6	6
H^5	3	3	3	4	4	4	4	5	5	5	5	6	6	6	6
G^6	3	3	4	4	4	4	5	5	5	5	6	6	6	6	7
H^6	3	4	4	4	4	5	5	5	5	6	6	6	6	7	7
G^7	4	4	4	4	5	5	5	5	6	6	6	6	7	7	7

Figure 2: Crossing bands A_i^7 for $G \square P_7$ and $i = 0, \ldots, 7$, visually highlighted as colored stripes. A colored cell represents crossings between an edge of the subgraph corresponding to the row and the column, respectively. By symmetry, it suffices to focus on the darker shaded part of the matrix. The number "i" in a cell indicates that these crossings are contained in A_i^7. — Observe that, in contrast to this, the restriction of an a-restricted drawing w.r.t. copy G^i sums over all crossings in row and, by symmetry, column G^i. This type of restrictions does not form a partition of the crossings. As an example, a bold outline marks this for G^3; this is very different from A_3^7.

Now, consider the graph $G \square P_d$ for some $d \geq 2$. The arguably most natural approach when discussing Cartesian products with paths is to try to consider the graph "copy-wise", i.e., look at the graph in chunks of subgraphs $G_i \cup H_i$, possibly akin to how we define our a-restrictions. However, it turns out that a different way to group crossings is more useful in our proofs. We partition the possible crossings $C(G \square P_d)$ into $d+1$ *crossing bands* A_i^d, as follows.

Crossings in $G^i \otimes (G^j \cup H^j)$	are assigned to	A_k^d, where $k = \lceil (i+j)/2 \rceil$,
Crossings in $H^i \otimes H^j$	are assigned to	A_k^d, where $k = \lceil (i+j+1)/2 \rceil$.

Observe that the crossing bands indeed form a partition, since each edge belongs to either G^i or H^i for some $0 \leq i \leq d$, our case distinction captures all possible edge pairings, and each crossing is assigned to precisely one crossing band. Intuitively, the crossing band index of a crossing is the average of the origins of the involved edges (their respective i's). The partition of crossings into the crossing bands is visualized in Fig. 2 for the case when $d = 7$. Of course, there are other ways the crossings could have been partitioned, and indeed, we experimented with many other choices. The partitioning method we present here is the one we found to be the most effective in obtaining results. For the sake of convenience, in the following discussions we will often refer to a drawing as having crossing bands; technically, this refers to the crossing bands of the underlying graph.

We will show that each crossing band contributes a specific minimum number of crossings to the overall crossing number of $G \square P_d$. Intuitively speaking, one may think about a natural drawing, like Figure 1a: Essentially each G-copy is drawn identically and requires a certain number of crossings, except for the left- and right-most G-copy ("front" and "end") which may require less crossings.

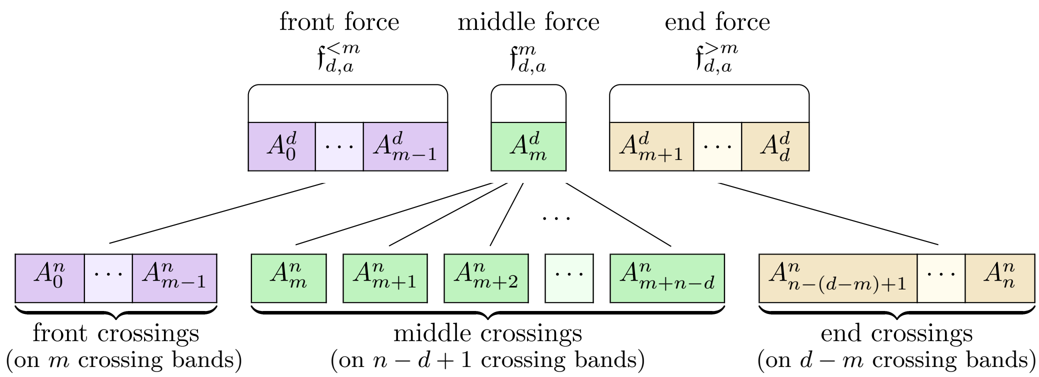


Figure 3: **(top)** Crossing bands considered in a force triple $(\mathfrak{f}^{<m}_{d,a}, \mathfrak{f}^{m}_{d,a}, \mathfrak{f}^{>m}_{d,a})$ and **(bottom)** where the forces are used as lower bounds for the number of crossings in an a-restricted drawing D_n.

However, for more complex G, the special front and end may consist of multiple G-copies each. All path-products for which the crossing number is known allow such a "natural" optimal drawing, however, our proof may of course not assume that this would always be the case. Nonetheless, this pattern allows us to establish certain minimal numbers of crossings that are *forced* along the crossing bands.

Let $d \geq 2$ and $0 < m < d$. We use the term *middle force* to refer to the minimum possible number of crossings from A^d_m in any a-restricted drawing of $G \square P_d$. Likewise, the term *front force* (*end force*) refers to the minimum possible number of crossings from the first m (final $d-m$, respectively) crossing bands in any a-restricted drawing of $G \square P_d$ (cf. Fig. 3 top):

$$\begin{array}{llcll} \textit{front force} & \mathfrak{f}^{<m}_{d,a} & := & \mathrm{cr}_a(G, d, & A^d_0 \cup \cdots \cup A^d_{m-1} \quad), \\ \textit{middle force} & \mathfrak{f}^{m}_{d,a} & := & \mathrm{cr}_a(G, d, & \qquad A^d_m \qquad), \\ \textit{end force} & \mathfrak{f}^{>m}_{d,a} & := & \mathrm{cr}_a(G, d, & A^d_{m+1} \cup \cdots \cup A^d_d \quad). \end{array}$$

Together they define the *force triple* $(\mathfrak{f}^{<m}_{d,a}, \mathfrak{f}^{m}_{d,a}, \mathfrak{f}^{>m}_{d,a})$. The rationale for considering the force triple is revealed in the following lemma.

Lemma 2. *Let $(\mathfrak{f}^{<m}_{d,a}, \mathfrak{f}^{m}_{d,a}, \mathfrak{f}^{>m}_{d,a})$ be a force triple of G. Let D_n, with $n \geq d$, be an a-restricted drawing of $G \square P_n$. Then there are*

(a) at least $\mathfrak{f}^{<m}_{d,a}$ crossings in D_n from the first m crossings bands;

(b) at least $\mathfrak{f}^{m}_{d,a}$ crossings in D_n from the $(m+i)$-th crossing band for every $0 \leq i \leq n-d$; and

(c) at least $\mathfrak{f}^{>m}_{d,a}$ crossings in D_n from the final $d-m$ crossing bands.

Proof: First, note that D_n contains $n-d+1$ subdrawings of $G \square P_d$. In particular, we denote the j-th subdrawing as D^j_n, for $j \in \{0, \ldots, n-d\}$; it is the one containing copies G^j through to G^{j+d}. Since D_n is an a-restricted drawing, each of these subdrawings is also a-restricted. We can obtain lower bounds on how many crossings in D_n come from each crossing band, by looking at the forced number of crossings in corresponding subdrawings.

Consider D_n^0, and recall that it is a-restricted. Then, by definition, the number of crossings in D_n^0 from its first m crossing bands is at least $\mathfrak{f}_{d,a}^{<m}$. However, the set of potential crossings contained in its first m crossing bands is a subset of the potential crossings contained in the first m crossing bands of D_n. Hence, item (a) is true. Analogous arguments hold for item (b) by considering the m-th crossing band in D_n^i for $i \in \{0, \dots, n-d\}$, and for item (c) by considering the final $d-m$ crossing bands in D_n^{n-d}. □

Lemmas 1 and 2 lead to our first main statement.

Theorem 2. *Let* $(\mathfrak{f}_{d,a}^{<m}, \mathfrak{f}_{d,a}^{m}, \mathfrak{f}_{d,a}^{>m})$ *be a force triple of* G*. If*

$$\operatorname{cr}(G\square P_{d-1}) \geq a(d-1)-b, \qquad \mathfrak{f}_{d,a}^{<m} + \mathfrak{f}_{d,a}^{>m} \geq a(d-1)-b, \qquad \text{and} \qquad \mathfrak{f}_{d,a}^{m} \geq a,$$

then for all $n \geq d$ *we also have*

$$\operatorname{cr}(G\square P_n) \geq an-b.$$

Proof: We will prove by induction, where the base case is known to be true according to the assumptions. Then, suppose that there is some value $n \geq d$ such that $\operatorname{cr}(G\square P_{n-1}) \geq a(n-1)-b$, but $\operatorname{cr}(G\square P_n) < an-b$. Let D_n be a crossing-minimal drawing of $G\square P_n$. By Lemma 1, D_n is a-restricted, and we note that it has fewer than $an-b$ crossings. Then, by Lemma 2, D_n must contain at least $\mathfrak{f}_{d,a}^{<m} + \mathfrak{f}_{d,a}^{>m} + (n-d+1)\mathfrak{f}_{d,a}^{m} \geq an-b$ crossings, contradicting our initial assumption and completing the proof. □

As will be demonstrated in Section 5, Theorem 2 is already sufficient to provide the desired lower bounds in several cases. However, in many cases the force triple values are not large enough to meet the conditions of Theorem 2. In Section 3.1 we briefly discuss why this might occur, and then propose a more sophisticated way of considering the forces which leads to significantly improved results.

3.1 Plus-forces and star-forces

The above results are obtained by partitioning all possible crossings of $G\square P_n$ into various crossing bands, and then considering lower bounds on the number of crossings from those crossing bands. If each of these individual lower bounds is sufficiently large, we obtain the desired result. However, in practice, it is often possible to draw a graph such that there are relatively few crossings in one crossing band, at the cost of a higher number of crossings in nearby crossing bands. Theorem 2 does not take this into account and pessimistically assumes that the minima for each crossing band could be attained simultaneously. Whenever this is not the case, the conditions of Theorem 2 are not met and it cannot be applied.

To remedy this, we propose a more sophisticated way of measuring forces. When calculating the above force values by solving a BCCNP instance, the objective function only considers crossings from certain crossing band(s); except for insisting on an a-restricted drawing, there are no further restrictions on the other crossing bands. This leads to the issue mentioned above, where additional crossings are able to "hide" in the other, largely unrestricted, crossing bands. With this in mind, we introduce *plus-forces*, which are illustrated in Fig. 4. Each plus-force is defined the same as its corresponding force, but with additional restrictions that are placed on the number of crossings in other crossing bands. In particular, for the middle and end plus-forces, restrictions are placed on

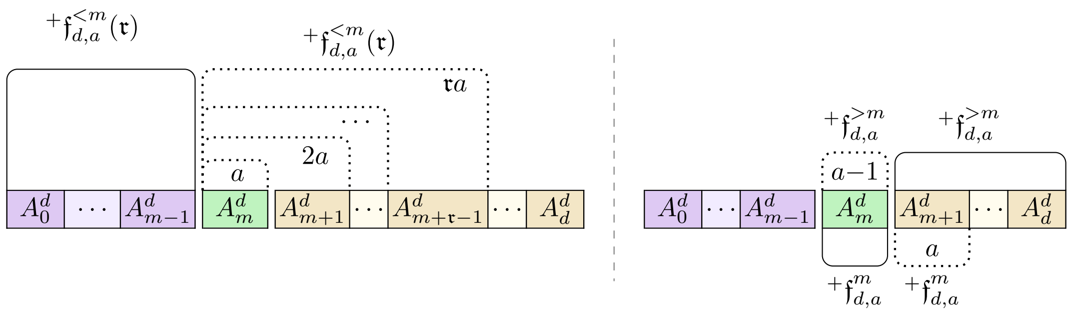


Figure 4: Crossing bands that are considered for counting (solid) and that have required crossings (dotted) in the plus-forces: **(left)** front force, **(right)** middle and end force.

exactly one, adjacent, crossing band. For the front plus-force, restrictions are placed on $\mathfrak{r}$ crossing bands, for $\mathfrak{r} \in \{0, \ldots, d-m+1\}$. Hence, we use the terms $^+\mathfrak{f}^{<m}_{d,a}(\mathfrak{r})$, $^+\mathfrak{f}^{m}_{d,a}$, $^+\mathfrak{f}^{>m}_{d,a}$ to denote the front, middle, and end plus-forces, respectively. We again require $d \geq 2$ and $0 < m < d$:

The *middle plus-force* $^+\mathfrak{f}^{m}_{d,a}$ is the minimum number of crossings from A^d_m in any a-restricted drawing, such that there are at most a crossings from A^d_{m+1}. This can be computed analogously to $\mathfrak{f}^m_{d,a}$, by simply adding the one extra capacity constraint $\sum_{c \in A^d_{m+1}} x^D_c \leq a$.

The *end plus-force* $^+\mathfrak{f}^{>m}_{d,a}$ is the minimum number of crossings from the final $d-m$ crossing bands in any a-restricted drawing, such that there are at most $a-1$ crossings from A^d_m. This can be computed in the same way as $\mathfrak{f}^{>m}_{d,a}$ by adding $\sum_{c \in A^d_m} x^D_c \leq a-1$. We specifically clarify that the right-hand side here is really $a-1$, not a as in the middle plus-force, or a multiple of a as in the front plus-force below.

For the *front plus-force* $^+\mathfrak{f}^{<m}_{d,a}(\mathfrak{r})$, we need to specify $\mathfrak{r} \in \{0, \ldots, d-m+1\}$, which is the number of crossing bands (starting from A^d_m) that restrictions should be placed on. Hence, we define $^+\mathfrak{f}^{<m}_{d,a}(0) \coloneqq \mathfrak{f}^{<m}_{d,a}$, and for $\mathfrak{r} > 0$, $^+\mathfrak{f}^{<m}_{d,a}(\mathfrak{r})$ is the minimum number of crossings from the first m crossing bands in any a-restricted drawing, such that, for every $r \in \{0, 1, \ldots, \mathfrak{r}-1\}$, there are at most $(r+1) \cdot a$ crossings in $\bigcup_{i=m}^{m+r} A^d_i$. Hence, $^+\mathfrak{f}^{<m}_{d,a}(\mathfrak{r})$ can be computed in the same way we compute $\mathfrak{f}^{<m}_{d,a}$ by adding $\mathfrak{r}$ additional capacity constraints:

$$\sum_{i=m}^{m+r} \sum_{c \in A^d_i} x^D_c \leq (r+1) \cdot a, \qquad \forall r \in \{0, 1, \ldots, \mathfrak{r}-1\}.$$

Since the plus-forces are simply more restricted versions of the forces, each plus-force is at least as large as its corresponding force. Whether or not they coincide tells us something about a-restricted drawings of the underlying graph. For example, if $\mathfrak{f}^m_{d,a} < {}^+\mathfrak{f}^m_{d,a}$, it tells us that in order for an a-restricted drawing of $G \square P_d$ to have only $\mathfrak{f}^m_{d,a}$ crossings from A^d_m, it must have more than a crossings from A^d_{m+1}.

To see why such an observation could be valuable, recall that for Theorem 2, we require $\mathfrak{f}^m_{d,a} \geq a$. This requirement ensures there are at least a crossings in each of the "middle" crossing bands (those after the first m and before the final $d-m$ crossing bands), hence ensuring that, for sufficiently large n, drawings of $G \square P_{n+1}$ must have at least a more crossings than drawings of $G \square P_n$. However, the situation could arise where one (or more) of the middle crossing bands has $a-1$ crossings, and yet we would still end up with the expected number of crossings as long as another middle crossing band has at least $a+1$ crossings. Indeed, if $\mathfrak{f}^{m}_{d,a} = a-1$ and ${}^{+}\mathfrak{f}^{m}_{d,a} \geq a$, it tells us that whenever there are $a-1$ crossings in a middle crossing band, the subsequent crossing band (if one exists) must have at least $a+1$ crossings. This observation permits us to adopt a broader condition; we want either (a) $\mathfrak{f}^{m}_{d,a} \geq a$, or (b) $\mathfrak{f}^{m}_{d,a} = a-1$ and ${}^{+}\mathfrak{f}^{m}_{d,a} \geq a$.

However, there are some awkward edge-cases to handle here. For example, what if it is the last middle crossing band that has $a-1$ crossings? Since there is no subsequent middle crossing band (the next crossing band is part of the final $d-m$ crossing bands), we cannot ensure that it has $a+1$ crossings simply by referring to the middle plus-force. Such edge-cases need to be handled delicately. In order to do so, we define the following *star-forces*, in a manner that will make the upcoming proofs neater.

$$\begin{array}{lll} \textit{front star-force} & {}^{*}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r}) & := \min\{\ {}^{+}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r}),\ \mathfrak{f}^{<m}_{d,a}+1\ \} \\ \textit{middle star-force} & {}^{*}\mathfrak{f}^{m}_{d,a} & := \min\{\ {}^{+}\mathfrak{f}^{m}_{d,a},\ \mathfrak{f}^{m}_{d,a}+1\ \} \\ \textit{end star-force} & {}^{*}\mathfrak{f}^{>m}_{d,a} & := \min\{\ {}^{+}\mathfrak{f}^{>m}_{d,a}-1,\ \mathfrak{f}^{>m}_{d,a}\ \} \end{array}$$

In the upcoming Theorem 3, the star-forces, as defined above, will allow us to handle the situation where $\mathfrak{f}^{m}_{d,a} = a-1$ and ${}^{+}\mathfrak{f}^{m}_{d,a} \geq a$, while also taking care of the edge-cases. First, however, we establish a set of intermediate results in Lemma 3. In particular, we give bounds on the number of crossings that must occur in the first m and final $d-m$ crossing bands, and also discuss the manner in which crossings may occur in the middle crossing bands.

Lemma 3. *Let $({}^{*}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r}), {}^{*}\mathfrak{f}^{m}_{d,a}, {}^{*}\mathfrak{f}^{>m}_{d,a})$ be a star-force triple of G, and suppose that ${}^{*}\mathfrak{f}^{m}_{d,a} \geq a$. Let D_n be an a-restricted drawing of $G \square P_n$ for $n \geq d$. Then:*

(a) D_n contains at least ${}^{*}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r}) - 1$ crossings from the first m crossing bands;

(b) D_n contains at least $a-1$ crossings from A^{n}_{i+m} for $i \in \{0, \dots, n-d\}$;

(c) if D_n contains exactly $a-1$ crossings from A^{n}_{i+m} for $i \in \{0, \dots, n-d-1\}$, then D_n contains at least $a+1$ crossings from A^{n}_{i+m+1}; and

(d) Let $z = 1$ if D_n contains exactly $a-1$ crossings from A^{n}_{n-d+m}, and $z = 0$ otherwise. Then, D_n contains at least ${}^{*}\mathfrak{f}^{>m}_{d,a} + z$ crossings from the final $d-m$ crossing bands.

Proof: Item (a): As previously noted, D_n contains $n-d+1$ subdrawings of $G \square P_d$, which we denote as D^{j}_{n} for $j \in \{0, \dots, n-d\}$. From Lemma 2(a), we have that D_n contains at least $\mathfrak{f}^{<m}_{d,a}$ crossings from the first m crossing bands. From the definition of ${}^{*}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r})$, it is clear that $\mathfrak{f}^{<m}_{d,a} \geq {}^{*}\mathfrak{f}^{<m}_{d,a}(\mathfrak{r}) - 1$. Hence, item (a) is true.

Item (b): By Lemma 2(b), we have that D_n contains at least $\mathfrak{f}^{m}_{d,a}$ crossings from A^{n}_{i+m} for $i \in \{0, \dots, n-d\}$. Now consider ${}^{*}\mathfrak{f}^{m}_{d,a}$, which by definition is either equal to ${}^{+}\mathfrak{f}^{m}_{d,a}$ or $\mathfrak{f}^{m}_{d,a}+1$. Since ${}^{+}\mathfrak{f}^{m}_{d,a} \geq \mathfrak{f}^{m}_{d,a}$ and both are integers, there are only two possibilities; either $\mathfrak{f}^{m}_{d,a} = {}^{*}\mathfrak{f}^{m}_{d,a}$, or $\mathfrak{f}^{m}_{d,a} = {}^{*}\mathfrak{f}^{m}_{d,a} - 1$, since, by assumption, ${}^{*}\mathfrak{f}^{m}_{d,a} \geq a$. Either way, item (b) is true.

Item (c): Now, suppose that D_n contains exactly $a-1$ crossings from A^{n}_{i+m}, which is only possible if $\mathfrak{f}^{m}_{d,a} = {}^{*}\mathfrak{f}^{m}_{d,a} - 1 \geq a-1$. This, in turn, is only possible if ${}^{+}\mathfrak{f}^{m}_{d,a} > \mathfrak{f}^{m}_{d,a}$, and so ${}^{+}\mathfrak{f}^{m}_{d,a} \geq a$. According to the definition of ${}^{+}\mathfrak{f}^{m}_{d,a}$, it is hence impossible to draw $G \square P_d$ with fewer than a crossings from A^{d}_{m} and fewer than $a+1$ crossings from A^{d}_{m+1}. However, in the subdrawing D^{i}_{n} there are fewer than a crossings from A^{d}_{m}. Hence, there must be at least $a+1$ crossings in D^{i}_{n} from A^{d}_{m+1},

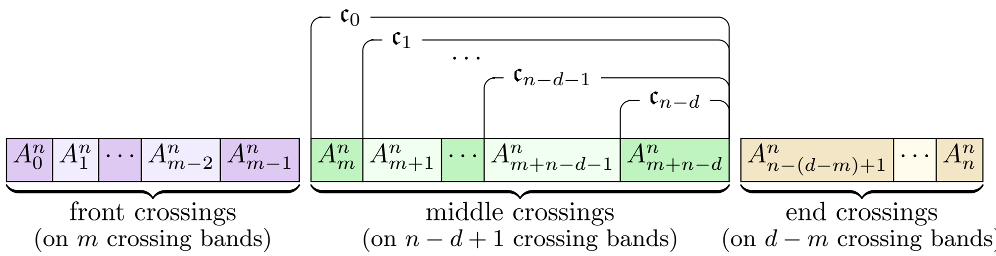


Figure 5: Counting crossings in D_n.

which implies that there are at least $a+1$ crossings in D_n from A_{i+m+1}^n. Therefore item (c) is true.

Item (d): Finally, by Lemma 2(c), we have that D_n contains at least $\mathfrak{f}_{d,a}^{>m}$ crossings from the final $d-m$ crossing bands. Note that by definition we have $\mathfrak{f}_{d,a}^{>m} \geq {}^*\mathfrak{f}_{d,a}^{>m}$, so if $z=0$ then item (d) is true. Suppose that $z=1$, and item (d) is false. This could only be the case if $\mathfrak{f}_{d,a}^{>m} = {}^*\mathfrak{f}_{d,a}^{>m}$, which implies that D_n must have exactly ${}^*\mathfrak{f}_{d,a}^{>m}$ crossings from the final $d-m$ crossing bands, i.e., one crossing less than suggested by item (d). However, since $z=1$, D_n contains exactly $a-1$ crossings from A_{n-d+m}^n by definition. As A_{n-d+m}^n is the m-th crossing band of D_n^{n-d}, the subdrawing D_n^{n-d} has exactly $a-1$ crossings in this band. Since D_n^{n-d} is an a-restricted drawing, by the definition of ${}^+\mathfrak{f}_{d,a}^{>m}$, it must have at least ${}^+\mathfrak{f}_{d,a}^{>m}$ crossings from its final $d-m$ crossing bands. This implies that D_n must contain at least ${}^+\mathfrak{f}_{d,a}^{>m}$ crossings from its final $d-m$ crossing bands. However, by definition we have ${}^+\mathfrak{f}_{d,a}^{>m} \geq {}^*\mathfrak{f}_{d,a}^{>m} + 1 = {}^*\mathfrak{f}_{d,a}^{>m} + z$. Hence item (d) is true. □

Lemma 3 enables us to establish a stronger version of Theorem 2.

Theorem 3. *Let* $({}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}), {}^*\mathfrak{f}_{d,a}^{m}, {}^*\mathfrak{f}_{d,a}^{>m})$ *be a star-force triple of* G *and* $\mathfrak{r}' := \max(\mathfrak{r}, 1)$. *If*

$$\mathrm{cr}(G\square P_{d+\mathfrak{r}'-2}) \geq a(d+\mathfrak{r}'-2)-b, \qquad {}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}) + {}^*\mathfrak{f}_{d,a}^{>m} \geq a(d-1)-b, \qquad \textit{and} \quad {}^*\mathfrak{f}_{d,a}^{m} \geq a$$

then for $n \geq d+\mathfrak{r}'-1$,

$$\mathrm{cr}(G\square P_n) \geq an-b.$$

Proof: Suppose that there is some value $n \geq d+\mathfrak{r}'-1$ such that $\mathrm{cr}(G\square P_{n-1}) \geq a(n-1)-b$, but $\mathrm{cr}(G\square P_n) < an-b$. Let n be minimal with this property. Let D_n be a crossing-minimal drawing of $G\square P_n$, then D_n contains fewer than $an-b$ crossings, and from Lemma 1 it is an a-restricted drawing. We want to count the number of crossings in D_n, cf. Figure 5. We will use the term *front crossings* to refer to the crossings in D_n from the first m crossing bands, *end crossings* to refer to the crossings in D_n from the final $d-m$ crossing bands, and *middle crossings* to refer to all remaining crossings of D_n. To assist in calculating these, let $\mathfrak{c}_i^j$ be the number of crossings in D_n from $A_{m+i}^n \cup \ldots \cup A_{m+j}^n$. To simplify the notation, let $\mathfrak{c}_i := \mathfrak{c}_i^{n-d}$. Note that the number of middle crossings in D_n is equal to $\mathfrak{c}_0$. We seek to compute a lower bound for $\mathfrak{c}_i$ for $i \in \{0, \ldots, n-d\}$. As in Lemma 3 it will be convenient to set $z=1$ if D_n has exactly $a-1$ crossings from A_{n-d+m}^n, and $z=0$ otherwise.

Recall from Lemma 3(b) that $\mathfrak{c}_i^i \geq a-1$, for $i \in \{0, \ldots, n-d\}$. If equality holds for any $i < n-d$, then from Lemma 3(c) we know that $\mathfrak{c}_i^{i+1} \geq (a-1)+(a+1) = 2a$. Essentially, for every middle crossing band except the last, a deficit in crossings will be immediatly evened out by the subsequent crossing band. Inductively then, it is clear that $\mathfrak{c}_i^j \geq (j-i+1)a-1$ for $i,j \in \{0,\dots,n-d\}$ and $i \leq j$, and if equality is met it implies that D_n contains exactly $a-1$ crossings from A_{j+m}^n. Hence, by setting $j = n-d$, we obtain

$$\mathfrak{c}_i \geq (n+1-d-i)a - z, \qquad \forall i \in \{0,\dots,n-d\}. \tag{1}$$

In particular, setting $i = 0$ in (1) tells us that there are at least $\mathfrak{c}_0 = (n+1-d)a - z$ middle crossings, while Lemma 3(a) and Lemma 3(d) give us lower bounds on the front and end crossings respectively. Combining all of these, we can see that D_n contains at least $({}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}) - 1) + \mathfrak{c}_0 + ({}^*\mathfrak{f}_{d,a}^{>m} + z) \geq an - b - 1$ crossings. Since by assumption D_n has fewer than $an - b$ crossings, it must have exactly $an - b - 1$ crossings. In particular, D_n has exactly ${}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}) - 1$ front crossings. However, from the definition of ${}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r})$, this is only possible if ${}^+\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}) > \mathfrak{f}_{d,a}^{<m}$ and $\mathfrak{r} \geq 1$. Hence, D_n must have fewer than ${}^+\mathfrak{f}_{d,a}^{<m}(\mathfrak{r})$ front crossings, and so from the definition of ${}^+\mathfrak{f}_{d,a}^{<m}(\mathfrak{r})$, there must be some positive integer $j \leq \mathfrak{r}$ such that $\mathfrak{c}_0^{j-1} \geq aj+1$. Note that since $n \geq d+\mathfrak{r}'-1$, and $\mathfrak{r} \leq \mathfrak{r}'$, we have $j \leq n-d+1$. If $j = n-d+1$ then $\mathfrak{c}_0 = \mathfrak{c}_0^{j-1} \geq a(n-d+1)+1$. Alternatively, if $j < n-d+1$, then from (1) we have $\mathfrak{c}_j \geq (n+1-d-j)a - z$, and so $\mathfrak{c}_0 = \mathfrak{c}_0^{j-1} + \mathfrak{c}_j \geq (n-d+1)a - z + 1$. Either way, adding $\mathfrak{c}_0$ to $({}^*\mathfrak{f}_{d,a}^{<m}(\mathfrak{r}) - 1) + ({}^*\mathfrak{f}_{d,a}^{>m} + z)$ implies that D_n contains at least $an - b$ crossings, contradicting our initial assumption and completing the proof. □

As will be shown in Section 5, Theorem 3 is much stronger than Theorem 2, and will enable us to established the desired lower bounds for many more graph classes.

4 Upper Bounds, Base Cases, and Computing Forces by Solving BCCNP

Upper bounds. An important ingredient of the crossing number proofs is establishing an upper bound matching the lower bound. In practice, in the case of $G\square P_n$, this turns out to be surprisingly simple, as simple enumeration schemes for simple drawings suffice. In fact, in most cases all but the very first and the very last copy of G can be drawn identically, and these drawings are easy to figure out. Thus, we will not discuss this further in the following, but state that whenever we could prove a lower bound, we also identified a construction for the matching upper bound. The tables in Section 5.2 show the corresponding subdrawings (and explain their interpretation).

Base cases and force computation. To apply our above theorems, each graph class $G\square P_n$ requires us to establish (a) $\mathrm{cr}(G\square P_n)$ for small values of n as the base cases, and (b) G-specific force values: sound proofs of BCCNP for small values d, m. Each thereby considered graph is relatively small (typically below 50 edges), but since our goal is to solve many different graph classes, there are too many such instances to reasonably do them all by hand.

It was shown in Buchheim et al. [10] that CNP can be formulated as an integer linear program (ILP). Subsequently, Chimani et al. [11,13] implemented this formulation as a practical algorithm, which requires sophisticated column generation techniques and other speed-up heuristics, to be tractable for "real-world" graphs with up to 100 edges. As this implementation is generally not easy to understand and validate, Chimani and Wiedera [14] presented a proof and validation framework that extracts a crossing number proof from the ILP computation; this certificate can be independently validated (with comparably simple methods). This tool has since regularly been used to establish base cases or to validate results, e.g., [8,9,15,16,21,31,38–41]. The modifications necessary to tackle BCCNP instead of CNP are relatively straight-forward: analogously to the case of the *simultaneous crossing number* [12], when restricting the objective function to disregard certain crossings, these crossing variables are considered with a small enough coefficient ε instead of 0, to avoid issues with non-simple drawings. The capacity constraints can directly be added to the ILP computation and the proof validator as well. We call this algorithm the *BCCNP solver*; all corresponding certificates for the crossing number proofs required below (as well as the proof validator), can be found at [5].

5 Calculations

Theorems 2 and 3 are only useful when the conditions are met, and checking if they are met requires us to compute multiple forces using the BCCNP solver for some setting of the parameters. Clearly, there are three possible outcomes to this approach. The first outcome is that the values computed by the BCCNP solver satisfy the requirements of at least one of the two theorems, and the desired result is obtained. The second outcome is that the values computed by the BCCNP solver do not satisfy the requirements of either theorem, and no result is obtained. The third outcome is that the BCCNP solver is unable to compute the values, as the graph is too large or complex to be tackled with reasonable computing power[1].

To test the efficacy of the two proposed approaches, we ran them both on all 21 non-isomorphic connected graphs on five vertices, and all 112 non-isomorphic connected graphs on six vertices. Since Theorems 2 and 3 attempt to show that $\mathrm{cr}(G \square P_n) \geq an - b$, we began by assuming that $\mathrm{cr}(G \square P_n) = an - b$ for some integers a, b. We computed $\mathrm{cr}(G \square P_n)$ for some small values of n, and used these to predict the values of a, b. Then, we checked to see if they matched the upper bound yielded from our simple enumeration schemes. Indeed, in all tested cases, the values of a, b matched.

Then, armed with an appropriate choice of a, b, we computed the forces for the (d, m) pair $(2, 1)$, and checked to see if they satisfy the relevant conditions for Theorem 2. If not, we then proceeded to compute the plus-forces for the same (d, m) pair and $\mathfrak{r} = d - m + 1$. Then, we checked if the relevant conditions of Theorem 3 were satisfied first for $\mathfrak{r} = 0$ (by using ${}^{+}\mathfrak{f}^{<m}_{d,a}(0) = \mathfrak{f}^{<m}_{d,a}$), and if not, then for $\mathfrak{r} = d - m + 1$. If any of these checks were successful, we computed the relevant base cases to settle the instance; note that checking for $\mathfrak{r} = 0$ first is worthwhile because the corresponding base case is considerably smaller than for $\mathfrak{r} = n - d + 1$. If none of the checks were successful, then we then repeated the above for the (d, m) pairs $(3, 1)$, and $(4, 2)$, only declaring failure if all of these proved unsuccessful.

5.1 Overview on the results

We summarize the results of our calculations in Table 1. In particular, we note that we obtained a successful result for all 21 graphs on five vertices, and 107 (out of 112) graphs on six vertices. The successful instances include 60 graphs on six vertices which, according to the recent survey by

[1] All computations were conducted on an Intel Xeon Gold 6134 with a time limit of 2 hours. Nearly 3/4 of the calculations finished in under a minute and less than 0.05% needed more than half an hour but less than two hours. This indicates that neither more computation power nor a higher time limit is likely to yield significant improvements.

$G \square P_n$, where G are	**Theorem 2**		**Theorem 3** with $\mathfrak{r} = 0$		**Theorem 3** with $\mathfrak{r} = d - m + 1$		**total**	
5-vertex graphs	19	90%	21	100%	21	100%	**21**	**100%**
6-vertex graphs	91	81%	104	93%	105	94%	**107**	**96%**
7-vertex graphs	529	62%	573	67%	675	79%	**676**	**79%**

Table 1: The number of instances (as well as percentage) for which Theorems 2 and 3 yield positive results for at least one (d, m) pair $(2, 1)$, $(3, 1)$, or $(4, 2)$.

Clancy et al. [15], have not previously been handled in the literature[2].

Concisely, we establish the following results w.r.t. 6-vertex graphs:

Theorem 4. *Let i, in G_i^6, be the graph indices from [15] to name all 6-vertex graphs[3]. For large enough n (in most cases $n \geq 1$), the crossing number $\mathrm{cr}(G_i^6 \square P_n)$ is given by Table 2. The table includes all 112 connected 6-vertex graphs except for $i = 31, 48, 73, 142, 156$.*

In Section 5.2, we list all these graphs, together with their upper bound construction and the required values for $d, m, \mathfrak{r}$. As mentioned above, the computer generated proof files for the BCCNP base cases (together with a proof validator) can be found at [5]. It is worth noting that in 78 of the 112 cases the smallest setting, $d = 2$ and $m = 1$, already suffices to prove the claim with Theorem 3 and in 66 cases, even with Theorem 2.

We note that, of the five graphs on six vertices we were unable to obtain a solution, four have already been handled in the literature. In particular, the graph with index $i = 31$ was handled in Bokal [7], the graphs with indices $i = 48, 73$ were handled in Klešč and Petrillová [30], and the graph with index $i = 156$ was handled in Zheng et al. [43]. As such, there is only one graph remaining from the six-vertex set that is yet to be handled in the literature at all, specifically the graph with index $i = 142$. It is equivalent to K_5 with a pendant edge attached to one of the vertices, and based on our experiments it appears the crossing number should be $9n - 3$. We discuss some possible reasons why our approach may have proved unsuccessful for these five graphs in Section 6.

Finally, we also tackled the Cartesian products of paths with any connected 7-vertex graph. There are few existing crossing number results on these graph classes (nine, according to [15]). Our goal here is to determine how many of these graph classes are in fact relatively simple—in terms of solvability using our framework—and thus do not merit specific treatment in the literature. On the other hand, we hope that (failings of) our approach may point at graphs which require us to identify new lower bound arguments. Overall, we see that already with only small values of $d, m, \mathfrak{r}$, our approach generates proofs for nearly 4/5 of all $G^7 \square P_n$ graph classes. Again, see the subsequent Section 5.2 for more details.

5.2 Details on the results

Section 5.2 list the smallest (d, m) pair for which at least one of Theorems 2 and 3 establishes the lower bounds for each successfully solved graph. The background color intensity reflects how

[2] This includes instances which have not been handled at all in the literature, and also instances which have only been handled in journals or periodicals which Clancy et al. [15] determined "impose no peer review, or that which does occur is inadequate." They note that such results "should be revisited and submitted to thorough peer review". As such, we treat such results as new in this manuscript.

[3] The index scheme from [15] includes the 44 disconnected 6-vertex graphs, and hence goes up to 156.

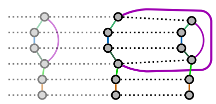
Table 2: Results for 6-vertex graphs, see Theorem 4.

$\mathrm{cr}(G_i^6 \square P_n)$	i (new results)	i (previously established, reproved)
$0n + 0$		$25, 40$
$1n - 1$		$26, 28, 41, 43, 46, 60$
$2n - 4$		42
$2n - 2$	$49,$	$27, 29, 44, 45, 47, 53, 54, 59, 63, 64, 66, 74, 77, 83, 94$
$2n + 0$		$61, 75, 85$
$3n - 5$		62
$3n - 3$	$67, 76, 78, 92,$	$51, 65, 70, 89, 90, 120$
$3n - 1$	$84, 95, 98, 110$	$68, 71, 86, 87, 91, 111$
$4n - 4$		$31, 72, 79, 80, 104, 113$
$4n - 2$	$82, 88, 99, 100, 101, 108, 115, 116, 118, 133$	
$4n$	$93, 109, 112, 121, 130, 137$	
$5n - 3$	$81, 102, 106, 107, 114, 122, 123, 124, 127,$	125
$5n - 1$	$131, 138, 146$	
$6n - 4$	134	
$6n - 2$	$105, 126, 128, 129, 132, 140, 141, 143,$	103
$6n$	152	
$7n - 1$	$119, 135, 139, 145, 148$	
$8n - 2$	$144, 147, 151$	
$9n - 3$	150	
$9n - 1$	154	
$10n$	$149, 153$	
$12n$	155	

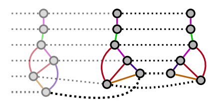
high the values are. The `d,m`* indicates that Theorem 2 does not suffice and that Theorem 3 with $\mathtt{r} = 0$ is needed. `d,m`$^*_*$ indicates that Theorem 3 with $\mathtt{r} = d - m + 1$ is needed. The blueness of the background also reflects this. Proof files for each of the obtained values can be found at [5] along with a tool that can be used to verify the results.

Then, all that remains is to handle the upper bounds. Also contained in Section 5.2 are small figures corresponding to each graph, indicating a drawing procedure which establishes the desired upper bound. The small figures should be interpreted as follows. To draw $G \square P_n$ with the indicated number of crossings, $n + 1$ copies of G should be drawn side by side, with the path edges forming horizontal lines. We call the first and last copies of G the *end copies*, and the remaining $n - 1$ copies are the *middle copies*. In the small figures from Section 5.2, we show two drawings of G. The end copies should be drawn identically to the right-most drawing in the small figure (taking the mirror image for the first copy). All of the middle copies should be drawn identically to the left-most drawing in the small figure. The edge colors are chosen arbitrarily, but are consistent between the copies. Any edges which end up stubs at the top and bottom are routed around the graph and connected according to the symbol of the stub and the edge color. The figures are presented in such a way that routing the edges around the graph never needs to introduce any additional crossings. If, in the small figure, the left-most drawing contains c_l crossings, and the right-most drawing contains c_r crossings, then an upper bound of $c_l(n - 1) + 2c_r$ is established for $\mathrm{cr}(G \square P_n)$. For the graphs marked by a ! between its two copies in Section 5.2, we need to draw two neighboring copies together either at the ends or in the middle for a sufficient upper bound.

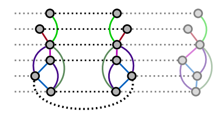
Essentially, we use one (or more) of the following three strategies:



Two last copies (G^6_{42})



Two last copies (G^6_{63})

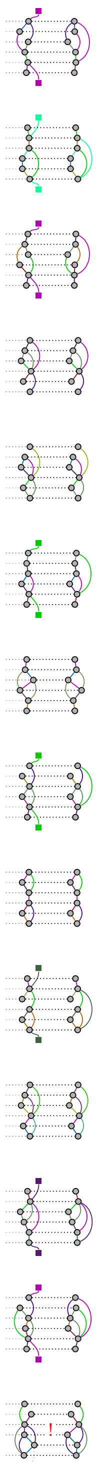


Two middle copies (G^6_{82})

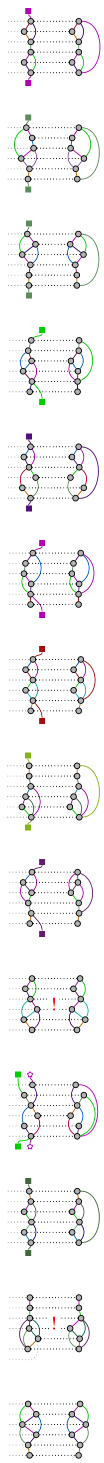
We use the following symbols to indicate the status of the results:

i		G_i^5	cr	d,m
2	✔		$2n-2$	3,1*
3	✔		$1n-1$	2,1
4	✔		$1n-1$	2,1
5	✔		$1n-1$	2,1
6	✔		$2n-2$	3,1*
7	✔		$1n-1$	2,1
9	✔		$2n-2$	2,1
10	✔		$2n+0$	2,1
11	✔		$2n-2$	2,1
12	✔		$2n-2$	2,1
13	✔		$1n-1$	2,1
14	✔		$2n-2$	2,1
15	✔		$3n-1$	2,1
16	✔		$3n-1$	2,1
17	✔		$2n+0$	2,1
18	✔		$3n-1$	2,1
19	✔		$3n-1$	2,1
20	✔		$4n+0$	2,1
21	✔		$6n+0$	2,1

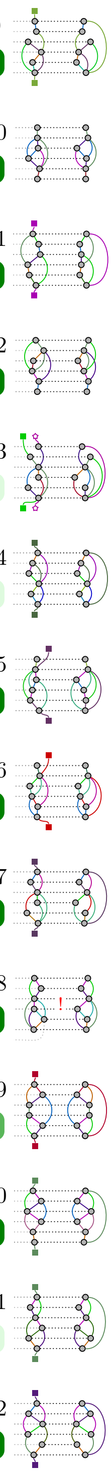
i		G_i^6	cr	d,m
25	✔		$0n+0$	2,1
26	✔		$1n-1$	2,1
27	✔		$2n-2$	3,1*
28	✔		$1n-1$	2,1
29	✔		$2n-2$	3,1*
31	✔		$4n-4$	
40	✔		$0n+0$	2,1
41	✔		$1n-1$	2,1
42	✔		$2n-4$	4,2
43	✔		$1n-1$	2,1
44	✔		$2n-2$	2,1
45	✔		$2n-2$	2,1
46	✔		$1n-1$	2,1
47	✔		$2n-2$	3,1*
48	✔		$4n-4$	
49	+		$2n-2$	2,1
51	✔		$3n-3$	3,1*
53	✔		$2n-2$	3,1*
54	✔		$2n-2$	2,1
59	✔		$2n-2$	2,1
60	✔		$1n-1$	2,1
61	✔		$2n+0$	2,1
62	✔		$3n-5$	4,2*
63	✔		$2n-2$	2,1
64	✔		$2n-2$	2,1
65	✔		$3n-3$	2,1
66	✔		$2n-2$	2,1
67	+		$3n-3$	3,1



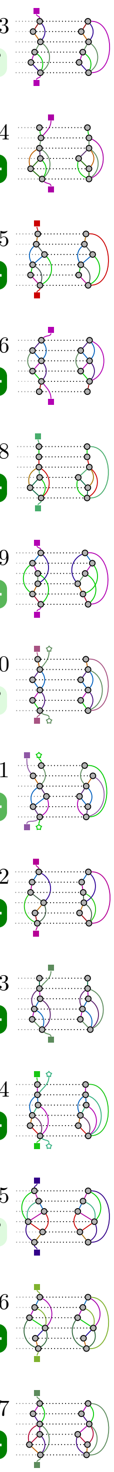
i		G_i^6	cr	d,m
68	✓		$3n-1$	2,1
70	✓		$3n-3$	3,1
71	✓		$3n-1$	2,1
72	✓		$4n-4$	$3,1^*$
73	✓ (grey)		$4n-4$	
74	✓		$2n-2$	2,1
75	✓		$2n+0$	$3,1^*_*$
76	✚		$3n-3$	2,1
77	✓		$2n-2$	$3,1^*$
78	✚		$3n-3$	$3,1^*$
79	✓		$4n-4$	$3,1^*$
80	✓		$4n-4$	$2,1^*$
81	✚		$5n-3$	3,1
82	✚		$4n-2$	$3,1^*$



i		G_i^6	cr	d,m
83	✓		$2n-2$	2,1
84	✚		$3n-1$	2,1
85	✓		$2n+0$	2,1
86	✓		$3n-1$	2,1
87	✓		$3n-1$	$2,1^*_*$
88	✚		$4n-2$	3,1
89	✓		$3n-3$	2,1
90	✓		$3n-3$	2,1
91	✓		$3n-1$	2,1
92	✚		$3n-3$	2,1
93	✚ (light)		$4n+0$	2,1
94	✓		$2n-2$	2,1
95	✚		$3n-1$	2,1
98	✚		$3n-1$	$3,1^*_*$



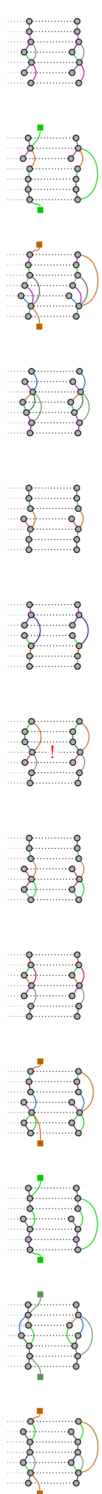
i		G_i^6	cr	d,m
99	✚		$4n-2$	3,1
100	✚		$4n-2$	2,1
101	✚		$4n-2$	2,1
102	✚		$5n-3$	$3,1^*$
103	✓		$6n-2$	$2,1^*_*$
104	✓		$4n-4$	$2,1^*$
105	✚		$6n-2$	$3,1^*_*$
106	✚		$5n-3$	3,1
107	✚		$5n-3$	$2,1^*_*$
108	✚		$4n-2$	2,1
109	✚ (light)		$4n+0$	2,1
110	✚		$3n-1$	2,1
111	✓		$3n-1$	2,1
112	✚		$4n+0$	2,1



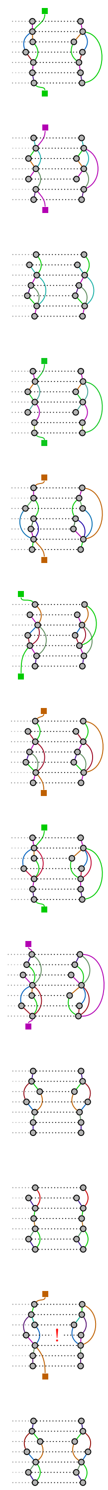
i		G_i^6	cr	d,m
113	✓		$4n-4$	2,1
114	✚		$5n-3$	3,1
115	✚		$4n-2$	2,1
116	✚		$4n-2$	2,1
118	✚		$4n-2$	2,1
119	✚ (light)		$7n-1$	$2,1^*_*$
120	✓		$3n-3$	2,1
121	✚ (light)		$4n+0$	2,1
122	✚		$5n-3$	2,1
123	✚		$5n-3$	3,1
124	✚		$5n-3$	2,1
125	✓		$5n-3$	2,1
126	✚		$6n-2$	$2,1^*_*$
127	✚		$5n-3$	$2,1^*_*$

i	G_i^6	cr	d,m
128		$6n{-}2$	$2,1^*_*$
129		$6n{-}2$	3,1
130		$4n{+}0$	2,1
131		$5n{-}1$	2,1
132		$6n{-}2$	3,1
133		$4n{-}2$	2,1
134		$6n{-}4$	3,1
135		$7n{-}1$	2,1
137		$4n{+}0$	2,1
138		$5n{-}1$	2,1
139		$7n{-}1$	$2,1^*_*$
140		$6n{-}2$	$2,1^*_*$
141		$6n{-}2$	2,1
142		$9n{-}3$	
143		$6n{-}2$	2,1
144		$8n{-}2$	2,1
145		$7n{-}1$	2,1
146		$5n{-}1$	2,1
147		$8n{-}2$	3,1
148		$7n{-}1$	2,1
149		$10n{+}0$	$2,1^*_*$
150		$9n{-}3$	2,1
151		$8n{-}2$	2,1
152		$6n{+}0$	2,1
153		$10n{+}0$	2,1
154		$9n{-}1$	2,1
155		$12n{+}0$	$2,1^*$
156		$15n{+}3$	

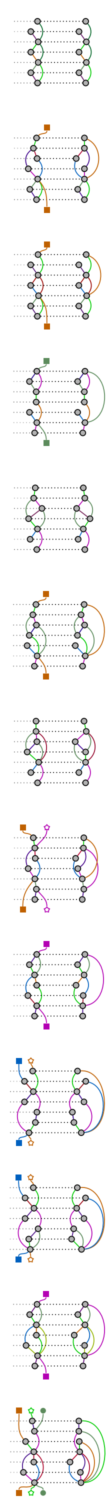
i	G_i^7	cr	d,m
4		$2n{-}2$	$3,1^*$
6		$4n{-}4$	$3,1^*$
9		$2n{-}2$	$3,1^*$
10		$3n{-}3$	2,1
11		$3n{-}3$	$3,1^*$
12		$5n{-}3$	$3,1^*_*$
15		$4n{-}2$	$2,1^*_*$
16		$6n{-}2$	$2,1^*_*$
17		$6n{-}4$	$3,1^*_*$
21		$2n{-}2$	$3,1^*$
24		$1n{-}1$	2,1
25		$3n{-}3$	3,1



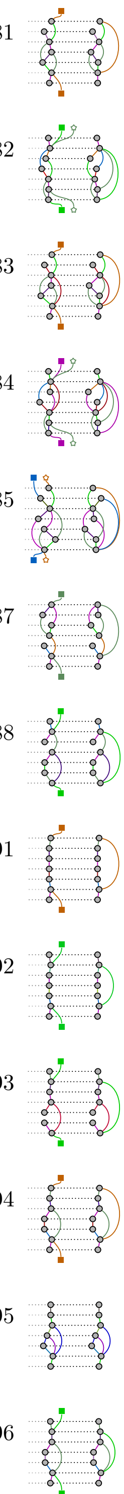
i	cr	d,m
26	$2n-2$	2,1
27	$2n-2$	2,1
29	$4n-4$	3,1
30	$4n-4$	2,1
32	$1n-1$	2,1
33	$2n-2$	$3,1^{*}$
34	$3n-5$	$4,2^{*}$
35	$2n-2$	$3,1^{*}$
36	$2n-2$	2,1
38	$4n-4$	$3,1^{*}$
45	$2n-2$	2,1
46	$4n-2$	$2,1^{*}_{*}$
47	$3n-3$	2,1



i	cr	d,m
48	$3n-1$	2,1
49	$4n-4$	$3,1^{*}$
50	$4n-4$	$3,1^{*}$
51	$3n-3$	2,1
52	$5n-3$	$2,1^{*}_{*}$
53	$6n-4$	$3,1^{*}_{*}$
54	$5n-5$	$3,1^{*}$
55	$5n-3$	$3,1^{*}_{*}$
56	$7n-5$	$4,2^{*}_{*}$
57	$2n-1$	$3,1^{*}_{*}$
58	$2n-2$	$3,1^{*}$
59	$4n-4$	4,2
60	$3n-1$	$3,1^{*}_{*}$

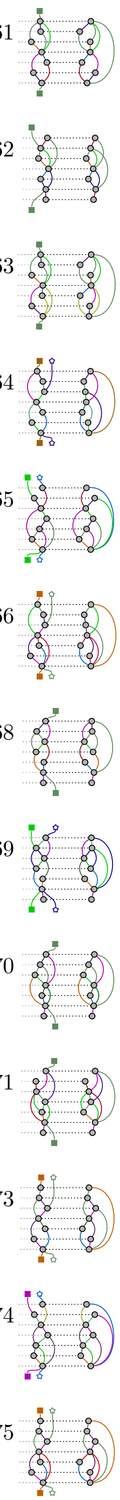


i	cr	d,m
62	$3n-3$	$3,1^{*}$
63	$5n-3$	$3,1^{*}_{*}$
65	$5n-5$	$3,1^{*}$
67	$3n-3$	2,1
68	$3n-1$	$3,1^{*}_{*}$
69	$5n-3$	$3,1^{*}_{*}$
71	$5n-3$	$4,2^{*}_{*}$
73	$6n-2$	$2,1^{*}_{*}$
74	$4n-2$	$2,1^{*}_{*}$
75	$5n-1$	$2,1^{*}_{*}$
76	$6n-2$	$2,1^{*}_{*}$
78	$6n-4$	$2,1^{*}_{*}$
80	$8n-4$	$3,1^{*}_{*}$

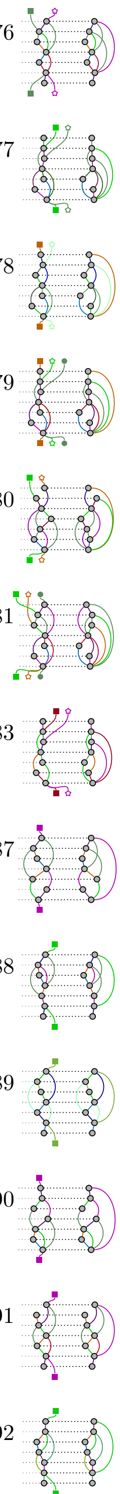


i	cr	d,m
81	$4n-2$	2,1
82	$6n-2$	$2,1^{*}_{*}$
83	$6n-4$	$3,1^{*}_{*}$
84	$8n-4$	$4,2^{*}_{*}$
85	$8n+0$	$2,1^{*}_{*}$
87	$4n-2$	$2,1^{*}_{*}$
88	$4n-4$	$3,1^{*}$
91	$1n-1$	2,1
92	$2n-2$	2,1
93	$3n-3$	2,1
94	$3n-3$	2,1
95	$3n-3$	2,1
96	$4n-4$	3,1

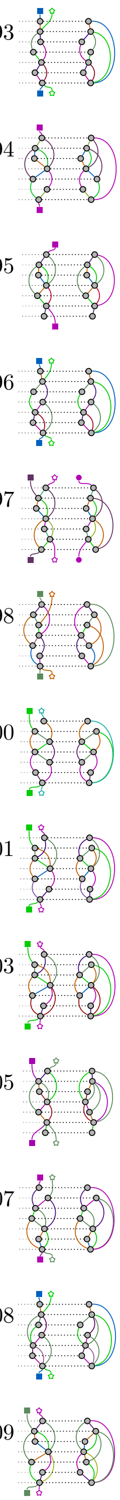
i	G_i^7	cr	d,m
97		$2n-2$	2,1
98		$4n-2$	$3,1^*_*$
99		$3n-3$	2,1
100		$4n-4$	2,1
101		$5n-5$	$3,1^*$
102		$4n-4$	2,1
103		$6n-6$	$3,1^*$
104		$2n-4$	4,2
105		$1n-1$	2,1
107		$2n-2$	2,1
108		$4n-4$	$3,1^*$
109		$3n-5$	$4,2^*$
110		$3n-3$	3,1
112		$2n-2$	2,1
114		$4n-4$	$3,1^*$
116		$4n-2$	$2,1^*_*$
117		$5n-3$	2,1
118		$5n-3$	$3,1^*_*$
119		$4n-2$	2,1
120		$6n-4$	$3,1^*_*$
121		$6n-2$	4,2
122		$6n-4$	$2,1^*_*$
124		$6n-4$	$3,1^*_*$
125		$8n-4$	$4,2^*_*$
126		$2n+0$	2,1
127		$2n-2$	2,1
128		$3n-1$	2,1
129		$4n-2$	$2,1^*_*$
130		$4n-4$	3,1
131		$5n-3$	$3,1^*_*$
132		$3n-1$	2,1
133		$4n-4$	2,1
135		$4n-2$	$3,1^*_*$
136		$4n-2$	$3,1^*_*$
137		$5n-3$	$2,1^*_*$
138		$6n-4$	$3,1^*_*$
139		$5n-5$	$2,1^*$
141		$5n-3$	$2,1^*_*$
142		$7n-5$	$3,1^*_*$
143		$4n-2$	2,1
147		$4n-2$	2,1
148		$4n-2$	$4,2^*_*$
149		$5n-3$	$3,1^*_*$
150		$3n-3$	2,1
152		$5n-3$	$2,1^*_*$
153		$6n-4$	$3,1^*_*$
154		$5n-5$	2,1
155		$7n-5$	$2,1^*_*$
156		$4n+0$	2,1
157		$5n-3$	$3,1^*_*$
158		$4n-2$	$2,1^*_*$
160		$4n-4$	2,1



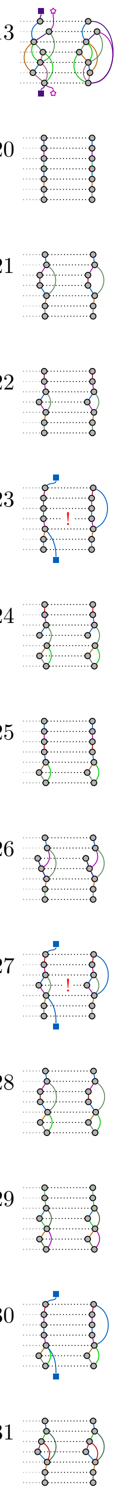
i	G_i^7	cr	d,m
161		$6n-2$	$2,1^*_*$
162		$6n-4$	$3,1^*_*$
163		$8n-4$	$4,2^*_*$
164		$6n-2$	$2,1^*_*$
165		$7n-1$	$2,1^*_*$
166		$8n-4$	$3,1^*_*$
168		$5n-1$	4,2
169		$7n-3$	4,2
170		$7n-3$	3,1
171		$8n-4$	$4,2^*_*$
173		$5n-1$	4,2
174		$6n-2$	$3,1^*_*$
175		$7n-3$	$3,1^*_*$



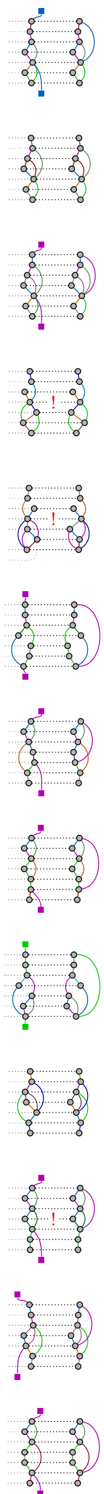
i	G_i^7	cr	d,m
176		$7n-5$	$3,1^*_*$
177		$8n-4$	$4,2^*_*$
178		$6n-4$	$2,1^*_*$
179		$8n-4$	$3,1^*_*$
180		$8n+0$	$2,1^*_*$
181		$10n-2$	$3,1^*_*$
183		$6n-2$	$2,1^*_*$
187		$6n-2$	2,1
188		$6n-4$	$3,1^*_*$
189		$6n-2$	$3,1^*_*$
190		$5n-3$	$2,1^*_*$
191		$6n-4$	$3,1^*_*$
192		$5n-3$	$3,1^*_*$



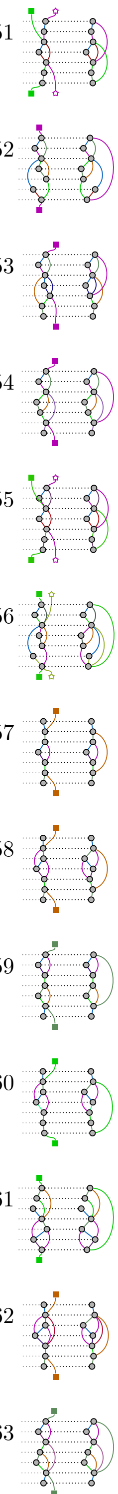
i	G_i^7	cr	d,m
193		$4n-2$	$2,1^*_*$
194		$7n-3$	4,2
195		$7n-3$	4,2
196		$6n-4$	$2,1^*_*$
197		$8n-4$	4,2
198		$9n-1$	$2,1^*_*$
200		$6n+0$	2,1
201		$9n-1$	$2,1^*_*$
203		$11n-3$	$2,1^*_*$
205		$9n-3$	$4,2^*_*$
207		$8n-2$	$2,1^*_*$
208		$8n-4$	4,2
209		$10n-2$	$3,1^*_*$



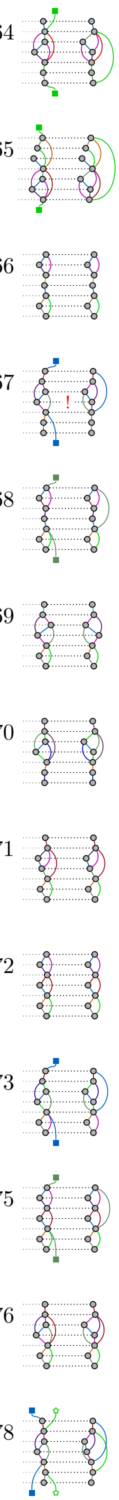
i	G_i^7	cr	d,m
213		$11n-1$	$2,1^*_*$
220		$0n+0$	2,1
221		$2n-2$	2,1
222		$1n-1$	2,1
223		$2n-4$	4,2
224		$2n-2$	$3,1^*$
225		$1n-1$	2,1
226		$3n-3$	$3,1^*$
227		$3n-5$	4,2
228		$3n-3$	$3,1^*$
229		$2n-2$	2,1
230		$3n-3$	3,1
231		$3n-3$	2,1



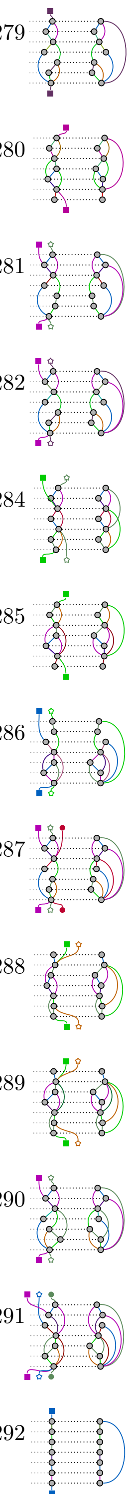
i	G_i^7	cr d,m	
233		$4n-4$	3,1
235		$4n-4$	2,1
236		$6n-6$	$3,1^*$
237		$3n-3$	$3,1^*$
240		$4n-2$	$3,1^*_*$
242		$3n-1$	2,1
243		$5n-3$	$3,1^*_*$
244		$3n-3$	2,1
245		$4n-2$	$2,1^*_*$
246		$5n-3$	$3,1^*_*$
248		$4n-6$	$4,2^*$
249		$5n-3$	$3,1^*_*$
250		$4n-4$	3,1



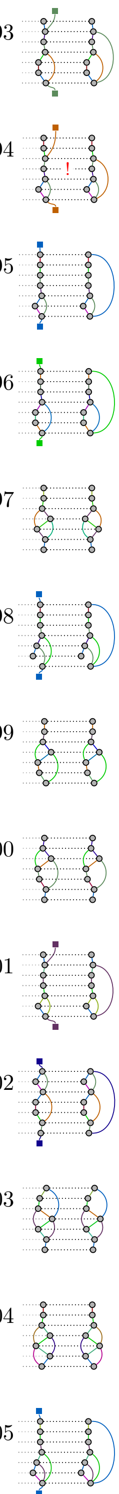
i	G_i^7	cr d,m	
251		$5n-3$	$3,1^*_*$
252		$5n-3$	$2,1^*_*$
253		$6n-4$	$3,1^*_*$
254		$5n-5$	2,1
255		$6n-4$	$3,1^*_*$
256		$7n-5$	$2,1^*_*$
257		$3n-3$	2,1
258		$4n-2$	2,1
259		$3n-3$	2,1
260		$3n-1$	2,1
261		$5n-3$	$2,1^*_*$
262		$6n-4$	$3,1^*_*$
263		$5n-5$	2,1



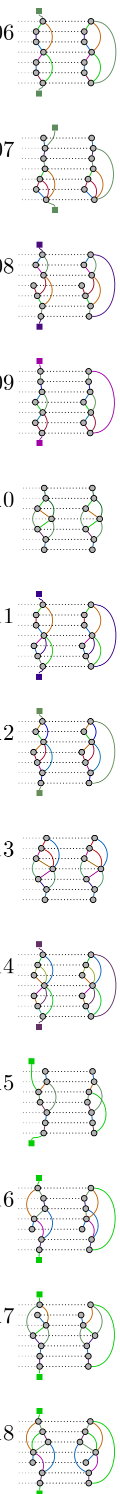
i	G_i^7	cr d,m	
264		$5n-3$	$3,1^*_*$
265		$7n-5$	$3,1^*_*$
266		$2n-2$	$3,1^*$
267		$4n-4$	4,2
268		$4n-4$	3,1
269		$3n-1$	$3,1^*_*$
270		$4n-2$	$2,1^*_*$
271		$4n-4$	$3,1^*$
272		$3n-3$	$3,1^*$
273		$5n-3$	$3,1^*_*$
275		$5n-5$	$3,1^*$
276		$5n-3$	$4,2^*_*$
278		$6n-2$	$2,1^*_*$



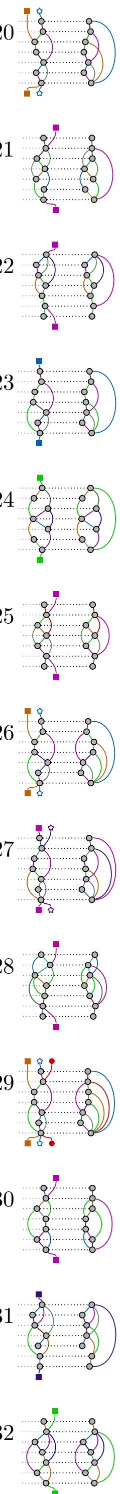
i	G_i^7	cr d,m	
279		$5n-3$	$2,1^*_*$
280		$4n-2$	2,1
281		$5n-1$	$3,1^*_*$
282		$6n-2$	4,2
284		$7n-5$	$3,1^*_*$
285		$6n-4$	$2,1^*_*$
286		$7n-3$	$3,1^*_*$
287		$8n-4$	$3,1^*_*$
288		$6n-2$	$3,1^*_*$
289		$8n-4$	4,2
290		$8n+0$	$2,1^*_*$
291		$10n-2$	$3,1^*_*$
292		$0n+0$	2,1



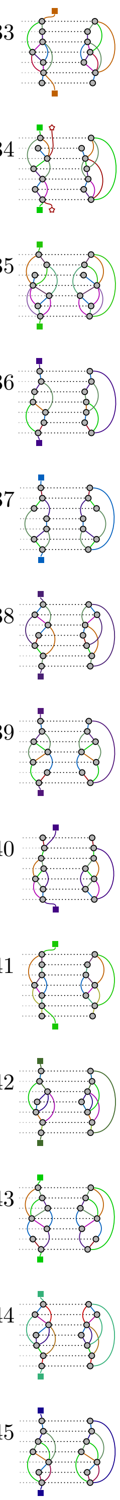
i	G_i^7	cr	d,m
293		$3n-3$	2,1
294		$3n-5$	4,2
295		$1n-1$	2,1
296		$2n-2$	2,1
297		$2n+0$	3,1$^*_*$
298		$3n-3$	3,1
299		$3n-2$	3,1$^*_*$
300		$3n-1$	3,1$^*_*$
301		$2n-2$	2,1
302		$3n-3$	2,1
303		$4n-2$	3,1$^*_*$
304		$3n-1$	3,1$^*_*$
305		$3n-3$	2,1



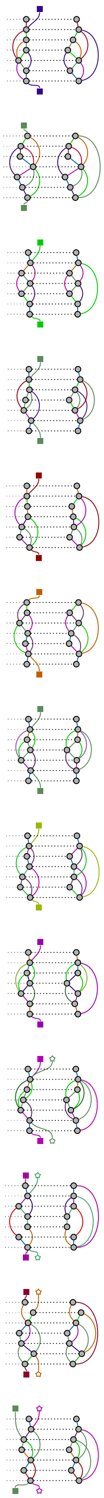
i	G_i^7	cr	d,m
306		$4n-4$	2,1
307		$4n-4$	3,1
308		$4n-4$	3,1
309		$2n-2$	2,1
310		$3n-1$	3,1$^*_*$
311		$5n-5$	3,1
312		$4n-4$	2,1
313		$5n-3$	3,1$^*_*$
314		$6n-6$	2,1
315		$4n-2$	3,1$^*_*$
316		$5n-3$	3,1$^*_*$
317		$4n-2$	3,1$^*_*$
318		$6n-2$	4,2$^*_*$

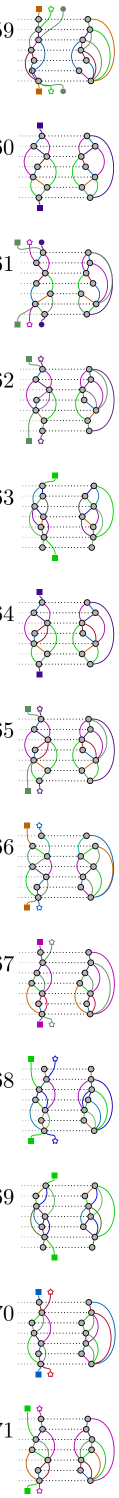


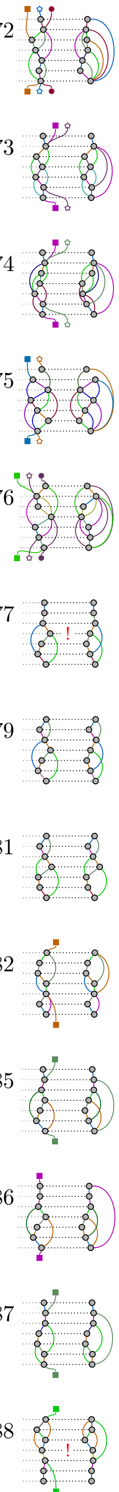
i	G_i^7	cr	d,m
320		$5n-3$	2,1
321		$5n-1$	3,1$^*_*$
322		$5n-3$	3,1$^*_*$
323		$5n-3$	2,1$^*_*$
324		$5n-1$	2,1
325		$5n-3$	2,1
326		$7n-3$	3,1$^*_*$
327		$6n-4$	2,1$^*_*$
328		$7n-3$	4,2
329		$8n-4$	3,1$^*_*$
330		$5n-1$	2,1
331		$6n-4$	3,1$^*_*$
332		$7n-3$	4,2$^*_*$

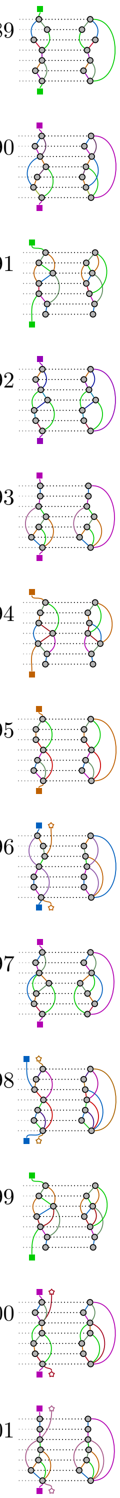


i	G_i^7	cr	d,m
333		$7n-3$	4,2
334		$7n-3$	4,2
335		$9n-3$	4,2$^*_*$
336		$4n-2$	2,1
337		$4n+0$	2,1
338		$5n-1$	2,1$^*_*$
339		$4n+0$	2,1
340		$4n-2$	2,1
341		$5n-3$	3,1$^*_*$
342		$5n-3$	2,1
343		$6n-2$	4,2
344		$6n-2$	4,2
345		$6n-2$	2,1$^*_*$



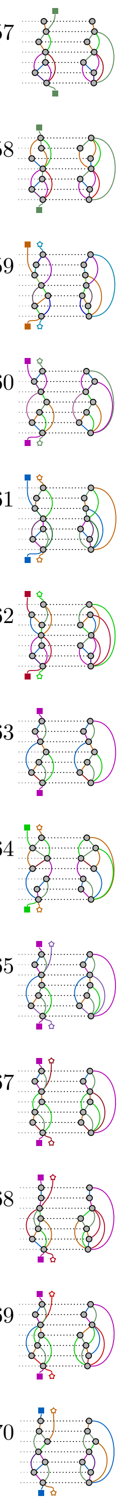
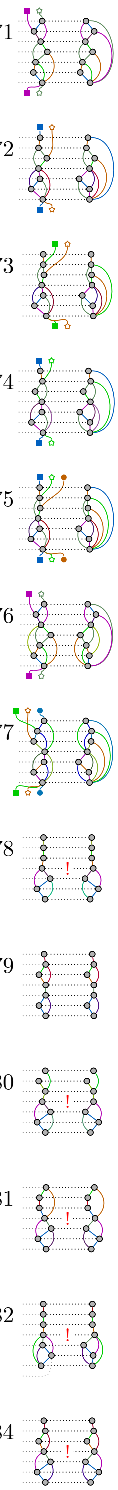
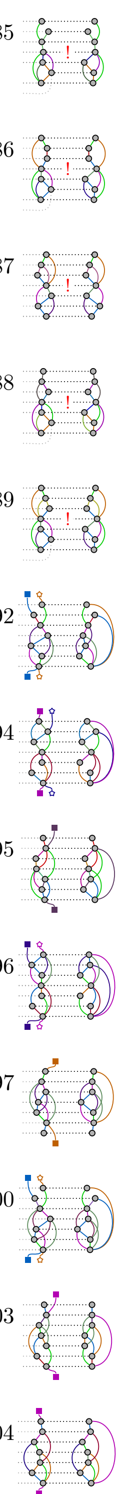
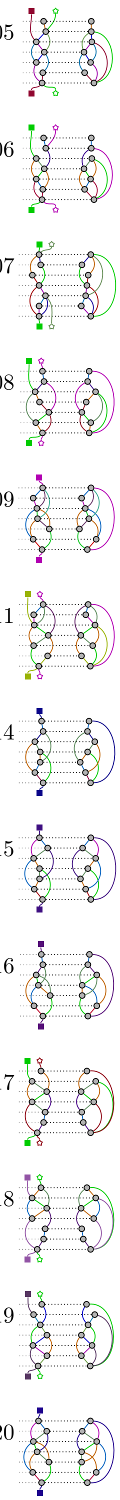






i	G_i^7	cr d,m	i	G_i^7	cr d,m	i	G_i^7	cr d,m	i	G_i^7	cr d,m
346		$6n-4$ $2,1^*_*$	359		$9n-3$ $4,2^*_*$	372		$9n-3$ $2,1^*_*$	389		$3n-1$ $2,1$
347		$8n-4$ $2,1^*_*$	360		$6n+0$ $2,1^*_*$	373		$7n-3$ $2,1$	390		$5n-3$ $2,1^*_*$
348		$4n-2$ $2,1$	361		$7n-1$ $2,1^*_*$	374		$9n-3$ $2,1^*_*$	391		$6n-2$ $4,2$
349		$7n-5$ $3,1^*_*$	362		$8n-2$ $2,1^*_*$	375		$9n-1$ $2,1$	392		$4n-2$ $2,1^*_*$
350		$6n-4$ $3,1^*_*$	363		$7n-3$ $4,2$	376		$11n-1$ $3,1^*_*$	393		$5n-1$ $2,1$
351		$6n-4$ $2,1^*_*$	364		$7n-1$ $2,1^*_*$	377		$4n-4$ $4,2$	394		$6n-2$ $4,2$
352		$7n-3$ $4,2$	365		$9n-3$ $2,1^*_*$	379		$5n-3$ $3,1^*_*$	395		$5n-5$ $3,1$
353		$7n-5$ $3,1^*_*$	366		$7n-3$ $4,2$	381		$4n-2$ $3,1^*_*$	396		$6n-4$ $2,1^*_*$
354		$6n-2$ $3,1$	367		$7n-3$ $4,2$	382		$6n-4$ $3,1^*_*$	397		$6n-2$ $4,2$
355		$8n-4$ $4,2$	368		$9n-3$ $4,2^*_*$	385		$5n-3$ $2,1$	398		$7n-5$ $4,2$
356		$5n-1$ $3,1^*_*$	369		$8n-4$ $4,2$	386		$4n-2$ $2,1^*_*$	399		$7n-3$ $3,1^*_*$
357		$7n-3$ $4,2$	370		$7n-5$ $2,1^*_*$	387		$4n-2$ $2,1$	400		$6n-4$ $2,1^*_*$
358		$7n-5$ $2,1^*_*$	371		$8n-2$ $3,1^*_*$	388		$4n-4$ $4,2$	401		$7n-3$ $3,1^*_*$

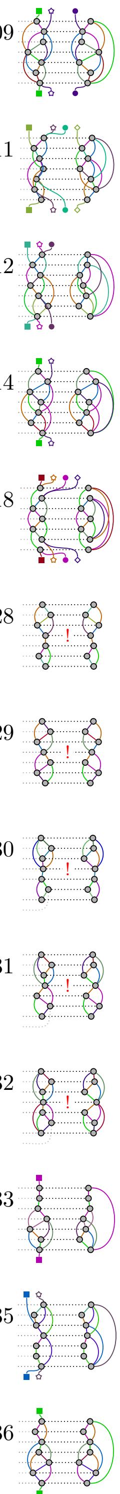
i	G_i^7	cr	d,m
402		$8n-4$	4,2
403		$3n-3$	2,1
404		$4n-4$	3,1
405		$2n-2$	2,1
406		$3n-1$	2,1$^*_*$
407		$4n-2$	2,1
408		$4n-4$	2,1
409		$5n-3$	3,1$^*_*$
410		$3n-1$	2,1$^*_*$
411		$5n-3$	3,1$^*_*$
412		$4n-2$	2,1$^*_*$
413		$5n-3$	2,1$^*_*$
414		$6n-4$	3,1$^*_*$
415		$6n-6$	3,1*
416		$6n-2$	4,2$^*_*$
417		$5n-3$	2,1$^*_*$
418		$7n-5$	3,1$^*_*$
419		$4n-2$	2,1
420		$5n-3$	3,1$^*_*$
421		$3n-3$	2,1
422		$4n-2$	2,1
423		$5n-1$	4,2
424		$4n-2$	4,2$^*_*$
425		$4n-4$	2,1
426		$5n-3$	2,1$^*_*$
427		$5n-1$	4,2
428		$5n-5$	2,1
429		$5n-5$	3,1*
430		$6n-4$	3,1$^*_*$
431		$4n-4$	2,1
432		$5n-3$	2,1$^*_*$
433		$6n-4$	3,1$^*_*$
434		$7n-5$	3,1$^*_*$
435		$6n-2$	4,2
436		$5n-5$	2,1
437		$6n-4$	2,1$^*_*$
438		$7n-5$	2,1$^*_*$
439		$5n-3$	3,1$^*_*$
440		$5n-5$	4,2
441		$4n-2$	3,1$^*_*$
442		$5n-3$	3,1$^*_*$
443		$6n-4$	3,1$^*_*$
444		$6n-4$	3,1$^*_*$
446		$6n-4$	3,1$^*_*$
448		$4n+0$	2,1
449		$5n-3$	3,1$^*_*$
450		$4n-2$	2,1
451		$5n-3$	2,1$^*_*$
452		$5n-3$	3,1$^*_*$
454		$6n-2$	3,1$^*_*$
455		$6n-2$	2,1$^*_*$
456		$7n-5$	3,1$^*_*$

i	G^7_i	cr	d,m
457		$6n-4$	$3,1^*_*$
458		$8n-4$	$4,2^*_*$
459		$6n-2$	$2,1^*_*$
460		$7n-1$	$2,1^*_*$
461		$8n-4$	$3,1^*_*$
462		$9n-3$	$3,1^*_*$
463		$6n-2$	$3,1^*_*$
464		$7n-3$	$4,2$
465		$8n-4$	$4,2^*_*$
467		$7n-5$	$2,1^*_*$
468		$8n-2$	$2,1^*_*$
469		$9n-3$	$4,2$
470		$5n-1$	$2,1$
471		$6n-2$	$4,2$
472		$7n-3$	$3,1^*_*$
473		$8n-4$	$4,2$
474		$6n-4$	$2,1^*_*$
475		$8n-4$	$2,1^*_*$
476		$8n+0$	$2,1^*_*$
477		$10n-2$	$3,1^*_*$
478		$2n-2$	$2,1$
479		$2n-2$	$2,1$
480		$3n-3$	$3,1^*$
481		$4n-4$	$3,1^*$
482		$3n-1$	$2,1^*_*$
484		$3n-3$	$2,1$
485		$4n-2$	$3,1^*_*$
486		$5n-3$	$3,1^*_*$
487		$5n-5$	$3,1^*$
488		$4n-2$	$2,1^*_*$
489		$6n-4$	$3,1^*_*$
492		$7n-1$	$2,1^*_*$
494		$7n-3$	$2,1^*_*$
495		$8n-4$	$4,2$
496		$9n-3$	$2,1^*_*$
497		$9n-3$	$3,1^*$
500		$10n-2$	$2,1^*_*$
503		$8n-2$	$3,1^*_*$
504		$8n-2$	$3,1^*_*$
505		$7n-3$	$4,2$
506		$7n-3$	$3,1^*_*$
507		$7n-3$	$2,1^*_*$
508		$7n-1$	$2,1^*_*$
509		$9n-3$	$3,1^*_*$
511		$8n-2$	$4,2$
514		$7n-1$	$2,1^*_*$
515		$8n-2$	$2,1^*_*$
516		$7n-1$	$2,1$
517		$6n-2$	$2,1$
518		$6n-2$	$3,1$
519		$5n-1$	$2,1$
520		$8n-2$	$2,1$

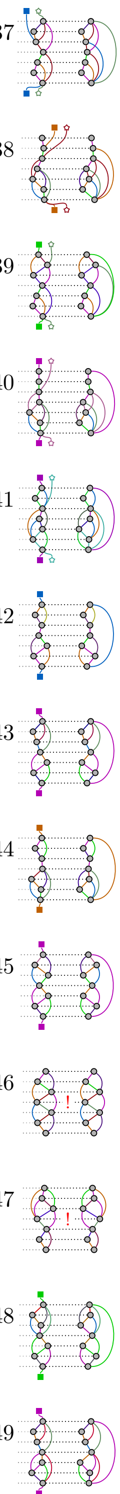
i	G_i^7	cr	d,m
521		$9n-3$	3,1$^*_*$
522		$7n-3$	2,1
523		$10n-4$	2,1$^*_*$
524		$8n+0$	2,1
525		$9n-1$	2,1
526		$10n-1$	3,1$^*_*$
527		$10n-2$	3,1$^*_*$
528		$11n-3$	2,1$^*_*$
529		$10n-4$	3,1$^*_*$
534		$8n-2$	2,1$^*_*$
535		$8n-2$	2,1
537		$8n-4$	4,2
538		$10n-2$	2,1$^*_*$
539		$9n-3$	2,1$^*_*$
541		$13n-1$	2,1$^*_*$
542		$6n-2$	2,1$^*_*$
543		$6n-2$	3,1*
544		$6n-2$	2,1
545		$5n-1$	2,1
546		$6n-4$	2,1$^*_*$
547		$6n-2$	3,1
548		$5n-3$	2,1$^*_*$
549		$6n-2$	4,2
550		$6n-4$	3,1$^*_*$
551		$5n-3$	2,1$^*_*$
552		$6n-2$	3,1
553		$5n-1$	2,1
554		$5n-1$	3,1$^*_*$
555		$7n-1$	2,1$^*_*$
556		$7n-3$	2,1
557		$7n-3$	2,1$^*_*$
558		$7n-3$	4,2
559		$7n-3$	2,1$^*_*$
560		$7n-3$	3,1$^*_*$
561		$9n-3$	2,1$^*_*$
562		$6n-2$	2,1$^*_*$
563		$9n-3$	3,1$^*_*$
564		$8n-2$	3,1$^*_*$
567		$7n-3$	4,2
568		$6n-4$	3,1$^*_*$
569		$4n-2$	2,1
571		$7n-3$	2,1$^*_*$
572		$6n-4$	2,1
573		$8n-4$	4,2$^*_*$
574		$8n-4$	3,1$^*_*$
575		$9n-1$	2,1$^*_*$
576		$10n-2$	3,1$^*_*$
577		$10n-4$	3,1$^*_*$
578		$11n-3$	2,1$^*_*$
579		$10n-4$	3,1$^*_*$
582		$11n-3$	2,1$^*_*$
583		$13n-1$	2,1$^*_*$

i	G_i^7	cr	d,m
584		$12n{+}0$	$2,1^*_*$
585		$11n{-}1$	$2,1^*_*$
586		$11n{-}1$	2,1
589		$10n{-}2$	$2,1^*_*$
591		$9n{-}1$	$2,1^*_*$
592		$10n{-}4$	$3,1^*_*$
593		$10n{-}4$	$3,1^*_*$
595		$9n{-}3$	3,1
599		$4n{-}2$	2,1
600		$2n{+}0$	$2,1^*_*$
601		$3n{-}1$	2,1
602		$4n{-}2$	$2,1^*_*$
603		$3n{-}1$	2,1
604		$5n{-}3$	2,1
605		$5n{-}3$	$2,1^*_*$
606		$7n{-}5$	$2,1^*_*$
607		$6n{-}2$	2,1
608		$5n{-}1$	2,1
609		$6n{+}0$	$2,1^*_*$
610		$6n{-}2$	2,1
611		$5n{-}3$	2,1
612		$7n{-}3$	$3,1^*_*$
613		$7n{-}1$	2,1
614		$7n{-}3$	4,2
615		$9n{-}3$	$2,1^*_*$
616		$5n{-}1$	2,1
617		$6n{-}4$	4,2
618		$7n{-}3$	4,2
619		$8n{-}4$	4,2
620		$8n{-}2$	$2,1^*_*$
621		$8n{-}4$	4,2
622		$9n{-}3$	$2,1^*_*$
623		$5n{-}1$	2,1
624		$4n{+}0$	2,1
625		$5n{-}1$	2,1
626		$4n{+}0$	2,1
627		$4n{-}2$	2,1
628		$6n{-}2$	2,1
629		$6n{-}4$	$2,1^*_*$
630		$6n{-}2$	2,1
631		$7n{-}3$	$2,1^*_*$
632		$6n{-}2$	2,1
633		$6n{-}4$	2,1
634		$8n{-}4$	$3,1^*_*$
635		$8n{-}2$	2,1
636		$9n{-}3$	$2,1^*_*$
637		$8n{-}2$	$2,1^*_*$
638		$8n{-}2$	$2,1^*_*$
639		$8n{-}4$	4,2
640		$8n{-}2$	$2,1^*_*$
641		$10n{-}2$	$2,1^*_*$
642		$7n{-}3$	2,1

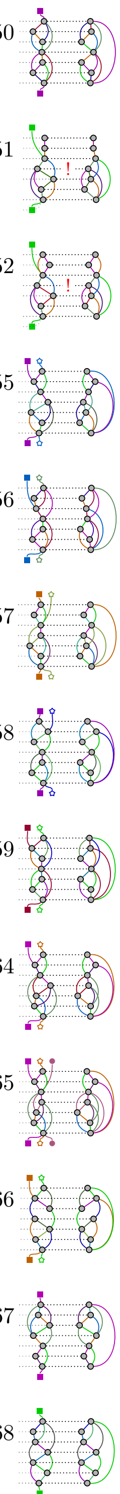
i	G_i^7	cr	d,m
643		$9n-3$	$2,1^*_*$
644		$9n-5$	$4,2$
645		$9n-3$	$2,1^*_*$
646		$5n-1$	$2,1$
647		$7n-1$	$2,1$
648		$7n-1$	$2,1$
649		$7n-3$	$4,2$
650		$8n+0$	$4,2$
651		$9n-3$	$2,1^*_*$
652		$9n-1$	$2,1$
653		$11n-1$	$2,1^*_*$
654		$9n-1$	$2,1^*_*$
655		$9n-1$	$2,1^*_*$
657		$10n-2$	$2,1^*_*$
658		$10n-2$	$4,2^*_*$
659		$12n-2$	$3,1^*_*$
660		$7n-1$	$2,1^*_*$
661		$7n-3$	$4,2$
662		$7n-3$	$2,1$
664		$8n-2$	$3,1$
665		$8n-2$	$3,1^*$
666		$8n-2$	$3,1^*_*$
667		$8n-2$	$2,1^*_*$
668		$10n-2$	$3,1^*_*$
670		$9n-3$	$3,1$
672		$9n-1$	$2,1$
673		$10n-2$	$2,1$
676		$11n-3$	$2,1^*_*$
677		$12n-2$	$2,1^*_*$
680		$11n-3$	$3,1$
682		$9n-1$	$2,1^*_*$
683		$12n-2$	$2,1^*_*$
687		$11n-1$	$2,1^*_*$
688		$10n-4$	$2,1^*_*$
690		$12n-2$	$2,1^*_*$
692		$8n-2$	$2,1$
693		$7n-1$	$2,1$
694		$8n+0$	$3,1^*_*$
695		$6n+0$	$2,1$
696		$9n-3$	$3,1^*_*$
697		$6n+0$	$2,1$
698		$10n-2$	$3,1^*_*$
699		$9n-3$	$2,1$
700		$9n-3$	$2,1$
701		$8n-2$	$2,1$
702		$11n-3$	$2,1^*_*$
703		$10n-2$	$2,1$
704		$11n-1$	$2,1^*_*$
705		$10n-2$	$2,1^*_*$
706		$12n-2$	$2,1^*_*$
707		$9n-1$	$2,1$
708		$9n-1$	$2,1^*_*$



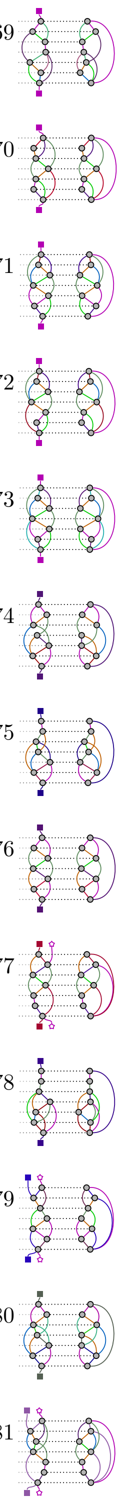
i	G_i^7	cr	d,m
709		$11n-1$	$2,1^*_*$
711		$13n-1$	2,1
712		$12n+0$	2,1
714		$12n-2$	2,1
718		$10n-2$	2,1
728		$3n-3$	3,1
729		$4n-4$	2,1
730		$4n-2$	$3,1^*_*$
731		$5n-3$	$2,1^*_*$
732		$6n-2$	$2,1^*_*$
733		$5n-1$	2,1
735		$7n-5$	$3,1^*_*$
736		$6n-2$	2,1



i	G_i^7	cr	d,m
737		$8n-6$	$3,1^*_*$
738		$8n-4$	$4,2^*_*$
739		$6n-4$	2,1
740		$7n-3$	4,2
741		$8n-4$	$3,1^*_*$
742		$3n-1$	$2,1^*_*$
743		$5n-3$	$2,1^*_*$
744		$5n-3$	$2,1^*_*$
745		$4n+0$	2,1
746		$5n-3$	$3,1^*_*$
747		$6n-4$	$4,2^*_*$
748		$6n-2$	4,2
749		$7n-5$	$3,1^*_*$



i	G_i^7	cr	d,m
750		$8n-4$	4,2
751		$6n-2$	3,1
752		$7n-3$	$4,2^*_*$
755		$7n-1$	2,1
756		$9n-5$	$4,2^*_*$
757		$8n-4$	4,2
758		$7n-3$	$2,1^*_*$
759		$9n-3$	$2,1^*_*$
764		$10n-2$	$2,1^*_*$
765		$12n-4$	$2,1^*_*$
766		$6n+0$	2,1
767		$6n-2$	2,1
768		$6n-2$	2,1



i	G_i^7	cr	d,m
769		$7n-1$	2,1
770		$7n-5$	$3,1^*_*$
771		$7n-1$	2,1
772		$8n-4$	$2,1^*_*$
773		$9n-3$	$2,1^*_*$
774		$8n-2$	2,1
775		$7n-1$	2,1
776		$5n-1$	2,1
777		$6n-2$	2,1
778		$8n-2$	$3,1^*$
779		$5n-1$	2,1
780		$8n-2$	2,1
781		$9n-3$	3,1

i	G_i^7	cr	d,m
782		$7n-3$	2,1
783		$10n-4$	$2,1^*_*$
784		$8n-2$	3,1
785		$6n+0$	2,1
786		$9n-1$	2,1
787		$10n+0$	$4,2^*_*$
788		$11n-3$	$2,1^*_*$
789		$10n-4$	$3,1^*_*$
791		$10n-4$	$2,1^*_*$
794		$8n-2$	2,1
795		$10n-2$	$2,1^*_*$
796	!	$10n-2$	$2,1^*_*$
797		$9n-3$	2,1
798		$12n-2$	$2,1^*_*$
799		$13n-1$	$3,1^*_*$
800		$12n+0$	2,1
801		$11n-1$	2,1
802		$14n-2$	$2,1^*$
804		$8n-2$	2,1
805		$8n-2$	2,1
806		$11n-3$	$3,1^*_*$
807		$9n-3$	2,1
808		$10n-2$	2,1
809		$11n-3$	$2,1^*_*$
810		$10n+0$	2,1
811		$13n-1$	$2,1^*_*$
812		$8n+0$	2,1
813		$9n-1$	2,1
814		$7n-1$	2,1
815		$10n-2$	2,1
817		$8n+0$	2,1
818		$10n-2$	2,1
819		$11n-3$	2,1
820		$13n-3$	$2,1^*_*$
821		$10n+0$	2,1
822		$12n-4$	$3,1^*_*$
823		$13n-1$	$2,1^*_*$
825		$13n-1$	2,1
826		$12n+0$	2,1
827		$15n-1$	2,1
830		$13n-1$	2,1
831		$11n-1$	$2,1^*_*$
833		$13n+1$	$2,1^*_*$

6 Conclusions

The potency of the approach described in this manuscript is demonstrated by the number of new results that have been established, particularly given that this field of research has been saturated for decades with papers painstakingly establishing one or two results at a time.

In addition to the volume of new results, another benefit of this approach is that it lets us identify the truly difficult graphs, compared to graphs which just require more exhaustive versions of existing arguments. These difficult graphs can then be given individual attention. As shown in the previous section, our approach was successful for all but five graphs from the six-vertex set. Four of these five graphs were previously handled in the literature. Two of them ($i = 31, 156$) required specialised arguments applicable only to those graphs, while the other two ($i = 48, 73$) were corollaries. Researchers in this field can now focus on the one remaining case ($i = 142$), potentially handling not only this graph, but perhaps a family of graphs which emerge from complete graphs with pendant edges added.

Our star-forces, as strengthenings of our basic forces, have essentially accounted for cases where one crossing was able to "hide" in a nearby crossing band. Although this proved to be powerful, it stands to reason that there could be instances where multiple crossings are able to hide, and an accordingly stronger definition and theorem statement would be required. Indeed, an analysis of the $i = 31$ instance shows this to be exactly the case; in this graph, two crossings can hide. Given this context, our approach could perhaps be viewed as the first step in a more general approach that seeks to account for cases where arbitrarily many crossings can hide. Furthermore, it seems likely that the BCCNP framework can be used to attack other families of graphs, particularly those resulting from graph products. In particular, we anticipate that it should be possible to obtain new results for Cartesian products with cycles, although this will likely require further fine-tuning of the crossing bands and corresponding forces. Other common families such as Cartesian products with stars, as well as other kinds of graph products such as strong products, should also be considered in this framework.

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