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On Relations between Neighborhoods of Threshold and Ferrers Digraphs

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Abstract. Ferrers digraphs have linearly nested in- and out-neighborhoods that define two rankings on the vertices. Removing the loops turns Ferrers digraphs into threshold digraphs in which the neighborhoods are no longer nested, in general. We show that the two vertex rankings of a Ferrers digraph must preserve two strict vertex preorders of its underlying threshold digraph, which are equivalent to vertex relations similar to neighborhood inclusion. The vertex relations obtained in this way offer the possibility of completing the classification of threshold digraphs based on neighborhood-inclusion criteria. In line with existing criteria, we introduce medial and hierarchical Ferrers digraphs, whose rankings are identical (up to reversal). We characterize threshold digraphs where either all or none of the corresponding Ferrers digraphs of a single threshold digraph are medial or hierarchical. These characterizations are found to be consistent with previous work on uniquely ranked threshold digraphs. Our results therefore affirm the validity of the neighborhood-inclusion criteria defined for simple digraphs.

1 Introduction

In network analysis, identifying structurally important nodes is a common objective and is most often addressed by applying a centrality index. By associating a numerical value with every node, centrality indices immediately provide a complete ranking of the nodes and thus constitute a deductive approach. In contrast, an inductive approach through network positions has been suggested, where a node's centrality is assessed via positional dominance [2]. That is, the dyadic relations between the nodes that constitute the network are compared to determine which nodes dominate other nodes in their network position. In graph-theoretic terms, this approach translates into the comparison of neighborhoods of vertices. Accordingly, for simple undirected graphs, the neighborhood-inclusion preorder has been identified as the common basis of centrality indices [19]. Threshold graphs [12] are a class of simple undirected graphs where the neighborhoods of vertices

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are linearly nested, i.e., where the vicinal preorder [9] on the vertices is total. This makes threshold graphs idealized representations of centered networks, on which centrality rankings necessarily agree [20].

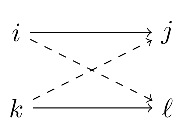
The idea of neighborhood inclusion as the core axiom for centrality has recently been extended to simple directed graphs [3, 14, 15]. The proposed directed neighborhood-inclusion criteria were found to be total on subclasses of threshold digraphs [4], termed uniquely ranked threshold digraphs, akin to the case of undirected threshold graphs. However, not all threshold digraphs satisfy at least one of the criteria, which raises the question of additional notions of centrality or other types of hidden ranking. Unless directed graphs are required to be simple, Ferrers digraphs [17] have been considered a generalization of threshold graphs because of their linearly nested in- and out-neighborhoods [5]. Both threshold and Ferrers digraphs are closely related, since the former correspond to the latter with loops removed [16]. Motivated by threshold digraphs missing a neighborhood-inclusion ranking, we therefore examine the correspondence of threshold and Ferrers digraphs from the neighborhood-inclusion perspective. This examination also allows us to assess the construct validity of the same notions of centrality across different digraph classes.

The remainder is organized as follows. After introducing the notation in Section 2, we present known characterizations of threshold and Ferrers digraphs and their link in Section 3. Section 4 proves necessary and sufficient conditions under which the vertices of a threshold digraph must or must not have a loop in the corresponding Ferrers digraph(s). In Section 5, we express strict vertex preorders introduced by Reilly et al. [16] as relations similar to neighborhood inclusions. By means of this expression, we show that the vertex rankings of a Ferrers digraph must preserve the strict vertex preorders of its underlying threshold digraph. Section 6 characterizes threshold digraphs for which the two rankings of their corresponding Ferrers digraph(s) are either always or never the same (up to reversal). In Section 7, these subclasses are compared with uniquely ranked threshold digraphs. We conclude with future directions in Section 8.

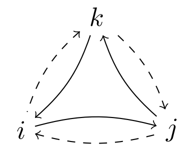
2 Notation

We consider directed graphs (digraphs) $D_\circ = (V, E)$ consisting of a finite set V of n vertices and a set $E \subseteq (V \times V)$ of directed edges. For an edge $(i, j) \in E$, we call i the source and j the target and also write $i \longrightarrow j$; for $(i, j) \notin E$ we also write $i \dashrightarrow j$. If $(i, j) \in E$ or $(j, i) \in E$, we call i and j adjacent, if both $(i, j) \in E$ and $(j, i) \in E$, we call i and j reciprocally adjacent and otherwise nonadjacent. Edges $(i, j) \in E$ with $i = j$ are called loops. A directed graph (V, E) where loops are not allowed, i.e., where $E \subseteq (V \times V) \setminus \{(i, i) : i \in V\}$ is called simple. For differentiation, unless otherwise stated, we denote by D a simple digraph. For a digraph $D_\circ = (V, E)$, we denote the in- and out-neighborhood of a vertex $i \in V$ by $\Gamma^+_{D_\circ}(i) := \{v \in V : (i, v) \in E\}$ and $\Gamma^-_{D_\circ}(i) := \{v \in V : (v, i) \in E\}$. Note that since $E \subseteq V \times V$, it is possible that $i \in \Gamma^+_{D_\circ}(i), \Gamma^-_{D_\circ}(i)$. For a simple digraph $D = (V, E)$, we denote the in- and out-neighborhood of a vertex $i \in V$ by $N^-_D(i) := \{j : (j, i) \in E\}$ and $N^+_D(i) := \{j : (i, j) \in E\}$ with the correspondingly closed neighborhoods $N^-_D[i] := N^-_D(i) \cup \{i\}$ and $N^+_D[i] := N^+_D(i) \cup \{i\}$. If it is obvious from the context to which D or $D_\circ$ refers, we omit the subscript and write $N^+(i), N^-(i), N^+[i], N^-(i)$, and $\Gamma^+(i), \Gamma^-(i)$. This notation, which uses different characters for the neighborhoods according to whether a digraph is simple or not (N versus Γ), is intended to avoid subscripts. A walk of length ℓ in a digraph is a sequence of vertices $v_0 v_1 \dots v_\ell$ with $(v_i, v_{i+1}) \in E$ for all $i = 0, \dots, \ell - 1$. In case all v_i, $i = 0, \dots, \ell$, are distinct, a walk is called a *path*. Visually, we will also write $v_0 \longrightarrow v_1 \longrightarrow \dots \longrightarrow v_\ell$. If there exists a walk from i to j, we say i reaches j and write $i \longrightarrow^* j$.

A binary relation $\leqslant$ over a set V is a subset of the Cartesian product $\leqslant \subseteq V \times V = \{(i, j) : i, j \in$



(a) 2-switch



(b) induced directed 3-cycle

Figure 1: Forbidden configurations for threshold digraphs. Solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

$V\}$. For $(i,j) \in \leqslant$ we write $i \leqslant j$ in infix notation. A reflexive and transitive $\leqslant$ is called a preorder or partial ranking. A binary relation is total (or complete) if $i \leqslant j$ or $j \leqslant i$ for any $i, j \in V$. A total preorder is also called (complete) ranking. An irreflexive and transitive binary relation is called a strict preorder or a strict partial order and is denoted by $<$. A reflexive, symmetric, and transitive binary relation is called an equivalence relation and is denoted by $\sim$.

3 Correspondence between Threshold and Ferrers Digraphs

In this section, known characterizations of threshold and Ferrers digraphs are presented through which the two digraph classes can be linked. For threshold digraphs, four equivalent characterizations were provided by Cloteaux et al. [4, Theorem 1]. In this work, we use the one based on forbidden configurations that are shown in Figure 1.

Definition 1 (Threshold digraph)**.** *A simple digraph is called threshold if it contains neither a 2-switch nor an induced directed 3-cycle.*

We denote by $\mathcal{T}_n$ the set of threshold digraphs on n vertices. Since the number of vertices does not change in the present context, we also omit the subscript and write $\mathcal{T}$. In addition to the four aforementioned characterizations, Reilly et al. [16] presented three others, one of which is based on an auxiliary bipartite digraph representation defined as follows.

Definition 2 (Auxiliary bipartite digraph, [16, Definition 7])**.** *Let $D = (V, E)$ be a digraph. Let V^+, V^- be copies of V, i.e., for any $i \in V$ there is an $i^+ \in V^+$ and an $i^- \in V^-$. The auxiliary bipartite digraph B_D of D is defined by $B_D := (V^+ \cup V^-, E_D)$ where*

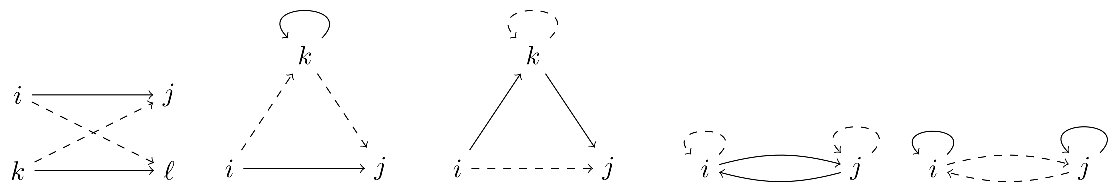
$$E_D := \big\{(i^+, j^-) : (i,j) \in E\big\} \cup \big\{(j^-, i^+) : (i,j) \notin E\big\}.$$

This definition of B_D entails that there is exactly one edge between any pair i^+, j^- with $i \neq j$, and that its direction encodes the presence or absence of the edge (i,j) in D. Moreover, for all $i \in V$, there is no edge between i^+ and i^- in B_D. An example can be seen in Figure 3, which we will use as a running example throughout the following sections. In the following, we will only recall threshold digraph characterizations that are relevant for our work.

Theorem 3 ([16, Theorem 8])**.** *Let $D = (V, E)$ be a digraph and B_D its associated auxiliary bipartite digraph. The following are equivalent:*

(i) D is a threshold digraph.

(ii) B_D has no directed cycles.



(a) 2-switch

(b) edge with a loop at its absent bypass of length 2

(c) intransitive path of length 2 with loopless middle

(d) pair of reciprocally adjacent vertices without loops

(e) pair of nonadjacent vertices with one loop each

Figure 2: Forbidden configurations in Ferrers digraphs [12, Fig. 2.2]; solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

(iii) There is a pair of functions $f, g : V \to [0,1]$ such that for $i \neq j \in V$

$$(i,j) \in E \iff f(i) + g(j) \geq 1 \ . \tag{1}$$

Ferrers digraphs [5] are graphical representations of Ferrers relations [17] and can be defined as follows.

Definition 4 (Ferrers digraph [12, Definition 2.1.1])**.** *A digraph $D_\circ = (V,E)$ is said to be a Ferrers digraph if it does* not *contain vertices $i, j, k, \ell \in V$ (not necessarily distinct), such that (i,j), $(k,\ell) \in E$ and $(i,\ell), (k,j) \notin E$.*

From all possible vertex identities in the definition, we obtain forbidden configurations for Ferrers digraphs that are depicted in Figure 2. We denote the set of Ferrers digraphs on n vertices by $\mathcal{F}_n$, but as the number of vertices does not change in the present context, we also omit the subscript and write $\mathcal{F}$. There are many equivalent characterizations of Ferrers digraphs [5, Theorem 1], of which the following two are relevant.

Theorem 5 ([12, Theorem 2.2.2])**.** *Let $D_\circ = (V,E)$ be a digraph. Then the following are equivalent:*

(i) $D_\circ$ is a Ferrers digraph.

(ii) There are two functions $w^+, w^- : V \to \mathbb{N}_0$ and a threshold $t \in \mathbb{N}_0$ such that for all $i,j \in V$

$$(i,j) \in E \Leftrightarrow w^+(i) + w^-(j) > t \ . \tag{2}$$

(iii) The out-neighborhoods $\Gamma^+(i), i \in V$ are nested, or, equivalently, the in-neighborhoods $\Gamma^-(i), i \in V$ are nested.

The above characterization (ii) of Ferrers digraphs can be reformulated. We define two functions $f, g : V \to [0,1]$ based on the weight functions w^+, w^- by

$$f(i) := \min\left\{1, \frac{w^+(i)}{t+1}\right\} \quad \text{and} \quad g(i) := \min\left\{1, \frac{w^-(i)}{t+1}\right\}$$

such that for all $i,j \in V$

$$(i,j) \in E \iff f(i) + g(j) \geq 1 \ . \tag{3}$$

$k <^+ j$
$i <^- j$

(a) D (b) B_D (c) $<^+, <^-$

Figure 3: A threshold digraph D, its associated auxiliary bipartite digraph B_D, and auxiliary strict preorders $<^+, <^-$.

Comparing the reformulated characterization with that for threshold digraphs via weight functions, we find the difference in the vertices. For Ferrers digraphs, Equivalence (3) applies to all $i, j \in V$, while, for threshold digraphs, Equivalence (1) applies only to $i \neq j \in V$. This means that removing the loops (if existent) of any Ferrers digraph yields a threshold digraph, and every threshold digraph (if not already a Ferrers digraph) can be modified into (at least) one Ferrers digraph by appropriately adding loops. With two concrete weight functions associated with a threshold digraph, it is straightforward to determine the loops that must be added: add a loop for any vertex i where $f(i) + g(i) \geq 1$. This also implies that $[D]_\sim \cap \mathcal{F} \neq \varnothing$ for every threshold digraph $D \in \mathcal{T}$. However, which loops are added depends on the specific definitions of the weight functions, which are not unique. Ferrers digraphs can, but do not have to, contain loops. Loopless Ferrers digraphs are exactly interval orders [12, Theorem 2.1.7]. Since interval orders are threshold digraphs as well [14, 15], they constitute exactly the intersection of Ferrers and threshold digraphs.

We can define the following equivalence relation so that threshold digraphs are representatives of equivalence classes consisting of Ferrers digraphs that differ only in their loops.

Definition 6. *Let $D = (V, E)$ and $D' = (V, E')$ be two digraphs with $D, D' \in \mathcal{T} \cup \mathcal{F}$. We define the binary relation $\sim \subseteq \big(\mathcal{T} \cup \mathcal{F}\big) \times \big(\mathcal{T} \cup \mathcal{F}\big)$ by*

$$D \sim D' \ :\Longleftrightarrow \ E \setminus \big\{(i,i) : i \in V\big\} = E' \setminus \big\{(i,i) : i \in V\big\} \ . \tag{4}$$

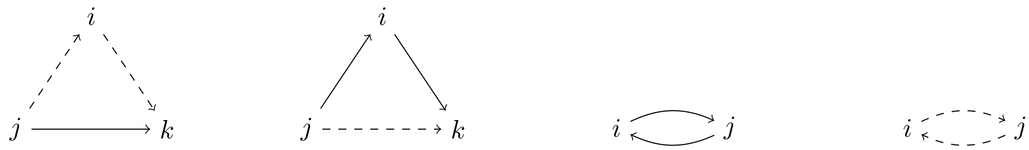
Note that, with a slight abuse of notation, we write D, D' in Definition 6 for a digraph that could have loops. The relation $\sim$ is reflexive, symmetric, and transitive, i.e., it is an equivalence relation. There are no two threshold digraphs that are equivalent to each other, since they differ in at least one edge that is not a loop. Therefore, $\mathcal{T}$ is a system of representatives for the equivalence classes induced by $\sim$. We denote the equivalence class of D under $\sim$ by

$$[D]_\sim := \big\{D' \in \mathcal{T} \cup \mathcal{F} : D \sim D'\big\} \ . \tag{5}$$

For example, Figure 5 shows all different Ferrers digraphs of the equivalence class $[D]_\sim$ of the running example D in Figure 3. Given a threshold digraph D, how can its equivalence class be generated? Determining the presence or absence of loops is discussed in the following section.

4 From Threshold to Ferrers Digraphs

In this section, we prove necessary and sufficient conditions for a vertex of a threshold digraph as to whether it must, or must not have a loop in every Ferrers digraph of its equivalence class.



(a) edge without a bypass of length 2 (b) intransitive path of length 2 (c) pair of reciprocally adjacent vertices (d) pair of nonadjacent vertices

Figure 4: Forbidden configurations of Ferrers digraphs as configurations in simple digraphs; solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

These conditions are established from the forbidden configurations of Ferrers digraphs and are alternatively expressed based on reachability in the auxiliary bipartite digraph. The alternative expressions are well-defined (in the sense that the conditions are mutually exclusive) because the auxiliary bipartite digraphs of the threshold digraphs are acyclic (Theorem 3).

Lemma 7. *Let $D = (V, E)$ be a threshold digraph and B_D its associated auxiliary bipartite digraph.*

(i) The following statements are equivalent for all $i \in V$:

- *(a)* $i^+ \longrightarrow^* i^-$
- *(b) i is in the middle of an intransitive path of length 2*
- *(c)* $\forall\, D_\circ \in [D]_{\sim} \cap \mathcal{F} : i \in \Gamma^+(i), \Gamma^-(i)$.

(ii) The following statements are equivalent for all $i \in V$:

- *(a)* $i^- \longrightarrow^* i^+$
- *(b) there is an edge without a bypass of length 2 with i in the middle of that absent bypass*
- *(c)* $\forall\, D_\circ \in [D]_{\sim} \cap \mathcal{F} : i \notin \Gamma^+(i), \Gamma^-(i)$.

Proof: We prove exemplarily (i) by proving (a) $\iff$ (b) and (b) $\iff$ (c); (ii) can be proven analogously.

(i) (a) $\implies$ (b): Let $i^+ \longrightarrow^* i^-$ in B_D. Then, there exists a shortest path $i^+ \longrightarrow k^- \longrightarrow j^+ \longrightarrow i^-$, where $k \in V \backslash \{i, j\}$ and $j \in V \backslash \{i, k\}$. Since there are no edges between any j^+ and j^- for all $j \in V$, there is no shorter path. And, any longer path can be shortened to such a shortest path: let $\ell^- \neq i^-$ be a vertex on a path from i^+ to i^- but not the target of the path's first edge. Because there must be an edge between any $i^+ \in V^+$ and $\ell^- \in V^-$ for $i \neq \ell$, the edge $i^+ \longrightarrow \ell^-$ must be present. Otherwise, since ℓ^- is on the path, a cycle would be closed, which contradicts that D is threshold (Theorem 3). Then, reaching i^- from i^+ via $i^+ \longrightarrow \ell^-$ is a shorter path. Thus, in B_D, there exists the shortest path $i^+ \longrightarrow k^- \longrightarrow j^+ \longrightarrow i^-$ which corresponds to an intransitive path of length 2 in D with i in the middle.

(b) $\implies$ (a): Let i be the middle of an intransitive path of length 2. Referring to the labeling as in Figure 4b, B_D contains the path $i^+ \longrightarrow k^- \longrightarrow j^+ \longrightarrow i^-$, i.e., $i^+ \longrightarrow^* i^-$.

(b) $\implies$ (c): Let i be the middle of an intransitive path of length 2 and let $D_\circ \in [D]_{\sim} \cap \mathcal{F}$. Assume, $i \notin \Gamma^+(i), \Gamma^-(i)$. Then, $D_\circ$ would contain an intransitive path of length 2 with a loopless middle (Figure 2c), which is forbidden in Ferrers digraphs and contradicts $D_\circ \in \mathcal{F}$. Therefore, $i \in \Gamma^+(i), \Gamma^-(i)$.

(c) $\implies$ (b): For all $D_\circ \in [D]_\sim \cap \mathcal{F}$, it is $i \in \Gamma^+(i), \Gamma^-(i)$. This implies $|V| \geq 3$, because there is no threshold digraph with $n \leq 2$ and $i \in \Gamma^+(i), \Gamma^-(i)$ for all $D_\circ \in [D]_\sim \cap \mathcal{F}$, as can be easily checked. Since $i \in \Gamma^+(i), \Gamma^-(i)$ for all $D_\circ \in [D]_\sim \cap \mathcal{F}$, removing the loop at i must destroy the nesting of the neighborhoods $\Gamma^+(\star), \Gamma^-(\star)$. The out-neighborhoods $\Gamma^+(\star)$ are linearly nested if and only if the in-neighborhoods $\Gamma^-(\star)$ are. Therefore, removing the loop of i implies that both nestings are no longer total. W.l.o.g. we consider the out-neighborhood $\Gamma^+(\star)$. Then, there is a $k \in V\backslash\{i\}$ with $\Gamma^+(k) \nsubseteq \Gamma^+(i)\backslash\{i\}$ and $\Gamma^+(i)\backslash\{i\} \nsubseteq \Gamma^+(k)$. The former entails there is an $\ell \in V$ with $\ell \in \Gamma^+(k)$ and $\ell \notin \Gamma^+(i)\backslash\{i\}$. The latter entails there is a $j \in V\backslash\{i\}$ with $j \in \Gamma^+(i)\backslash\{i\}$ but $j \notin \Gamma^+(k)$. Therefore, since $D_\circ \in \mathcal{F}$, it must hold $\Gamma^+(k) \subsetneq \Gamma^+(i)$ and $\ell = i$. We further differentiate according to whether $j \neq k$ or $j = k$. If $j \neq k$, i is in the middle of an intransitive path of length 2 from k to j, and the claim follows. If $j = k$, this corresponds to the situation, where i and k are reciprocally adjacent and $j = k \notin \Gamma^+(k)$ and, thus, $k \notin \Gamma^-(k)$ as well. Furthermore, this situation means that the out-neighborhoods of i and k are the same when their adjacency and loops are neglected:

$$\Gamma^+(i)\backslash\{i,k\} = \Gamma^+(k)\backslash\{i,k\} = \Gamma^+(k)\backslash\{i\} \ .$$

Because $k \in \Gamma^-(i)\backslash\Gamma^-(k)$, it must hold $\Gamma^-(k) \subseteq \Gamma^-(i)$. There are two possible cases: either there is an $h \in V\backslash\{i,k\}$ with $h \in \Gamma^-(i)\backslash\Gamma^-(k)$, or not. In the former case, i is in the middle of an intransitive path of length 2 from h to k, and the claim follows. The latter case means that the in-neighborhoods of i and k are also the same, when their adjacency and loops are neglected:

$$\Gamma^-(i)\backslash\{i,k\} = \Gamma^-(k)\backslash\{i,k\} = \Gamma^-(k)\backslash\{i\} \ .$$

Overall, i and k are reciprocally adjacent and share the same in- and out-neighbors otherwise, i.e., they are equivalent except for their loops. However, swapping the labeling of i and k means then there exists another $D'_\circ \in [D]_\sim \cap \mathcal{F}$ where $i \notin \Gamma^+_{D'_\circ}(i)$ and $k \in \Gamma^+_{D'_\circ}(k)$. This contradicts the assumption that $i \in \Gamma^+(i), \Gamma^-(i)$ for all $D_\circ \in [D]_\sim \cap \mathcal{F}$. □

For our running example D of Figure 3, the previous Lemma implies that in any Ferrers digraph, vertex j requires a loop (cf. also Figure 5). In contrast, both vertices i and k may have or do not have a loop. Note, however, that if neither of the above two conditions holds for a vertex i, it does not imply that i's loop is in general completely optional. If two vertices i and j are reciprocally adjacent, at *least* one of them requires a loop. On the other hand, if i and j are nonadjacent, at *most* one of them is allowed to have a loop. Otherwise, in both cases, the out- and in-neighborhoods $\Gamma^+(\star), \Gamma^-(\star)$ of i and j are not comparable in terms of set inclusion. Unless either (i) or (ii) of Lemma 7 applies to one of the two vertices, it is not uniquely determined which loop needs to be added.

5 From Strict Preorders to Rankings

In this section, we show that each of the vertex rankings given through the neighborhood inclusions in Ferrers digraph must preserve a strict preorder that is defined by a relation similar to neighborhood inclusion in the corresponding threshold digraph. Reilly et al. [16] pointed out that the auxiliary bipartite digraph encapsulates requirements for the order of weights. Using that B_D is acyclic for a threshold digraph D, they define an auxiliary strict partial order (or strict preorder) on the vertices by reachability in B_D to (efficiently) generate random threshold digraphs. To facilitate later interpretations of the notation for neighborhood inclusions, we divide the single auxiliary strict partial order [16] into two strict preorders.

Definition 8 (Auxiliary strict preorders, adapted from [16, Definition 10])**.** *Let $D = (V, E)$ be a threshold digraph and B_D its associated auxiliary bipartite digraph with the vertex set $V^+ \cup V^-$. We define two binary relations $<^+, <^-$ on V by*

$$\begin{aligned} i <^+ j \quad &:\Longleftrightarrow \quad j^+ \longrightarrow^* i^+ \text{ in } B_D \\ i <^- j \quad &:\Longleftrightarrow \quad i^- \longrightarrow^* j^- \text{ in } B_D \ . \end{aligned}$$

Both binary relations $<^+, <^-$ are irreflexive and transitive, i.e., they are strict preorders. For a digraph that is not threshold, $<^+$ and $<^-$ are in general not strict preorders because B_D is not necessarily acyclic, such that $<^+, <^-$ are no longer irreflexive. For digraphs in which loops are admitted, we define the following two neighborhood-inclusion preorders.

Definition 9. *Let $D_\circ = (V, E)$ be a digraph. We define two binary relations $\leqslant^\oplus, \leqslant^\ominus$ on V by*

$$\begin{aligned} i \leqslant^\oplus j \quad &:\Longleftrightarrow \quad \Gamma^+(i) \subseteq \Gamma^+(j) \\ i \leqslant^\ominus j \quad &:\Longleftrightarrow \quad \Gamma^-(i) \subseteq \Gamma^-(j) \ . \end{aligned}$$

Both binary relations $\leqslant^\oplus, \leqslant^\ominus$ are reflexive and transitive, i.e., they are preorders or partial rankings. Ferrers digraphs are exactly those digraphs on which both $\leqslant^\oplus$ and $\leqslant^\ominus$ are total such that $\leqslant^\oplus$ and $\leqslant^\ominus$ are rankings. The similar notation of $<^+, <^-$ and $\leqslant^\oplus, \leqslant^\ominus$ is intended to illustrate that the rankings $\leqslant^\oplus$ and $\leqslant^\ominus$ in a Ferrers digraph $D_\circ \in [D]_\sim$ must preserve $<^+$ and $<^-$, respectively, in the corresponding threshold digraph D. To prove this preservation, we use the fact that the auxiliary strict preorder is equivalent to a specific relation between open, closed, and pair-excluding neighborhoods, and combinations thereof (Lemma 13 and 14). The latter type of neighborhood is defined as follows.

Definition 10 (Pair-excluding neighborhood inclusion)**.** *Let $D = (V, E)$ be a digraph. For a pair of vertices $i, j \in V$, we define the pair-excluding out- and in-neighborhood inclusion by*

$$N^+(i)\backslash\{j\} \subseteq N^+(j)\backslash\{i\} \quad \textit{and} \quad N^-(i)\backslash\{j\} \subseteq N^-(j)\backslash\{i\} \ . \tag{6}$$

Lemma 11. *On threshold digraphs, the pair-excluding out- and in-neighborhood inclusion is a reflexive and total relation.*

Proof: Reflexivity of both the pair-excluding out- and in-neighborhood inclusions is trivial. We show that the inclusion relation is total by contradiction. We do so exemplarily for the outgoing version, but the same argument applies to the incoming variant. Assume that there is a pair of vertices $i \neq j \in V$ with $N^+(i)\backslash\{j\} \not\subseteq N^+(j)\backslash\{i\}$ and $N^+(j)\backslash\{i\} \not\subseteq N^+(i)\backslash\{j\}$. Then, there exist two vertices $k, \ell \in V\backslash\{i, j\}$ with $k \in N^+(i)\backslash\{j\}$ and $k \notin N^+(j)\backslash\{i\}$ and $\ell \in N^+(j)\backslash\{i\}$ and $\ell \notin N^+(i)\backslash\{j\}$. However, these edges form a 2-switch that does not occur in threshold digraphs. Therefore, it must be $N^+(i)\backslash\{j\} \subseteq N^+(j)\backslash\{i\}$ or $N^+(j)\backslash\{i\} \subseteq N^+(i)\backslash\{j\}$. □

Remark 12. *The pair-excluding out- and in-neighborhood inclusion relation is* not *transitive. Consider, for example, a path $k \longrightarrow i \longrightarrow j$. Then, it is $N^+(i)\backslash\{j\} = N^+(j)\backslash\{i\}$ and $N^+(j)\backslash\{k\} \subsetneq N^+(k)\backslash\{j\}$, but $N^+(i)\backslash\{k\} \not\subseteq N^+(k)\backslash\{i\}$.*

Lemma 13. *Let $D = (V, E)$ be a threshold digraph. Let B_D and $<^+, <^-$ be its associated auxiliary bipartite digraph and auxiliary strict preorders, respectively. The following equivalences hold for any $i \neq j \in V$:*

$$
\begin{array}{llll}
(i) & i <^+ j & \iff & \text{either } N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}, \text{ or,} \\
& & & N^+[i] = N^+(j) \text{ and } N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}\ , \\
(ii) & i <^- j & \iff & \text{either } N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}, \text{ or,} \\
& & & N^-[i] = N^-(j) \text{ and } N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}\ .
\end{array}
$$

Proof: We prove exemplarily (i). The proof of (ii) works analogously.

(i) $\Longrightarrow$: Let $i <^+ j$. This means that there is a directed path from j^+ to i^+ in B_D. For this path, there is either an $x^- \in V^-\backslash\{i^-, j^-\}$ on the path or not.

Case 1. Let there be an $x^- \in V^-\backslash\{i^-, j^-\}$ on the path from j^+ to i^+. Because $x \neq j$ there must be an edge between j^+ and x^-. Since $x^- \longrightarrow j^+$ would close a cycle, which contradicts that D is a threshold digraph (Theorem 3), it must be $j^+ \longrightarrow x^-$. With the same line of arguments, it must hold $x^- \longrightarrow i^+$. Therefore, it is $x \in N^+(j)\backslash N^+(i)$. Since the pair-excluding in-neighborhood inclusion is total (Lemma 11) and $x \in V\backslash\{i, j\}$ it must hold $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$.

Case 2. Let there be no $x^- \in V^-\backslash\{i^-, j^-\}$ on the path from j^+ to i^+. This entails, in particular, that there is no $x^- \in V^-\backslash\{i^-, j^-\}$ with $j^+ \longrightarrow x^-$ and $x^- \rightarrow i^+$. Because B_D is free of cycles, it is $j \not<^+ i$ such that there is also no $x^- \in V^-\backslash\{i^-, j^-\}$ with $i^+ \longrightarrow x^-$ and $x^- \longrightarrow j^+$. In other words, the outgoing neighborhoods of i and j are equal if their adjacency is not considered, i.e., $N^+(i)\backslash\{j\} = N^+(j)\backslash\{i\}$. Because there are no edges between i^+ and i^-, as well as between j^+ and j^-, the only possibility of having a path from j^+ to i^+ is then a path of length four of the kind $j^+ \longrightarrow i^- \longrightarrow y^+ \longrightarrow j^- \longrightarrow i^+$ with $y \in V\backslash\{i, j\}$. This path implies $i \in N^+(j)$ and $j \notin N^+(i)$ such that overall it is $N^+[i] = N^+(j)$. Moreover, $y \in N^-(j)\backslash N^-(i)$ and, therefore, since the pair-excluding in-neighborhood inclusion is total (Lemma 11), it must hold $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$.

$\Longleftarrow$: *Case 1.* Let $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$. Then, there is an $x \in V\backslash\{i, j\}$ with $x \in N^+(j)\backslash\{i\}$ but $x \notin N^+(i)\backslash\{j\}$. This entails the path $j^+ \longrightarrow x^- \longrightarrow i^+$ in B_D, i.e., $i <^+ j$.

Case 2. Let $N^+[i] = N^+(j)$ and $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$. The former induces $i \in N^+(j), j \notin N^+(i)$, i.e., $j^+ \longrightarrow i^-$ and $j^- \longrightarrow i^+$ in B_D. Due to the inclusion of the pair-excluding out-neighborhood, there is an $x \in V\backslash\{i, j\}$ with $x \in N^-(j)\backslash\{i\}$ but $x \notin N^-(i)\backslash\{j\}$. This entails the path $i^- \longrightarrow x^+ \longrightarrow j^-$ in B_D and therefore $j^+ \longrightarrow i^- \longrightarrow x^+ \longrightarrow j^- \longrightarrow i^+$, i.e., $i <^+ j$. □

Lemma 14. *Let $D = (V, E)$ be a threshold digraph. Let B_D and $<^+, <^-$ be its associated auxiliary bipartite digraph and auxiliary strict preorders, respectively. The following equivalences hold for any $i \neq j \in V$:*

$$
\begin{array}{llll}
(i) & i \not<^+ j \text{ and } j \not<^+ i & \iff & \text{either } N^+(i) = N^+(j), \\
& & & \text{or } N^+[i] = N^+[j], \\
& & & \text{or } N^+[i] = N^+(j) \text{ and } N^-(i) \supseteq N^-[j], \\
& & & \text{or } N^+(i) = N^+[j] \text{ and } N^-[i] \subseteq N^-(j)\ , \\
(ii) & i \not<^- j \text{ and } j \not<^- i & \iff & \text{either } N^-(i) = N^-(j), \\
& & & \text{or } N^-[i] = N^-[j], \\
& & & \text{or } N^-[i] = N^-(j) \text{ and } N^+(i) \supseteq N^+[j], \\
& & & \text{or } N^-(i) = N^-[j] \text{ and } N^+[i] \subseteq N^+(j)\ .
\end{array}
$$

Proof: We prove exemplarily (i). The proof of (ii) works analogously. We note that Lemma 13 provides the following equivalence for a threshold digraph:

$$
\begin{array}{lll}
i \not<^+ j & \iff & N^+(i)\backslash\{j\} \supseteq N^+(j)\backslash\{i\}, \text{ and,} \\
& & \text{either } N^+[i] \neq N^+(j) \text{ or } N^-(i)\backslash\{j\} \supseteq N^-(j)\backslash\{i\}\ .
\end{array} \tag{7}
$$

(i) $\Longrightarrow$: From $i \not<^+ j$ and $j \not<^+ i$ it follows with Equivalence (7) that $N^+(i)\backslash\{j\} = N^+(j)\backslash\{i\}$. Equality in the pair-excluding out-neighborhood inclusion entails, dependent on the adjacency of the dyad i and j, either $N^+(i) = N^+(j)$, $N^+[i] = N^+[j]$, $N^+[i] = N^+(j)$, or $N^+[j] = N^+[i]$. In the first two cases, nothing further needs to be proven. When $N^+[i] = N^+(j)$, since $i \not<^+ j$, then it must hold $N^-(i)\backslash\{j\} \supseteq N^-(j)\backslash\{i\}$. The equality $N^+[i] = N^+(j)$ implies, in particular, that $j \in N^-(i)$ and $i \notin N^-(j)$. Then, it is

$$N^-[j] = N^-(j) \cup \{j\} \overset{i \notin N^-(j)}{=} N^-(j)\backslash\{i\} \cup \{j\} \subseteq N^+(i)\backslash\{j\} \cup \{j\} \overset{j \in N^-(i)}{=} N^-(i) \ .$$

With the same line of arguments, it follows $N^-[i] \subseteq N^-(j)$ when $N^+(i) = N^+[j]$.

$\Longleftarrow$: We prove the claim for each case of neighborhood relation(s).

Case 1. Let $N^+(i) = N^+(j)$. This entails $N^+(i)\backslash\{j\} = N^+(j)\backslash\{i\}$. Equality in the open neighborhood implies, in particular, $N^+[i] \neq N^+(j)$ and $N^+[j] \neq N^+(i)$. Then, with Equivalence (7) it follows $i \not<^+ j$ and $j \not<^+ i$.

Case 2. Works analogously to Case 1.

Case 3. Let $N^+[i] = N^+(j)$ and $N^-(i) \supseteq N^-[j]$. The former implies $N^+(j)\backslash\{i\} = N^+(i) = N^+(i)\backslash\{j\}$, such that $j \not<^+ i$ with Equivalence (7). The latter implies $N^-(i)\backslash\{j\} \supseteq N^-(j) = N^-(j)\backslash\{i\}$, and therefore $i \not<^+ j$ as well.

Case 4. Works analogously to Case 3. □

Lemma 15. *Let $D = (V, E)$ be a threshold digraph and $<^+, <^-$ be its associated auxiliary strict preorders. Let $D_\circ \in [D]_\sim \cap \mathcal{F}$. Then, it holds*

$$i <^+ j \implies i \lneqq^\oplus j \quad \text{and} \quad i <^- j \implies i \lneqq^\ominus j \ .$$

The converse does not hold in general.

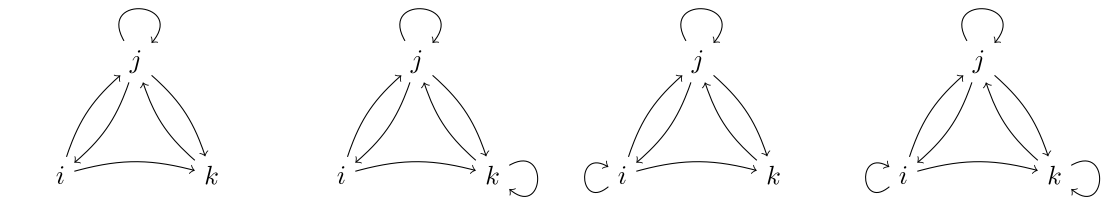
Proof: We prove the implication exemplarily for $<^+$. Let $i <^+ j$. By Lemma 13, it is either $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$, or, $N^+[i] = N^+(j)$ and $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$. The former case directly implies $\Gamma^+(i) \subsetneq \Gamma^+(j)$ for any $D_\circ \in [D]_\sim \cap \mathcal{F}$. In the latter case, $i \in N^+(j)$ and there is a $k \in V\backslash\{i, j\}$ with $k \in N^-(j)\backslash N^-(i)$. However, this means that j is in the middle of an intransitive path of length 2. Thus, by Lemma 7 (i), j must have a loop for any $D_\circ \in [D]_\sim \cap \mathcal{F}$. Simultaneously, the equality $N^+[i] = N^+(j)$ implies $j \notin N^+(i)$, i.e., $j \notin \Gamma^+(i)$. Therefore, in any case, $\Gamma^+(i) \subsetneq \Gamma^+(j)$ or $i \lneqq^\oplus j$.

The converse does not hold in general: if $i \lneqq^\oplus j$ due to a loop at j, while it is $N^+(i) = N^+(j)$, then i and j are not comparable with respect to $<^+$ (Lemma 14). □

In general, the two rankings $\leqslant^\oplus, \leqslant^\ominus$ do not have to be the same, i.e., the choice of direction may influence the position of a vertex in the ranking. However, there are Ferrers digraphs, where these rankings are the same in the sense that they are either identical or the reversal of each other. See, for example, Figures 5b or 5c, and 5d, respectively. Those Ferrers digraphs and their corresponding threshold digraphs are subject to the next section.

6 Purely Medial and Purely Hierarchical Threshold Digraphs

In this section, we characterize threshold digraphs whose equivalence class consists only of Ferrers digraphs in which the incoming neighborhoods are nested in the same (medial) or opposite (hierarchical) way as the outgoing neighborhoods. These neighborhood nestings reduce two rankings



$k \lneq^{\oplus} i \lneq^{\oplus} j$
$i \lneq^{\ominus} k \lneq^{\ominus} j$

(a) neither medial nor hierarchical

$i =^{\oplus} k \lneq^{\oplus} j$
$i \lneq^{\ominus} k =^{\ominus} j$

(b) medial

$k \lneq^{\oplus} i =^{\oplus} j$
$k =^{\ominus} i \lneq^{\ominus} j$

(c) medial

$k \lneq^{\oplus} i =^{\oplus} j$
$i \lneq^{\ominus} j =^{\ominus} k$

(d) hierarchical

Figure 5: All Ferrers digraphs $[D]_{\sim}$ of the running example D, cf. Figure 3, each with different neighborhood nestings as represented by $\leqslant^{\oplus}, \leqslant^{\ominus}$. Any of the preorders $\leqslant^{\oplus}, \leqslant^{\ominus}$ preserves the respective strict preorder $<^{+}, <^{-}$.

to a single one (up to reversal). Their threshold digraphs are interesting because the notion of centrality (medial or hierarchical) is fixed, while the vertices could be ranked differently. We term these threshold digraphs as purely medial or hierarchical, respectively. In the course of the characterization, we also provide necessary and sufficient conditions for threshold digraphs that have no medial or hierarchical Ferrers digraphs in their equivalence class. The nestings are named after neighborhood-inclusion criteria for assessing the centrality of vertices that have been proposed for simple digraphs [3, 14, 15]. We correspondingly define subclasses of Ferrers digraphs as follows.

Definition 16. *The set of hierarchical Ferrers digraphs $\mathcal{H} \subseteq \mathcal{F}$ is defined by*

$$\mathcal{H} := \Big\{ D_{\circ} \in \mathcal{F} : \forall i, j \in V : i \leqslant^{\oplus} j \text{ and } j \leqslant^{\ominus} i, \text{ or, } j \leqslant^{\oplus} i \text{ and } i \leqslant^{\ominus} j \Big\} .$$

The set of medial Ferrers digraphs $\mathcal{M} \subseteq \mathcal{F}$ is defined by

$$\mathcal{M} := \Big\{ D_{\circ} \in \mathcal{F} : \forall i, j \in V : i \leqslant^{\oplus} j \text{ and } i \leqslant^{\ominus} j, \text{ or, } j \leqslant^{\oplus} i \text{ and } j \leqslant^{\ominus} i \Big\} .$$

We denote by $\mathcal{T}_{\overline{\mathcal{M}}} := \{D \in \mathcal{T} : [D]_{\sim} \cap \mathcal{M} = \varnothing\}$ and $\mathcal{T}_{\overline{\mathcal{H}}} := \{D \in \mathcal{T} : [D]_{\sim} \cap \mathcal{H} = \varnothing\}$ the threshold digraphs whose equivalence class does *not* contain a single Ferrers digraph with a medial or hierarchical neighborhood inclusion, respectively. Accordingly, we denote with $\mathcal{T}_{\mathcal{M}} := \{D \in \mathcal{T} : [D]_{\sim} \cap \mathcal{F} \subseteq \mathcal{M}\}$ and $\mathcal{T}_{\mathcal{H}} := \{D \in \mathcal{T} : [D]_{\sim} \cap \mathcal{F} = \mathcal{H}\}$ the purely medial and hierarchical threshold digraphs, respectively. The subsets of threshold digraphs $\mathcal{T}_{\overline{\mathcal{H}}}$ ($\mathcal{T}_{\overline{\mathcal{M}}}$) and $\mathcal{T}_{\mathcal{H}}$ ($\mathcal{T}_{\mathcal{M}}$) only denote two extremes, between which lie threshold digraphs with multiple forms of neighborhood-nesting of their corresponding Ferrers digraphs (hierarchical and non-hierarchical, or, hierarchical and medial, etc.). The extremes are mutually exclusive ($\mathcal{T}_{\overline{\mathcal{H}}} \cap \mathcal{T}_{\mathcal{H}} = \varnothing$ and $\mathcal{T}_{\overline{\mathcal{M}}} \cap \mathcal{T}_{\mathcal{M}} = \varnothing$), but *not* complementary, i.e., $(\mathcal{T}_{\overline{\mathcal{H}}})^c \neq \mathcal{T}_{\mathcal{H}}$ and $(\mathcal{T}_{\overline{\mathcal{M}}})^c \neq \mathcal{T}_{\mathcal{M}}$ in general. This relationship is also reflected in the fact that negating the necessary and sufficient condition for $\mathcal{T}_{\overline{\mathcal{H}}}$ ($\mathcal{T}_{\overline{\mathcal{M}}}$) is only necessary but not sufficient for $\mathcal{T}_{\mathcal{H}}$ ($\mathcal{T}_{\mathcal{M}}$).

An overview of which subset of $\mathcal{T}$ is described by which theorem can be seen in Figure 6. It should be noted that the mapped sizes do not reflect the actual sizes of the subsets, which we do not yet know except for $\mathcal{T}_{\mathcal{H}}$.

We characterize the subsets by conditions on the auxiliary strict preorders. To prove the conditions, we use an equivalence relation (Lemma 17) between the auxiliary strict preorders and

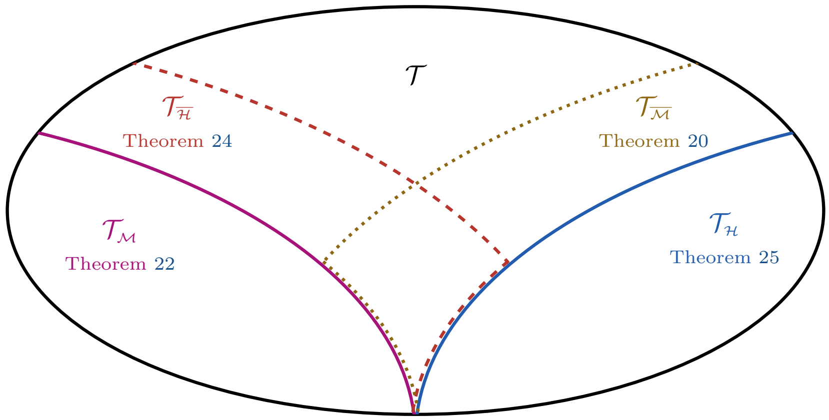
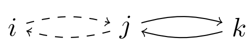
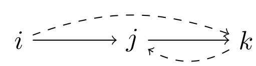


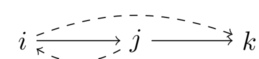
Figure 6: Subsets of threshold digraphs: threshold digraphs whose Ferrers digraphs are all medial ($\mathcal{T}_{\mathcal{M}}$) or hierarchical ($\mathcal{T}_{\mathcal{H}}$), and, threshold digraphs among whose corresponding Ferrers digraphs is not a single medial ($\mathcal{T}_{\overline{\mathcal{M}}}$) or hierarchical ($\mathcal{T}_{\overline{\mathcal{H}}}$) one; relation of subsets refers to the case where the number of vertices is larger than 2.



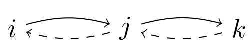
(a) triangle of a reciprocally adjacent and nonadjacent pair



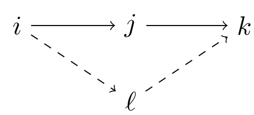
(b) intransitive path of length 2 without return to the middle



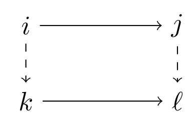
(c) intransitive path of length 2 without return from the middle



(d) one-way of length 2



(e) isolated path of length 2



(f) open square

Figure 7: Configurations in simple digraphs; solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

specific configurations, some of which have not yet occurred and are therefore defined in Figure 7. These configurations overlap with the cases that can arise from two vertices whose neighborhood inclusions contradict a medial or hierarchical nesting, respectively (cf. Figure 8 and 10). Other cases do not represent one of the configurations but imply them (see, for example, Lemma 18). Whether a threshold digraph is purely medial or hierarchical hinges on *all* the Ferrers digraphs in its equivalence class. Thus, for the characterization through the auxiliary strict preorders, some steps in the proofs require that we replace the Ferrers digraph under consideration with a very similar one that differs in a single loop. The fact that these replacements are still Ferrers digraphs is guaranteed by Lemma 19, 21, and 23.

Except for the subsequent Lemma, we handle in this section the medial before the hierarchical notion, since its characterization is slightly simpler.

Lemma 17. *Let D be a threshold digraph. Let B_D and $<^+, <^-$ be its associated auxiliary bipartite digraph and auxiliary strict preorders, respectively. The following equivalences hold.*

(i) $\exists i \neq j \in V : i <^- j$ *and* $i <^+ j$	$\Longleftrightarrow$	*D contains an isolated path of length 2, a triangle of a reciprocally adjacent and nonadjacent pair, or an intransitive path of length 2 without a return to or from the middle.*
(ii) $\exists i \neq j \in V : i <^- j$ *and* $j <^+ i$	$\Longleftrightarrow$	*D contains an open square or a one-way of length 2.*

Proof:

(i) $\Longrightarrow$: Let $i \neq j \in V$ with $i <^- j$ and $i <^+ j$. With the equivalences of Lemma 13, by each of the preorder relations $i <^- j$ and $i <^+ j$, there are two possibilities for the respective pair-excluding neighborhood inclusions. Combined, this results in three different possible cases.

Case 1. Let $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$ and $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$. The former implies a $k \in V\backslash\{i,j\}$ with $k \in N^-(j)\backslash N^-(i)$, and the latter implies an $\ell \in V\backslash\{i,j\}$ with $\ell \in N^+(j)\backslash N^+(i)$. If $k \neq \ell$, this corresponds to an isolated path of length 2 and, if $k = \ell$, this corresponds to a triangle of a reciprocally adjacent and nonadjacent pair.

Case 2. Let $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$ and $N^+(j) = N^+[i]$. The former implies a $k \in V\backslash\{i,j\}$ with $k \in N^-(j)\backslash N^-(i)$. The latter implies $i \in N^+(j)$ and $j \notin N^+(i)$. This corresponds to an intransitive path of length 2 without a return to the middle.

Case 3. Let $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$ and $N^-(j) = N^-[i]$. The former implies a $k \in V\backslash\{i,j\}$ with $k \in N^+(j)\backslash N^+(i)$. The latter implies $j \in N^+(i)$ and $i \notin N^+(j)$. This corresponds to an intransitive path of length 2 without a return from the middle.

$\Longleftarrow$: This direction can be shown by considering the corresponding path in B_D of a configuration. We do so exemplarily for an isolated path of length 2; for the other configurations, the claim can be shown accordingly. Referring to the labeling of Figure 7e, the paths $\ell^- \longrightarrow i^+ \longrightarrow j^-$ and $j^+ \longrightarrow k^- \longrightarrow \ell^+$ are present in B_D. The former entails $\ell <^- j$ and the latter $\ell <^+ j$ and thus the claim follows.

(ii) $\Longrightarrow$: Let $i \neq j \in V$ with $i <^- j$ and $j <^+ i$. With the equivalences of Lemma 13, for both relations $i <^- j$ and $j <^+ i$, there are two possibilities of what applies to the respective pair-excluding neighborhood inclusions. Combined, this results in only one possibility: $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$ and $N^+(j)\backslash\{i\} \subsetneq N^+(i)\backslash\{j\}$. Any other combination is not possible. If, for example, $i <^- j$ results from $N^-(j) = N^-[i]$ and $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$, the latter entails $i <^+ j$, which is in contrast to the given $j <^+ i$. Due to $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$, there is a $k \in V\backslash\{i,j\}$ with $k \in N^-(j)\backslash N^-(i)$. Analogously, because of $N^+(j)\backslash\{i\} \subsetneq N^-(i)\backslash\{j\}$, there is an $\ell \in V\backslash\{i,j\}$ with $\ell \in N^+(i)\backslash N^+(j)$. If $k \neq \ell$, this corresponds to an open square and, if $k = \ell$, this corresponds to a one-way length 2.

$\Longleftarrow$: Similar to (i), this direction can be shown considering the corresponding path in B_D of a configuration. We do so exemplarily for an open square; the one-way of length 2 works accordingly. Referring to the labeling of Figure 7f, the paths $k^- \longrightarrow i^+ \longrightarrow j^-$ and $k^+ \longrightarrow \ell^- \longrightarrow j^+$ are present in B_D. The former entails $k <^- j$ and the latter $j <^+ k$ and thus the claim follows. □

6.1 Purely Medial Threshold Digraphs

This section characterizes $\mathcal{T}_{\overline{\mathcal{M}}}$ and $\mathcal{T}_{\mathcal{M}}$. Both characterizations require, in their proofs, that all cases that can arise from two nodes whose neighborhoods contradict a medial nesting are considered. These cases are illustrated in Figure 8.

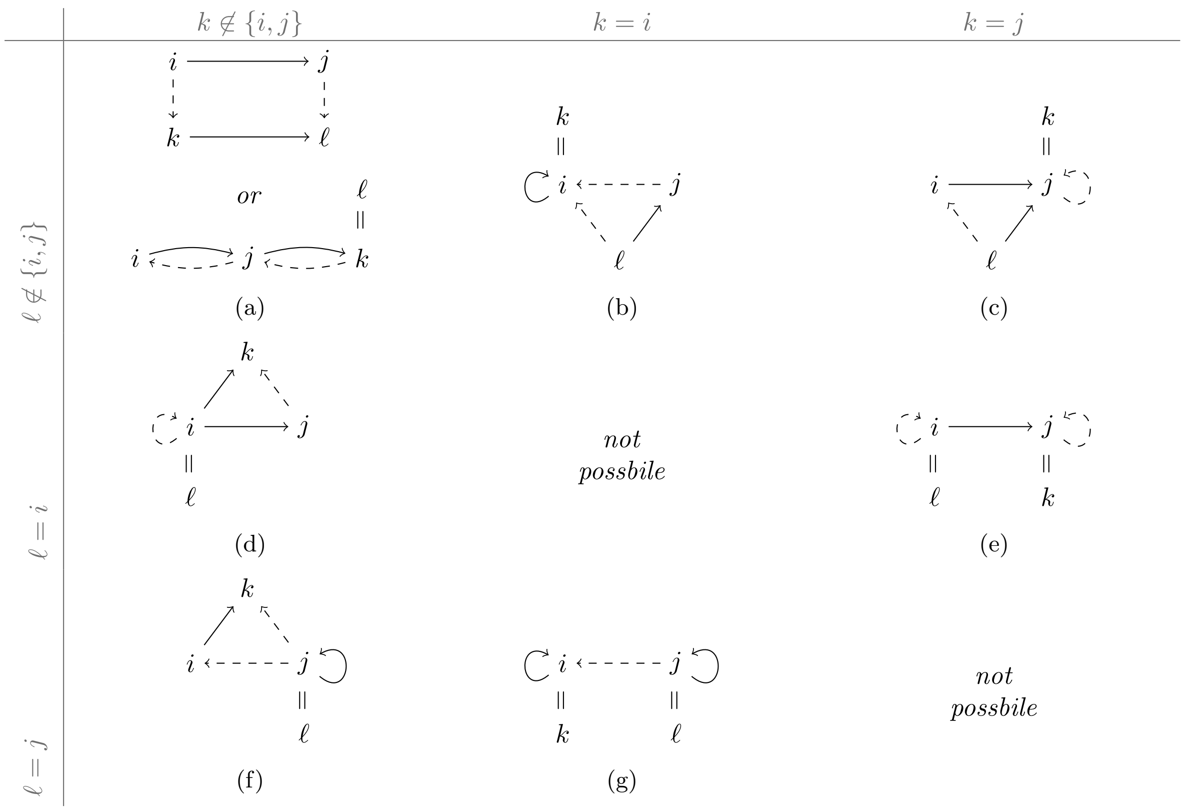


Figure 8: All cases that can result from a digraph $D_\circ = (V, E)$ having a pair of vertices whose neighborhood-inclusions contradict a medial nesting. That is, there are two vertices $i \neq j \in V$ with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(i) \subsetneq \Gamma^-(j)$. Vertices k and ℓ denote elements in the respective nonempty set difference, more precisely, $k \in \Gamma^+(i) \backslash \Gamma^+(j)$ and $\ell \in \Gamma^-(j) \backslash \Gamma^-(i)$. Note that, if $k = \ell = j$, the condition $k = j \in \Gamma^+(i) \backslash \Gamma^+(j)$ implies $k = j \notin \Gamma^-(j)$ which contradicts $\ell = j \in \Gamma^-(j)$, and thus, cannot occur. The same applies to $k = \ell = i$. Solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

Lemma 18. *Let $D = (V, E)$ be a threshold digraph and denote $<^+, <^-$ its associated auxiliary strict preorders. Let $D_\circ$ be a Ferrers digraph $D_\circ \in [D]_\sim \cap \mathcal{F}$ with $D_\circ \notin \mathcal{M}$. We refer with $i \neq j \in V$ to a pair of vertices whose neighborhoods are not medially nested, i.e., with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(i) \subsetneq \Gamma^-(j)$, and denote vertices induced by the proper subset relations with $k \in \Gamma^+(i) \backslash \Gamma^+(j)$ and $\ell \in \Gamma^-(j) \backslash \Gamma^-(i)$. If $k \in \{i,j\}$ and $\ell \notin \{i,j\}$, or, $\ell \in \{i,j\}$ and $k \notin \{i,j\}$, then $i <^- j$ and $j <^+ i$.*

Proof: We prove the claim exemplarily for $k \in \{i,j\}$ and $\ell \notin \{i,j\}$; the other case works analogously by swapping out- and in-neighborhoods, and the auxiliary strict preorders $<^-$ and $<^+$.

Let $k \in \{i,j\}$ and $\ell \notin \{i,j\}$, see Figure 8b and 8c. Since $\ell \in N^-(j) \backslash N^-(i)$, it is $i <^- j$. If $k = j$, reciprocating with an edge from j to i would make $D_\circ$ contain the forbidden intransitive path of length 2 with a loopless middle (Figure 2c). If $k = i$, having no edge from i to j would cause $D_\circ$ to contain the forbidden edge with a loop at its absent bypass of length 2 (Figure 2b).

Thus, in both cases, $j \in N^+(i)$ and $i \notin N^+(j)$.

There are three possibilities for the comparability of i and j w.r.t. $<^+$. Because $\Gamma^+(j) \subsetneq \Gamma^+(i)$, the inclusion $N^+(i)\backslash\{j\} \subsetneq N^+(j)\backslash\{i\}$ is not possible. Since $j \in N^+(i)$ and $i \notin N^+(j)$, it follows $N^+[i] \neq N^+(j)$. Then, the negation of the equivalence in Lemma 13 (i) provides $i \not<^+ j$. Due to the adjacency between i and j, the pair i and j could be only incomparable w.r.t. $<^+$, if $N^+(i) = N^+[j]$ and $N^-[i] \supseteq N^-(j)$ according to Lemma 14. However, because $\ell \in N^-(j)\backslash N^-(i)$ it follows $N^-[i] \not\supseteq N^-(j)$, such that i and j cannot be incomparable. Since we already derived $i \not<^+ j$, it must hold $j <^+ i$, and the claim follows. □

Lemma 19. *Let $D = (V, E)$ be a threshold digraph and $i \neq j \in V$ two vertices with $N^+[j] = N^+(i)$ and $N^-[i] = N^-(j)$. Let $D_\circ \in [D]_\sim \cap \mathcal{F}$, and define the pairs of vertices whose neighborhoods contradict a medial nesting in $D_\circ$ by*

$$\mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ} := \Big\{\{k,\ell\} : k,\ell \in V : \Gamma^+_{D_\circ}(k) \subsetneq \Gamma^+_{D_\circ}(\ell) \text{ and } \Gamma^-_{D_\circ}(\ell) \subsetneq \Gamma^-_{D_\circ}(k), \text{ or, } \Gamma^+_{D_\circ}(\ell) \subsetneq \Gamma^+_{D_\circ}(k) \text{ and } \Gamma^-_{D_\circ}(k) \subsetneq \Gamma^-_{D_\circ}(\ell)\Big\}.$$

The following statements hold.

(i) Let $D_\circ$ be such that $i \notin \Gamma^+_{D_\circ}(i)$ and $j \notin \Gamma^+_{D_\circ}(j)$. Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k) \cup \{k\} & k = i \\ \Gamma^+_{D_\circ}(k) & k \neq i\,. \end{cases}$$

Then $D'_\circ \in [D]_\sim \cap \mathcal{F}$, and $\mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$.

(ii) Let $D_\circ$ be such that $i \in \Gamma^+_{D_\circ}(i)$ and $j \in \Gamma^+_{D_\circ}(j)$. Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k)\backslash\{k\} & k = i \\ \Gamma^+_{D_\circ}(k) & k \neq i\,. \end{cases}$$

Then $D'_\circ \in [D]_\sim \cap \mathcal{F}$, and $\mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$.

Proof: We prove exemplarily (i); (ii) can be proven analogously by selecting the in-neighborhoods $\Gamma^-_{D'_\circ}(\star)$.

(i) First, we show $D'_\circ \in \mathcal{F}$ by proving that the out-neighborhoods $\Gamma^+_{D'_\circ}(\star)$ are linearly nested. Since only $\Gamma^+_{D_\circ}(i) \neq \Gamma^+_{D'_\circ}(i)$, it remains to show that $\Gamma^+_{D'_\circ}(h) \subseteq \Gamma^+_{D'_\circ}(i)$ or $\Gamma^+_{D'_\circ}(i) \subseteq \Gamma^+_{D'_\circ}(h)$, for any vertex $h \neq i \in V$. Let $h \neq i \in V$. We differentiate based on the inclusion of the out-neighborhoods of i and h in $D_\circ$.

$\underline{\Gamma^+_{D_\circ}(h) \subseteq \Gamma^+_{D_\circ}(i)}$: Since $i \notin \Gamma^+_{D_\circ}(i)$, it follows $i \notin \Gamma^+_{D_\circ}(h)$. Then, it is $\Gamma^+_{D'_\circ}(h) = \Gamma^+_{D_\circ}(h) \subseteq \Gamma^+_{D_\circ}(i) \subsetneq \Gamma^+_{D_\circ}(i) \cup \{i\} = \Gamma^+_{D'_\circ}(i)$.

$\underline{\Gamma^+_{D_\circ}(i) \subsetneq \Gamma^+_{D_\circ}(h)}$: We differentiate further according to whether $i \notin \Gamma^+_{D_\circ}(h)$ or $i \in \Gamma^+_{D_\circ}(h)$. If $i \in \Gamma^+_{D_\circ}(h)$, it follows $\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D_\circ}(i) \cup \{i\} \subseteq \Gamma^+_{D_\circ}(h) \cup \{i\} = \Gamma^+_{D_\circ}(h) = \Gamma^+_{D'_\circ}(h)$. Let $i \notin \Gamma^+_{D_\circ}(h)$. Since $j \in \Gamma^+_{D_\circ}(i)\backslash\Gamma^+_{D_\circ}(j)$, it is $\Gamma^+_{D_\circ}(j) \subsetneq \Gamma^+_{D_\circ}(i)$ which implies in particular $h \neq j$. Then, it follows $j \in \Gamma^+_{D_\circ}(i) \subsetneq \Gamma^+_{D_\circ}(h)$ which is equivalent to $h \in \Gamma^-_{D_\circ}(j)$. However, since $N^-[i] = N^-(j)$, this would entail $h \in N^-[i]$ which contradicts $i \notin \Gamma^+_{D_\circ}(h)$.

Summarizing, $D'_\circ \in \mathcal{F}$. Since $D_\circ$ and $D'_\circ$ differ only in the loop at i, we have $D'_\circ \in [D]_\sim \cap \mathcal{F}$. It remains to prove that $\mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$. Let $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ}$. If $i \notin \{k,\ell\}$, then $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$ since the neighborhoods of k and ℓ do not differ between $D_\circ$ and $D'_\circ$. Let $i \in \{k,\ell\}$, w.l.o.g. we assume $k = i$. Since $N^+[j] = N^+(i)$ and $N^-[i] = N^-(j)$, as well as $i \in \Gamma^+_{D'_\circ}(i)$ and $j \notin \Gamma^+_{D'_\circ}(j)$ it follows $\Gamma^+_{D'_\circ}(j) \subsetneq \Gamma^+_{D'_\circ}(i)$ and $\Gamma^-_{D'_\circ}(j) = \Gamma^-_{D'_\circ}(i)$ such that $\ell \neq j$. We prove $\{i,\ell\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$ by contradiction. Assume $\{i,\ell\} \notin \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$. From $\{i,\ell\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ}$ it is either $\Gamma^+_{D'_\circ}(i) \subsetneq \Gamma^+_{D'_\circ}(\ell)$ and $\Gamma^-_{D'_\circ}(\ell) \subsetneq \Gamma^-_{D'_\circ}(i)$, or, $\Gamma^+_{D'_\circ}(\ell) \subsetneq \Gamma^+_{D'_\circ}(i)$ and $\Gamma^-_{D'_\circ}(i) \subsetneq \Gamma^-_{D'_\circ}(\ell)$. We only prove the former case; the latter works analogously. From $\{i,\ell\} \notin \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$, it follows that neither $\Gamma^+_{D_\circ}(i) \subsetneq \Gamma^+_{D_\circ}(\ell)$ and $\Gamma^-_{D_\circ}(\ell) \subsetneq \Gamma^-_{D_\circ}(i)$, nor $\Gamma^+_{D_\circ}(\ell) \subsetneq \Gamma^+_{D_\circ}(i)$ and $\Gamma^-_{D_\circ}(i) \subsetneq \Gamma^-_{D_\circ}(\ell)$. This is logically equivalent to either $\Gamma^+_{D_\circ}(\ell) \subseteq \Gamma^+_{D_\circ}(i)$ and $\Gamma^-_{D_\circ}(\ell) \subseteq \Gamma^-_{D_\circ}(i)$, or, $\Gamma^+_{D_\circ}(i) \subseteq \Gamma^+_{D_\circ}(\ell)$ and $\Gamma^-_{D_\circ}(i) \subseteq \Gamma^-_{D_\circ}(\ell)$, or $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(\ell)$, or $\Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(\ell)$. However, the first two cases cannot apply, since $\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D_\circ}(i) \cup \{i\} \subsetneq \Gamma^+_{D_\circ}(\ell) = \Gamma^+_{D'_\circ}(\ell)$ contradicts $\Gamma^+_{D_\circ}(\ell) \subseteq \Gamma^+_{D_\circ}(i)$, and $\Gamma^-_{D'_\circ}(\ell) = \Gamma^-_{D_\circ}(\ell) \subsetneq \Gamma^-_{D_\circ}(i) \cup \{i\} = \Gamma^-_{D'_\circ}(i)$ contradicts $\Gamma^-_{D_\circ}(i) \subseteq \Gamma^-_{D_\circ}(\ell)$. Thus, either $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(\ell)$ or $\Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(\ell)$. Assume $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(\ell)$, the other case works accordingly. Then, it is $i \notin \Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(\ell)$ but $j \in \Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(\ell)$, which is equivalent to $\ell \in \Gamma^-_{D_\circ}(j)$ but $\ell \notin \Gamma^-_{D_\circ}(i)$ contradicting $N^-[i] = N^-(j)$. Overall, $\{i,\ell\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$ and since $\{i,j\} \in \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ} \setminus \mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ}$ we have the proper relation $\mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$. □

Theorem 20. *Let $D = (V,E)$ be a threshold digraph and $<^+, <^-$ be its associated auxiliary strict preorders, respectively. The following equivalence applies:*

$$D \in \mathcal{T}_{\overline{\mathcal{M}}} \iff \exists i \neq j \in V : i <^- j \text{ and } j <^+ i \ .$$

Proof: $\Longrightarrow$: Let $D \in \mathcal{T}_{\overline{\mathcal{M}}}$. Since $[D]_\sim \cap \mathcal{F} \neq \varnothing$, there is a $D_\circ \in [D]_\sim \cap \mathcal{F}$. W.l.o.g., assume that $D_\circ$ has the minimal cardinality of $\mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$ among all elements of $[D]_\sim \cap \mathcal{F}$. Since $D \in \mathcal{T}_{\overline{\mathcal{M}}}$, there are $i \neq j \in V$ with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(i) \subsetneq \Gamma^-(j)$. Because of the proper subset relations, there exist $k, \ell \in V$ with $k \in \Gamma^+(i) \setminus \Gamma^+(j)$ and $\ell \in \Gamma^-(j) \setminus \Gamma^-(i)$. We distinguish cases dependent on whether a proper subset relation is due to a loop or not, i.e., whether $k, \ell \in \{i,j\}$. We refer to the cases as labeled by Figure 8. In Case 8a, the claim follows from Lemma 17 (ii). In Case 8b and 8c, as well as Case 8d and 8f the claim is implied by Lemma 18. Thus, it remains to prove the claim for Case 8e and 8g.

Case 8e and 8g. Let $k \in \{i,j\}$ and $\ell \in \{i,j\} \setminus \{k\}$. We assume that k and ℓ are the only elements in which the neighborhoods differ, i.e., $\{k\} = \Gamma^+(i) \setminus \Gamma^+(j)$ and $\{\ell\} = \Gamma^-(j) \setminus \Gamma^-(i)$; otherwise, one of the above cases applies. If $k = j$ and $\ell = i$, it must hold $i \notin N^+(j)$, otherwise $D_\circ$ would contain the forbidden pair of reciprocally adjacent vertices without loops (Figure 2d). If $k = i$ and $\ell = j$, it must hold $j \in N^+(i)$, otherwise $D_\circ$ would contain the forbidden pair of nonadjacent vertices with one loop each (Figure 2e). Thus, in both cases, $N^+[j] = N^+(i)$ and $N^-[i] = N^-(j)$. Then, by Lemma 19, there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $\mathcal{D}^{\overline{\mathcal{M}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{M}}}_{D_\circ}$, which contradicts the minimality assumption of $D_\circ$.

$\Longleftarrow$: Let $i \neq j \in V$ with $i <^- j$ and $j <^+ i$, and let $D_\circ \in [D]_\sim \cap \mathcal{F}$. With Lemma 15 it follows $i \lneq^\ominus j$ and $j \lneq^\oplus i$, i.e., $D_\circ \notin \mathcal{M}$. Since $D_\circ$ was arbitrary, it follows $[D]_\sim \cap \mathcal{M} = \varnothing$ and thus $D \in \mathcal{T}_{\overline{\mathcal{M}}}$. □

Lemma 21. *Let $D = (V,E)$ be a threshold digraph and $i \neq j \in V$ two vertices with $N^+[j] = N^+(i)$ and $N^-[i] = N^-(j)$. Then, the following statements hold.*

(i) Let $D_\circ \in [D]_\sim \cap \mathcal{F}$ with $i \in \Gamma^+_{D_\circ}(i)$ and $j \notin \Gamma^+_{D_\circ}(j)$. Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k)\backslash\{k\} & k = i \\ \Gamma^+_{D_\circ}(k) & k \neq i \ . \end{cases}$$

Then, $D'_\circ \in [D]_\sim \cap \mathcal{F}$.

(ii) Let $D_\circ \in [D]_\sim \cap \mathcal{F}$ with $i \notin \Gamma^+_{D_\circ}(i)$ and $j \in \Gamma^+_{D_\circ}(j)$. Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k)\backslash\{k\} & k = j \\ \Gamma^+_{D_\circ}(k) & k \neq j \ . \end{cases}$$

Then, $D'_\circ \in [D]_\sim \cap \mathcal{F}$.

Proof: We prove exemplarily (i); (ii) can be proven analogously by selecting the in-neighborhoods $\Gamma^-_{D'_\circ}(\star)$.

(i) We show $D'_\circ \in \mathcal{F}$ by proving that the out-neighborhoods $\Gamma^+_{D'_\circ}(\star)$ are nested. Since only $\Gamma^+_{D_\circ}(i) \neq \Gamma^+_{D'_\circ}(i)$, it remains to show that $\Gamma^+_{D'_\circ}(h) \subseteq \Gamma^+_{D'_\circ}(i)$ or $\Gamma^+_{D'_\circ}(i) \subseteq \Gamma^+_{D'_\circ}(h)$, for any vertex $h \neq i \in V$. Let $h \neq i \in V$. We differentiate based on the inclusion of the out-neighborhoods of i and h in $D_\circ$.

$\underline{\Gamma^+_{D_\circ}(i) \subseteq \Gamma^+_{D_\circ}(h)}$: Since $i \in \Gamma^+_{D_\circ}(i)$, it follows $i \in \Gamma^+_{D_\circ}(h)$. Then, it is $\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D_\circ}(i)\backslash\{i\} \subsetneq \Gamma^+_{D_\circ}(i) \subseteq \Gamma^+_{D_\circ}(h) = \Gamma^+_{D'_\circ}(h)$.

$\underline{\Gamma^+_{D_\circ}(h) \subsetneq \Gamma^+_{D_\circ}(i)}$: We differentiate further according to whether $i \notin \Gamma^+_{D_\circ}(h)$ or $i \in \Gamma^+_{D_\circ}(h)$. If $i \notin \Gamma^+_{D_\circ}(h)$, it follows $\Gamma^+_{D'_\circ}(h) = \Gamma^+_{D_\circ}(h) = \Gamma^+_{D_\circ}(h)\backslash\{i\} \subseteq \Gamma^+_{D_\circ}(i)\backslash\{i\} = \Gamma^+_{D'_\circ}(i)$. Let $i \in \Gamma^+_{D_\circ}(h)$. Since $i \notin N^+(j)$, it is $h \neq j$ and we have a chain of strict inclusions $\Gamma^+_{D_\circ}(j) \subsetneq \Gamma^+_{D_\circ}(h) \subsetneq \Gamma^+_{D_\circ}(i)$. Due to $N^+[j] = N^+(i)$, it must hold $\Gamma^+(h)\backslash\Gamma^+(j) = \{i\}$ and $\Gamma^+(i)\backslash\Gamma^+(h) = \{j\}$. However, then it is $h \in N^-(i)$ but $h \notin N^-(j)$ which contradicts $N^-[i] = N^-(j)$.

Summarizing, $D'_\circ \in \mathcal{F}$. Since $D_\circ$ and $D'_\circ$ differ only in the loop at i, we have $D'_\circ \in [D]_\sim \cap \mathcal{F}$. $\square$

Theorem 22. *Let $D = (V, E)$ be a threshold digraph and $<^+, <^-$ be its associated auxiliary strict preorders, respectively. The following equivalence applies:*

$$D \in \mathcal{T}_\mathcal{M} \iff \begin{array}{l} \nexists i \neq j \in V : i <^- j \ \textit{and} \ j <^+ i, \ \textit{and} \\ \forall i \neq j \in V : i \mathrel{\substack{\not<^- \\ \not>_-}} j \ \textit{or} \ i \mathrel{\substack{\not<^+ \\ \not>_+}} j \implies i \mathrel{\substack{\dashrightarrow \\ \dashleftarrow}} j \ \textit{or} \ i \rightleftarrows j \ . \end{array}$$

Proof: $\implies$: Let $D \in \mathcal{T}_\mathcal{M}$. Then, in particular, $[D]_\sim \cap \mathcal{M} = [D]_\sim \cap \mathcal{F} \neq \varnothing$. Therefore, with the equivalence of Theorem 20, there are no $i \neq j \in V$ with $i <^- j$ and $j <^+ i$. It remains to prove that if i and j are not comparable w.r.t. $<^-$ or $<^+$, they are either nonadjacent or reciprocally adjacent. We assume that there is exactly one edge between i and j and prove the claim by contradiction. There are three cases for possible combinations of comparability w.r.t. $<^+$ and $<^-$.

Case 1. Vertices i and j are comparable with respect to $<^-$ but not to $<^+$. W.l.o.g. we assume $i <^- j$. The comparability $i <^- j$ can be due to two conditions (Lemma 13). However, it must hold the first condition $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$, because the second one would entail $i <^+ j$, which would make i and j comparable w.r.t. $<^+$. Further, because there is only one edge between i and j by assumption, we know with Lemma 14, it is either $N^+[i] = N^+(j)$ and $N^-(i) \subseteq N^-[j]$, or, $N^+[j] = N^+(i)$ and $N^-(j) \subseteq N^-[i]$. The latter case cannot apply since $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$. Therefore, the edge existing between i and j goes from j to i.

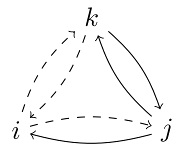


Figure 9: Triad *111U* of the triadic census [6]

Further, $N^-(i)\backslash\{j\} \subsetneq N^-(j)\backslash\{i\}$ implies that there exists a $k \in V\backslash\{i,j\}$ with $k \in N^-(j)\backslash N^-(i)$. Since $[D]_\sim \cap \mathcal{F} \subseteq \mathcal{M}$ and $j \in N^+(k)\backslash N^+(i)$, it must hold $\Gamma^+(i) \subseteq \Gamma^+(k)$ and $\Gamma^-(i) \subseteq \Gamma^-(k)$ for any $D_\circ \in [D]_\sim \cap \mathcal{F}$. In particular, this entails $j \in \Gamma^-(i) \subseteq \Gamma^-(k)$, i.e., $j \in N^-(k)$. Similarly, because $i \in N^+(j)\backslash N^+(k)$, it must hold $\Gamma^+(k) \subseteq \Gamma^+(j)$ and $\Gamma^-(k) \subseteq \Gamma^-(j)$. Since $i \notin \Gamma^-(j) \supseteq \Gamma^-(k)$, it follows $i \notin N^-(k)$. Overall, this means that D contains the subdigraph induced by the three vertices i, j, k as depicted in Figure 9. In particular, $k \in N^+(j)\backslash N^+(i)$ entailing $i <^+ j$ which is contradicting that i and j are not comparable w.r.t. $<^+$. Therefore, the assumption that there is exactly one edge between i and j must have been wrong. It follows the claim, that i and j are either nonadjacent or reciprocally adjacent.

Case 2. Vertices i and j are comparable with respect to $<^+$ but not to $<^-$; works analogously to Case 1.

Case 3. Vertices i and j are not comparable w.r.t. both $<^-$ and $<^+$. Since there is exactly one edge between i and j, it must hold either $N^+[i] = N^+(j)$ and $N^-(i) = N^-[j]$, or $N^+[j] = N^+(i)$ and $N^-(j) = N^-[i]$. We prove the claim for the former; the latter works accordingly. Let $N^+[i] = N^+(j)$ and $N^-(i) = N^-[j]$. Then, i and j cannot have both loops, because it would imply $\Gamma^+(i) \subsetneq \Gamma^+(j)$ and $\Gamma^-(j) \subsetneq \Gamma^-(i)$, thus, $[D]_\sim \cap \mathcal{F} \not\subseteq \mathcal{M}$. With the same argument, i and j cannot both have no loop. Therefore, for any $D_\circ \in [D]_\sim \cap \mathcal{F}$, either j must have a loop and i must not, or the other way around. In the former case, assume $j \in \Gamma^+_{D_\circ}(j)$ and $i \notin \Gamma^+_{D_\circ}(i)$. With Lemma 21 (i), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $j \notin \Gamma^+_{D'_\circ}(j)$ and $i \notin \Gamma^+_{D'_\circ}(i)$. In the latter case, assume $j \notin \Gamma^+_{D_\circ}(j)$ and $i \in \Gamma^+_{D_\circ}(i)$. With Lemma 21 (ii), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $j \notin \Gamma^+_{D'_\circ}(j)$ and $i \notin \Gamma^+_{D'_\circ}(i)$. In both cases, $D'_\circ \notin \mathcal{M}$ which is a contradiction to $D \in \mathcal{T}_\mathcal{M}$.

$\Longleftarrow$: We prove the claim by contraposition. This means that we assume $D \notin \mathcal{T}_\mathcal{M}$ and need to prove the existence of a pair $i \neq j \in V$ that meets $i <^- j$ and $j <^+ i$, or has exactly one edge between them when they are incomparable w.r.t. $<^-$ or $<^+$. Since $D \notin \mathcal{T}_\mathcal{M}$, there is a $D_\circ \in [D]_\sim \cap \mathcal{F}$ with $D_\circ \notin \mathcal{M}$. This means that there exist $i \neq j \in V$ with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(i) \subsetneq \Gamma^-(j)$. Due to the proper subset relations, there exist $k, \ell \in V$ with $k \in \Gamma^+(i)\backslash\Gamma^+(j)$ and $\ell \in \Gamma^-(j)\backslash\Gamma^-(i)$. We distinguish the cases dependent on whether the proper subset relation is due to a loop or not, and refer to the cases as labeled by Figure 8. Case 8a follows from Lemma 17 (ii). Cases 8b and 8c, as well as the Cases 8d and 8f, are all implied by Lemma 18. Thus, it remains to prove the claim for Case 8e and 8g.

Case 8e. Let $k = j$ and $\ell = i$. Since $D_\circ \in \mathcal{F}$, it must hold $j \dashrightarrow i$, or, equivalently, $i^- \longrightarrow j^+$. Otherwise, $D_\circ$ would contain the forbidden pair of reciprocally adjacent vertices without loops (Figure 2d). Therefore, between i and j is exactly one edge. It remains then to show that if i and j are comparable w.r.t. to both $<^+$ and $<^-$, it must be $j <^+ i$ and $i <^- j$, or the other way around. The claim is proven by contradiction.

Assume $j <^+ i$ and $j <^- i$. Then, because of the latter, $j^- \longrightarrow^* i^-$ and with $i^+ \longrightarrow j^-$ it follows $i^+ \longrightarrow^* i^-$. Lemma 7 (i) implies $i \in \Gamma^-(i)$ which contradicts $\ell = i \notin \Gamma^-(i)$. Assume $i <^+ j$ and $i <^- j$. Then, because of the former, $j^+ \longrightarrow^* i^+$ and with $i^+ \longrightarrow j^-$ in particular $j^+ \longrightarrow^* j^-$. Lemma 7 (i) implies $j \in \Gamma^-(j)$ which contradicts $k = j \notin \Gamma^-(j)$.

Case 8g. Let $k = i$ and $\ell = j$. Since $D_\circ \in \mathcal{F}$, it must hold $i \longrightarrow j$, otherwise $D_\circ$ would contain the forbidden pair of nonadjacent vertices with one loop each (Figure 2e). Therefore, between i and j is exactly one edge. It remains then to show that if i and j are comparable w.r.t. to both $<^+$ and $<^-$, it must be $j <^+ i$ and $i <^- j$, or the other way around. This is proven by contradiction.

Assume $j <^+ i$ and $j <^- i$. Then, because of the latter, $j^- \longrightarrow^* i^-$ and with $i^- \longrightarrow j^+$ it follows $j^- \longrightarrow^* j^+$. Lemma 7 (ii) implies $j \notin \Gamma^-(i)$ contradicting $\ell = j \in \Gamma^-(j)$. Assume $i <^+ j$ and $i <^- j$. Then, because of the former, $j^+ \longrightarrow^* i^+$ and with $i^- \longrightarrow j^+$ it follows $i^- \longrightarrow^* i^+$. Lemma 7 (ii) implies $i \notin \Gamma^-(i)$ contradicting $k = i \in \Gamma^-(i)$. ☐

6.2 Purely Hierarchical Threshold Digraphs

This section characterizes $\mathcal{T}_{\overline{\mathcal{H}}}$ and $\mathcal{T}_{\mathcal{H}}$. Both characterizations require, in their proofs, that all cases that can arise from two nodes whose neighborhoods contradict a hierarchical nesting are considered. These cases are illustrated in Figure 10.

Lemma 23. *Let $D = (V, E)$ be a threshold digraph and $i \neq j \in V$. Let $D_\circ \in [D]_\sim \cap \mathcal{F}$ where $i \in \Gamma^+_{D_\circ}(i)$ and $j \notin \Gamma^+_{D_\circ}(j)$. Define the pairs of dyads whose neighborhoods contradict a hierarchical nesting in $D_\circ$ by*

$$\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} := \Big\{\{k,\ell\} : k,\ell \in V : \Gamma^+_{D_\circ}(k) \subsetneq \Gamma^+_{D_\circ}(\ell) \text{ and } \Gamma^-_{D_\circ}(k) \subsetneq \Gamma^-_{D_\circ}(\ell), \text{ or, } \Gamma^+_{D_\circ}(\ell) \subsetneq \Gamma^+_{D_\circ}(k) \text{ and } \Gamma^-_{D_\circ}(\ell) \subsetneq \Gamma^-_{D_\circ}(k)\Big\} .$$

The following statements hold.

(i) Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k)\backslash\{k\} & k = i \\ \Gamma^+_{D_\circ}(k) & k \neq i \ . \end{cases}$$

If $N^+(i) = N^+(j)$ or $N^-(i) = N^-(j)$, then $D'_\circ \in [D]_\sim \cap \mathcal{F}$ and the set $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ is described by

$$\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} = \Big\{\{i,k\} : k \in V\backslash\{j\} : \Gamma^+_{D'_\circ}(i) \subsetneq \Gamma^+_{D'_\circ}(k) \text{ and } \Gamma^-_{D'_\circ}(i) \subsetneq \Gamma^-_{D'_\circ}(k), \text{ and, either } \Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k) \text{ or } \Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(k)\Big\} . \tag{8}$$

If $N^+(i) = N^+(j)$ and $N^-(i) = N^-(j)$, then $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ and thus $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} = \varnothing$.

(ii) Let $D'_\circ$ denote the digraph given by

$$\Gamma^+_{D'_\circ}(k) := \begin{cases} \Gamma^+_{D_\circ}(k) \cup \{k\} & k = j \\ \Gamma^+_{D_\circ}(k) & k \neq j \ . \end{cases}$$

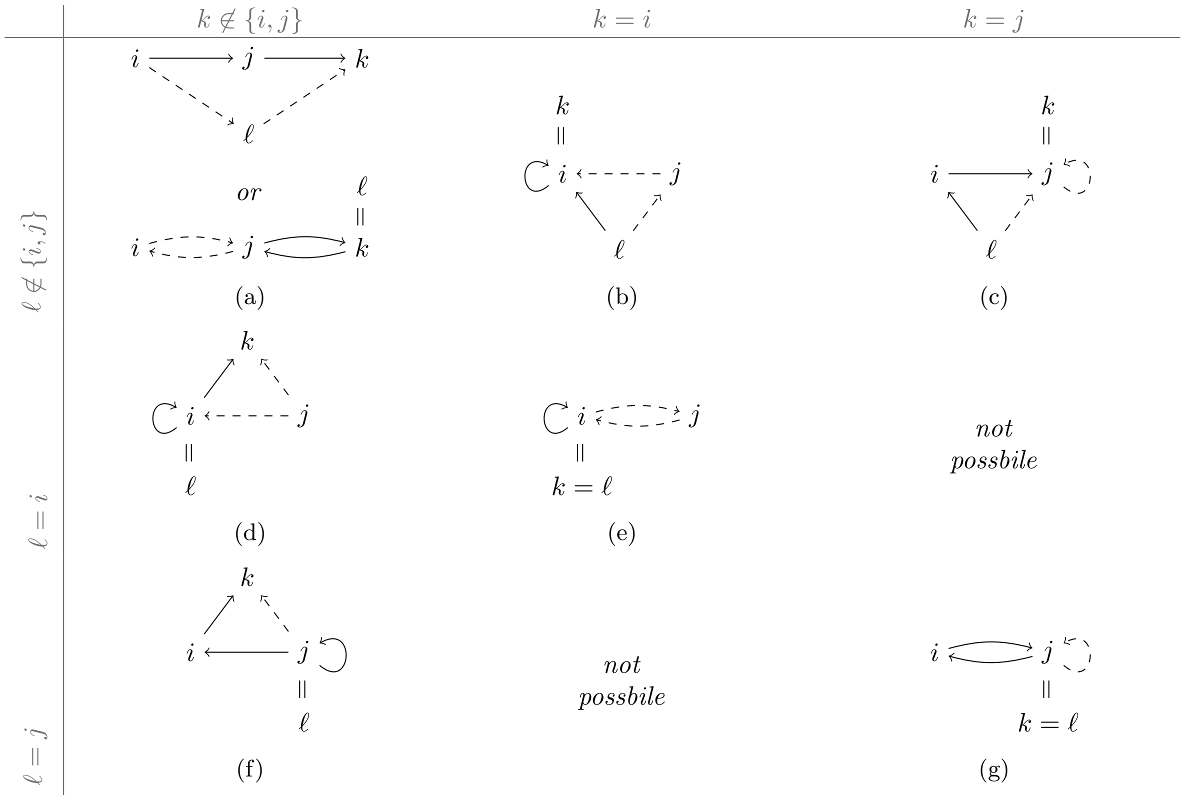


Figure 10: All cases that can result from a digraph $D_\circ = (V, E)$ having a pair of vertices whose neighborhood-inclusions contradict a hierarchical nesting. That is, there are two vertices $i \neq j \in V$ with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(i) \subsetneq \Gamma^-(j)$. Vertices k and ℓ denote elements in the respective nonempty set difference, more precisely, $k \in \Gamma^+(i)\backslash\Gamma^+(j)$ and $\ell \in \Gamma^-(j)\backslash\Gamma^-(i)$. Note that, if $k = i$ and $\ell = j$, the condition $k = i \in \Gamma^+(i)\backslash\Gamma^+(j)$ implies $i \notin \Gamma^+(j)$, which means $j \notin \Gamma^-(i)$ contradicting $\ell = j \in \Gamma^-(j)$, and, thus, cannot occur. The same applies to $k = j$ and $\ell = i$. Solid edges are present, dashed edges are absent, and no conditions are placed on missing edges.

If $N^+[i] = N^+[j]$ or $N^-[i] = N^-[j]$, then $D'_\circ \in [D]_\sim \cap \mathcal{F}$ and the set $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ is described by

$$\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} = \Big\{ \{j,k\} : k \in V\backslash\{i\} : \Gamma^+_{D'_\circ}(k) \subsetneq \Gamma^+_{D'_\circ}(j) \text{ and } \Gamma^-_{D'_\circ}(k) \subsetneq \Gamma^-_{D'_\circ}(j), \text{ and, } \\ \text{either } \Gamma^+_{D_\circ}(j) = \Gamma^+_{D_\circ}(k) \text{ or } \Gamma^-_{D_\circ}(j) = \Gamma^-_{D_\circ}(k) \Big\}. \qquad (9)$$

If $N^+[i] = N^+[j]$ and $N^-[i] = N^-[j]$, then $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ and thus $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} = \varnothing$.

Proof: We prove exemplarily (i); (ii) can be shown analogously.

(i) First, we show $D'_\circ \in \mathcal{F}$ by proving that the out-neighborhoods (or in-neighborhoods) are nested. Since only the neighborhoods of i changed, it remains to show that $\Gamma^+_{D'_\circ}(k) \subseteq \Gamma^+_{D'_\circ}(i)$ or $\Gamma^+_{D'_\circ}(i) \subseteq \Gamma^+_{D'_\circ}(k)$ (or, $\Gamma^-_{D'_\circ}(k) \subseteq \Gamma^-_{D'_\circ}(i)$ or $\Gamma^-_{D'_\circ}(i) \subseteq \Gamma^-_{D'_\circ}(k)$) for any vertex $k \neq i \in V$. However, if

$N^+(i) = N^+(j)$, it implies

$$\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D'_\circ}(j) = \Gamma^+_D(j)$$

such that the out-neighborhoods share an inclusion relation for any pair i and k since j and k do so. The same applies to the in-neighborhoods if $N^-(i) = N^-(j)$. Therefore, $D'_\circ \in \mathcal{F}$ and since $D_\circ$ and $D'_\circ$ differ only in the loop at i, we have $D'_\circ \in [D]_\sim \cap \mathcal{F}$.

Next, we prove the Equality (8). The inclusion "$\supseteq$" is straightforward to validate. For "$\subseteq$", let $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. If $i \notin \{k,\ell\}$, then $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ because the neighborhoods of k and ℓ do not differ between $D_\circ$ and $D'_\circ$. Thus, $i \in \{k,\ell\}$ and w.l.o.g. we assume $\ell = i$. Since i and j are nonadjacent, it follows with $i \in \Gamma^+_{D_\circ}(i)$ and $j \notin \Gamma^+_{D_\circ}(j)$ that $\{i,j\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$, such that $k \neq j$. From $\{i,k\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}$ it is either $\Gamma^+_{D'_\circ}(i) \subsetneq \Gamma^+_{D'_\circ}(k)$ and $\Gamma^-_{D'_\circ}(i) \subsetneq \Gamma^-_{D'_\circ}(k)$, or $\Gamma^+_{D'_\circ}(k) \subsetneq \Gamma^+_{D'_\circ}(i)$ and $\Gamma^-_{D'_\circ}(k) \subsetneq \Gamma^-_{D'_\circ}(i)$. However, because $\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D_\circ}(i) \backslash \{i\}$, the latter would imply $\{i,k\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ which contradicts $\{i,k\} \notin \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. Thus, the former must hold. From $\{i,k\} \notin \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$, it follows that neither $\Gamma^+_{D_\circ}(i) \subsetneq \Gamma^+_{D_\circ}(k)$ and $\Gamma^-_{D_\circ}(i) \subsetneq \Gamma^-_{D_\circ}(k)$, nor $\Gamma^+_{D_\circ}(k) \subsetneq \Gamma^+_{D_\circ}(i)$ and $\Gamma^-_{D_\circ}(k) \subsetneq \Gamma^-_{D_\circ}(i)$. This is logically equivalent to either $\Gamma^+_{D_\circ}(k) \subseteq \Gamma^+_{D_\circ}(i)$ and $\Gamma^-_{D_\circ}(i) \subseteq \Gamma^-_{D_\circ}(k)$, or $\Gamma^+_{D_\circ}(i) \subseteq \Gamma^+_{D_\circ}(k)$ and $\Gamma^-_{D_\circ}(k) \subseteq \Gamma^-_{D_\circ}(i)$, or $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$, or $\Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(k)$. However, the first two cases cannot apply since $\Gamma^+_{D'_\circ}(i) = \Gamma^+_{D_\circ}(i) \backslash \{i\} \subsetneq \Gamma^+_{D_\circ}(k) = \Gamma^+_{D_\circ}(k)$ contradicts $\Gamma^+_{D_\circ}(k) \subseteq \Gamma^+_{D_\circ}(i)$, and $\Gamma^-_{D'_\circ}(i) = \Gamma^-_{D_\circ}(i) \backslash \{i\} \subsetneq \Gamma^-_{D_\circ}(k) = \Gamma^-_{D'_\circ}(k)$ contradicts $\Gamma^-_{D_\circ}(k) \subseteq \Gamma^-_{D_\circ}(i)$. Thus, it is $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$ or $\Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(k)$.

It remains to prove that $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ applies when $N^+(i) = N^+(j)$ and $N^-(i) = N^-(j)$. Let $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}$. If $i \notin \{k,\ell\}$, then it follows that $\{k,\ell\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. Let $\{i,k\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. Then, it holds either $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$ or $\Gamma^-_{D_\circ}(i) = \Gamma^-_{D_\circ}(k)$. Assume $\Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$, the other case works accordingly. Then, we have $i \in \Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$ and $j \notin \Gamma^+_{D_\circ}(i) = \Gamma^+_{D_\circ}(k)$, which is equivalent to $k \in N^-(i)$ and $k \notin N^-(j)$, contradicting $N^-(i) = N^-(j)$. Thus, $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} = \varnothing$. Since $\{i,j\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}$ we have the proper inclusion relation $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. □

Theorem 24. *Let $D = (V,E)$ be a threshold digraph and $<^+, <^-$ be its associated auxiliary strict preorders, respectively. The following equivalence applies:*

$$D \in \mathcal{T}_{\overline{\mathcal{H}}} \iff \exists i \neq j \in V : i <^- j \text{ and } i <^+ j \;.$$

Proof: $\Longrightarrow$: Let $D \in \mathcal{T}_{\overline{\mathcal{H}}}$. Because $[D]_\sim \cap \mathcal{F} \neq \varnothing$, there is a $D_\circ \in [D]_\sim \cap \mathcal{F}$. W.l.o.g. we assume that $D_\circ$ has the minimal cardinality of $\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ among all elements of $[D]_\sim \cap \mathcal{F}$. Since $D \in \mathcal{T}_{\overline{\mathcal{H}}}$, there are $i \neq j \in V$ with $\Gamma^+(j) \subsetneq \Gamma^+(i)$ and $\Gamma^-(j) \subsetneq \Gamma^-(i)$. Because of the proper subset relations, there exist $k, \ell \in V$ with $k \in \Gamma^+(i) \backslash \Gamma^+(j)$ and $\ell \in \Gamma^-(i) \backslash \Gamma^-(j)$. We distinguish cases dependent on whether a proper subset relation is due to a loop or not, i.e., whether $k, \ell \in \{i,j\}$. We refer to the cases as labeled by Figure 10.

Case 10a. Let $k, \ell \notin \{i,j\}$. The claim follows with Lemma 17.

Case 10b. Let $k = i$ and $\ell \notin \{i,j\}$. Since $\ell \in N^-(i) \backslash N^-(j)$, it follows $j <^- i$. Because $i \in N^+(\ell) \backslash N^+(j)$, it holds $j <^+ \ell$. We differentiate based on whether $i \longrightarrow j$ or $i \dashrightarrow j$. If $i \longrightarrow j$, there is the path $i^+ \longrightarrow j^- \longrightarrow \ell^+ \longrightarrow i^- \longrightarrow j^+$ in B_D. Thus, $j <^+ i$ and the claim follows. If $i \dashrightarrow j$, we assume $N^+(i) = N^+(j)$, otherwise Case 10a applies. Further, $j \notin \Gamma^+_{D_\circ}(j)$, since otherwise $D_\circ$ would contain the forbidden pair of nonadjacent vertices with one loop each (Figure 2e). Then, by Lemma 23 (i), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $i \notin \Gamma^+_{D'_\circ}(i)$. If $|\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}| \lneq |\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}|$, we have a contradiction to the minimality assumption of $D_\circ$. Therefore, let $|\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}| \geq |\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}|$. Then, since $\{i,j\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ but $\{i,j\} \notin \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}$, there must be a pair of vertices $\{g,h\}$ with $\{g,h\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. From Equality (8), we know $\{g,h\} = \{i,h\}$ with $h \neq j$ and $\Gamma^+_{D'_\circ}(i) \subsetneq \Gamma^+_{D'_\circ}(h)$ and $\Gamma^-_{D'_\circ}(i) \subsetneq \Gamma^-_{D'_\circ}(h)$. Since $D_\circ \in \mathcal{F}$, these inclusions imply $\Gamma^+_{D_\circ}(i) \subseteq \Gamma^+_{D_\circ}(h)$ and $\Gamma^-_{D_\circ}(i) \subseteq \Gamma^-_{D_\circ}(h)$. With $i \in \Gamma^+_{D_\circ}(i)$ and $i \in \Gamma^-_{D_\circ}(i)$, it is then in particular $i \in \Gamma^+_{D_\circ}(h)$ and $i \in \Gamma^-_{D_\circ}(h)$. Thus, i and h are reciprocally adjacent, while i and j are nonadjacent, i.e., $D_\circ$ contains a triangle of a reciprocally adjacent and nonadjacent pair. The claim follows from Lemma 17 (i).

Case 10c. Let $k = j$ and $\ell \notin \{i,j\}$. Since $\ell \in N^-(i) \backslash N^-(j)$, it holds $j <^- i$. We differentiate based on whether $j \dashrightarrow i$ or $j \longrightarrow i$. If $j \dashrightarrow i$, there is the path $i^+ \longrightarrow j^- \longrightarrow \ell^+ \longrightarrow i^- \longrightarrow j^+$ in B_D. Thus, $j <^+ i$ and the claim follows. If $j \longrightarrow i$, we assume $N^+[i] = N^+[j]$, otherwise Case 10a applies. Since i is in the middle of an intransitive path of length 2, it follows $i \in \Gamma^+_{D_\circ}(i)$ (Lemma 7 (i)). By Lemma 23 (ii), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $j \in \Gamma^+_{D'_\circ}(j)$. If $|\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}| \lneq |\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}|$, we have a contradiction to the minimality assumption of $D_\circ$. Therefore, let $|\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}| \geq |\mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}|$. Then, since $\{i,j\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$ but $\{i,j\} \notin \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ}$, there must be a pair of vertices $\{g,h\}$ with $\{g,h\} \in \mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \backslash \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$. From Equality (9), we know $\{g,h\} = \{j,h\}$ with $h \neq i$ and $\Gamma^+_{D'_\circ}(h) \subsetneq \Gamma^+_{D'_\circ}(j)$ and $\Gamma^-_{D'_\circ}(h) \subsetneq \Gamma^-_{D'_\circ}(j)$. Since $D_\circ \in \mathcal{F}$, these inclusions imply $\Gamma^+_{D_\circ}(h) \subseteq \Gamma^+_{D_\circ}(j)$ and $\Gamma^-_{D_\circ}(h) \subseteq \Gamma^-_{D_\circ}(j)$. With $j \notin \Gamma^+_{D_\circ}(j)$ and $j \notin \Gamma^-_{D_\circ}(j)$, it is then in particular $j \notin \Gamma^+_{D_\circ}(h)$ and $j \notin \Gamma^-_{D_\circ}(h)$. Thus, j and h are nonadjacent, while i and j are reciprocally adjacent, i.e., $D_\circ$ contains a triangle of a reciprocally adjacent and nonadjacent pair. The claim follows by Lemma 17 (i).

Case 10d. Works with a similar line of arguments as Case 10b.

Case 10f. Works with a similar line of arguments as Case 10c.

Case 10e. Let $k = \ell = i$. We assume $\Gamma^+(i) \backslash \Gamma^+(j) = \{k\} = \{i\}$ and $\Gamma^-(i) \backslash \Gamma^-(j) = \{\ell\} = \{i\}$, otherwise Case 10d and 10b would apply respectively. Then, it holds in particular $N^+(i) = N^+(j)$ and $N^-(i) = N^-(j)$. By Lemma 23 (i), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$, which contradicts the minimality assumption of $D_\circ$.

Case 10g. Let $k = \ell = j$. We assume $\Gamma^+(i) \backslash \Gamma^+(j) = \{k\} = \{j\}$ and $\Gamma^-(i) \backslash \Gamma^-(j) = \{\ell\} = \{j\}$, otherwise Case 10f and 10c would apply respectively. Then, it applies in particular $N^+[i] = N^+[j]$ and $N^-[i] = N^-[j]$. By Lemma 23 (ii), there is a $D'_\circ \in [D]_\sim \cap \mathcal{F}$ with $\mathcal{D}^{\overline{\mathcal{H}}}_{D'_\circ} \subsetneq \mathcal{D}^{\overline{\mathcal{H}}}_{D_\circ}$, which contradicts the minimality assumption of $D_\circ$.

$\Longleftarrow$: Let $i \neq j \in V$ with $i <^- j$ and $i <^+ j$, and let $D_\circ \in [D]_\sim \cap \mathcal{F}$. With Lemma 15 it follows $i \lneq^\ominus j$ and $i \lneq^\oplus j$, i.e., $D_\circ \notin \mathcal{H}$. Since $D_\circ$ was arbitrary, it follows $[D]_\sim \cap \mathcal{H} = \varnothing$ and thus $D \in \mathcal{T}_{\overline{\mathcal{H}}}$. □

Theorem 25. *Let $D = (V,E)$ be a threshold digraph and $<^+, <^-$ be its associated auxiliary strict preorders, respectively. The following equivalence applies:*

$$\begin{aligned} D \in \mathcal{T}_{\mathcal{H}} \quad \Longleftrightarrow \quad & \nexists i \neq j \in V : i <^- j \text{ and } i <^+ j, \text{ and} \\ & \forall i \neq j \in V : N^+(i) = N^+(j) \text{ or } N^-(i) = N^-(j) \\ & \qquad \Longrightarrow i \notin \Gamma^+(i) \text{ and } j \notin \Gamma^+(j) \text{ for all } D_\circ \in [D]_\sim \cap \mathcal{F}, \text{ and} \\ & \forall i \neq j \in V : N^+[i] = N^+[j] \text{ or } N^-[i] = N^-[j] \\ & \qquad \Longrightarrow i \in \Gamma^+(i) \text{ and } j \in \Gamma^+(j) \text{ for all } D_\circ \in [D]_\sim \cap \mathcal{F}\ . \end{aligned}$$

Proof: $\Longrightarrow$: Let $D \in \mathcal{T}_{\mathcal{H}}$. Then, in particular, $[D]_{\sim} \cap \mathcal{H} = [D]_{\sim} \cap \mathcal{F} \neq \varnothing$. Therefore, with Theorem 24, there are no $i \neq j \in V$ with $i <^{+} j$ and $i <^{-} j$. It remains to prove that if i and j have the same open or closed in- or out-neighborhood, the vertices' loop must be absent or present, respectively. We prove both cases for the out-neighborhood by contradiction; the proofs for the in-neighborhoods work analogously. Let $i \neq j \in V$.

Case 1. Let $N^{+}(i) = N^{+}(j)$ and $D_{\circ} \in [D]_{\sim} \cap \mathcal{F}$. W.l.o.g. we assume $i \in \Gamma^{+}(i)$. Because $N^{+}(i) = N^{+}(j)$, i and j are nonadjacent. It follows $j \notin \Gamma^{+}(j)$, otherwise $D_{\circ}$ would contain the forbidden pair of nonadjacent vertices with one loop each (Figure 2e). However, this entails $\Gamma^{+}(j) \subsetneq \Gamma^{+}(i)$ and $\Gamma^{-}(j) \subsetneq \Gamma^{-}(i)$. This nesting of the neighborhoods contradicts $D_{\circ} \in \mathcal{H}$.

Case 2. Let $N^{+}[i] = N^{+}[j]$ and $D_{\circ} \in [D]_{\sim} \cap \mathcal{F}$. W.l.o.g. we assume $j \notin \Gamma^{+}(j)$. Because $N^{+}[i] = N^{+}[j]$, i and j are reciprocally adjacent. It follows $i \in \Gamma^{+}(i)$, otherwise $D_{\circ}$ would contain the forbidden pair of reciprocally adjacent vertices without loops (Figure 2d). However, this entails $\Gamma^{+}(j) \subsetneq \Gamma^{+}(i)$ and $\Gamma^{-}(j) \subsetneq \Gamma^{-}(i)$. This nesting of the neighborhoods contradicts $D_{\circ} \in \mathcal{H}$.

$\Longleftarrow$: We prove the claim by contraposition. Assume, that $D \notin \mathcal{T}_{\mathcal{H}}$. Then, there is a $D_{\circ} \in [D]_{\sim} \cap \mathcal{F}$ with $D_{\circ} \notin \mathcal{H}$. This entails that there exist $i \neq j \in V$ with $\Gamma^{+}(j) \subsetneq \Gamma^{+}(i)$ and $\Gamma^{-}(j) \subsetneq \Gamma^{-}(i)$. Because of the proper subset relations, there exist $k, \ell \in V$ with $k \in \Gamma^{+}(i) \backslash \Gamma^{+}(j)$ and $\ell \in \Gamma^{-}(i) \backslash \Gamma^{+}(j)$. Since $\Gamma^{+}(\star), \Gamma^{-}(\star)$ can contain loops which are not present in D, we distinguish cases dependent on whether the proper subset relation is due to a loop or not, i.e., whether $k, \ell \in \{i, j\}$. We refer to the cases as labeled by Figure 10.

Case 10a. Let $k, \ell \notin \{i, j\}$. The claim follows with Lemma 17.

Case 10b. Let $k = i$ and $\ell \notin \{i, j\}$. Since $\ell \in \Gamma^{-}(i) \backslash \Gamma^{-}(j)$, it is $\ell \in N^{-}(i) \backslash N^{-}(j)$ and, thus, it is $j <^{-} i$. We differentiate between the presence or absence of the edge from i to j. If $i \longrightarrow j$, in B_D, there is the path $i^{+} \longrightarrow j^{-} \longrightarrow \ell^{+} \longrightarrow i^{-} \longrightarrow j^{+}$. By definition, this means $j <^{+} i$, such that the claim follows. If $i \dashrightarrow j$, we can assume $\Gamma^{+}(i) \backslash \Gamma^{+}(j) = \{i\}$, otherwise Case 10a applies. However, then it is $N^{+}(i) = N^{+}(j)$ while $k = i \in \Gamma^{+}(i)$.

Case 10c. Let $k = j$ and $\ell \notin \{i, j\}$. Since $\ell \in \Gamma^{-}(i) \backslash \Gamma^{-}(j)$, it is $\ell \in N^{-}(i) \backslash N^{-}(j)$ and, thus, it is $j <^{-} i$. We differentiate between the presence or absence of the edge from j to i. If $j \longrightarrow i$, we can assume $\Gamma^{+}(i) \backslash \Gamma^{+}(j) = \{j\}$, otherwise Case 10a applies. However, then it is $N^{+}[i] = N^{+}[j]$ while $k = j \notin \Gamma^{+}(j)$. If $j \dashrightarrow i$, there is the path $i^{+} \longrightarrow j^{-} \longrightarrow \ell^{+} \longrightarrow i^{-} \longrightarrow j^{+}$. By definition, this means $j <^{+} i$, such that the claim follows.

Case 10d and 10f. Works by following the same line of arguments as in Case 10b and 10c but with swapping the out- and in-neighborhoods, and the auxiliary strict preorders.

Case 10e. Let $k = \ell = i$. We assume $\Gamma^{+}(i) \backslash \Gamma^{+}(j) = \{i\}$ and $\Gamma^{-}(i) \backslash \Gamma^{-}(j) = \{i\}$, otherwise Case 10d and 10b would apply respectively. Then, it follows $N^{+}(i) = N^{+}(j)$ while $i \in \Gamma^{+}(i)$.

Case 10g. Let $k = \ell = j$. We assume $\Gamma^{+}(i) \backslash \Gamma^{+}(j) = \{j\}$ and $\Gamma^{-}(i) \backslash \Gamma^{-}(j) = \{j\}$, otherwise Case 10f and 10c would apply respectively. Then, it follows $N^{+}[i] = N^{+}[j]$ while $j \notin \Gamma^{+}(j)$. $\square$

6.3 Summary

Through the equivalences between configurations and strict preorders (Lemma 17), as well as between noncomparability in the strict preorders and equality in the neighborhoods (Lemma 14), the following corollary is directly implied and summarizes the findings.

Corollary 26. *Let $D = (V, E)$ be a threshold digraph. The following equivalences hold:*

(i) $D \in \mathcal{T}_{\overline{\mathcal{M}}}$ $\iff$ *D contains an open square or a one-way of length 2.*

(ii) $D \in \mathcal{T}_{\mathcal{M}}$ $\iff$ *D does not contain an open square or a one-way of length 2, and, whenever two vertices $i \neq j \in V$ have the same pair-excluding in- or out-neighborhood, they are either nonadjacent or reciprocally adjacent.*

(iii) $D \in \mathcal{T}_{\overline{\mathcal{H}}}$ $\iff$ *D contains an isolated path of length 2, a triangle of a reciprocally adjacent and nonadjacent pair or, an intransitive path of length 2 without a return to or from the middle.*

(iv) $D \in \mathcal{T}_{\mathcal{H}}$ $\iff$ *D does not contain an isolated path of length 2, a triangle of a reciprocally adjacent and nonadjacent pair, or, an intransitive path of length 2 without a return to or from the middle, and, whenever two vertices $i \neq j \in V$ have the same open or closed in- or out-neighborhood, both are not allowed to, or must have loops, respectively.*

7 Purely Medial and Hierarchical versus Uniquely Ranked Threshold Digraphs

Uniquely ranked threshold digraphs are simple digraphs that are total on at least one of the multiple neighborhood-inclusion criteria, which have been proposed for simple digraphs [14, 15]. Among those criteria are the medial and hierarchical preorders. Simple digraphs in which the preorders are total are characterized by forbidden characterizations [15, Lemma 4 & 5] that largely coincide with those appearing in Corollary 26. The overlap suggests that purely medial (hierarchical) threshold digraphs are consistent with, but not equivalent to, the medially (hierarchically) ranked threshold digraphs. This raises the question of how the different subsets of threshold digraphs relate to each other, which is addressed in this section. The subset relations found in this way make it possible to relate the same notions of centrality of two different digraph classes.

Definition 27 ([15, Definition 1])**.** *Let $D = (V, E)$ be a digraph.*

(i) The strong and weak hierarchical downwards neighborhood-inclusion relations $\leqslant^{(\downarrow)}, \leqslant^{[\downarrow]} \subseteq V \times V$ are respectively defined by

$$i \leqslant^{(\downarrow)} j \quad :\iff \quad \begin{cases} & N^+(i) \subseteq N^+(j) \\ \wedge & N^-(i) \supseteq N^-(j) \end{cases}$$

$$i \leqslant^{[\downarrow]} j \quad :\iff \quad \begin{cases} & N^+[i] \subseteq N^+[j] \\ \wedge & N^-[i] \supseteq N^-[j] \end{cases}$$

(ii) The medial neighborhood-inclusion relation $\leqslant^{\bowtie} \subseteq V \times V$ is defined by

$$i \leqslant^{\bowtie} j \;:\iff \begin{cases} & N^+(i) \subseteq N^+[j] \\ \wedge & N^-(i) \subseteq N^-[j] \\ \wedge & N^-(i) \subseteq N^+[j] \quad \text{if } (i,j) \in E \\ \wedge & N^+(i) \subseteq N^-[j] \quad \text{if } (j,i) \in E \;. \end{cases}$$

(a) (b)

Figure 11: Threshold digraphs that serve as counterexamples in the analysis of the relation between purely medial (hierarchical) and medially (hierarchically) ranked threshold digraphs.

Lemma 28. *Let D be a threshold digraph and let $\leqslant^{\bowtie}$ denote the medial and $\leqslant^{(\downarrow)}$, $\leqslant^{[\downarrow]}$ the hierarchical neighborhood inclusions. The following implications do or do not hold.*

(i) $D \in \mathcal{T}_{\overline{\mathcal{M}}}$ $\quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad$ $\leqslant^{\bowtie}$ *is not total on* D

(ii) $D \in \mathcal{T}_{\mathcal{M}}$ $\quad \overset{\not\Longrightarrow}{\not\Longleftarrow} \quad$ $\leqslant^{\bowtie}$ *is total on* D

(iii) $D \in \mathcal{T}_{\overline{\mathcal{H}}}$ $\quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad$ $\leqslant^{(\downarrow)}$ *and* $\leqslant^{[\downarrow]}$ *are not total on* D

(iv) $D \in \mathcal{T}_{\mathcal{H}}$ $\quad \Longleftrightarrow \quad$ $\leqslant^{(\downarrow)}$ *and* $\leqslant^{[\downarrow]}$ *are total on* D

Proof:

(i) $\Longrightarrow$: Let $D \in \mathcal{T}_{\overline{\mathcal{M}}}$. With Corollary 26 we know D contains an open square or a one-way of length 2. The negation of [15, Lemma 5] implies that D is not total on $\leqslant^{\bowtie}$.

$\not\Longleftarrow$: Let D be the threshold digraph consisting of a single converging triangle [15, Fig. 5k]. Then, $\leqslant^{\bowtie}$ is not total on D. However, $D_\circ \in [D]_{\sim} \cap \mathcal{F}$ with loops at both vertices of the reciprocally adjacent pair is a medial Ferrers digraph. Therefore, $D \notin \mathcal{T}_{\overline{\mathcal{M}}}$.

(ii) $\not\Longrightarrow$: Consider the threshold digraph D given in Figure 11a for which $D \in \mathcal{T}_{\mathcal{M}}$ applies. However, since D contains a converging triangle induced by the vertices i, j, k, the medial preorder $\leqslant^{\bowtie}$ cannot be total on D [15, Lemma 5].

$\not\Longleftarrow$: Let D be the threshold digraph given in Figure 3a (the running example) on which the medial preorder $\leqslant^{\bowtie}$ is total ($i =^{\bowtie} k \leqslant^{\bowtie} j$). However, i and k have the same pair-excluding out-neighborhood and are neither nonadjacent nor reciprocally adjacent. Thus, Corollary 26 implies $D \notin \mathcal{T}_{\mathcal{M}}$. Exemplary Ferrers digraphs in $[D]_{\sim}$ that are not medially nested are depicted in Figure 5a and 5d.

(iii) $\Longrightarrow$: Let $D \in \mathcal{T}_{\overline{\mathcal{H}}}$. With Corollary 26, we know D contains an isolated path of length 2, a triangle of a reciprocally adjacent and nonadjacent pair, or an intransitive path of length 2 without a return to or from the middle. The negation of [15, Lemma 4] implies that D is not total on $\leqslant^{(\downarrow)}$ and $\leqslant^{[\downarrow]}$. Note that an intransitive path of length 2 without a return to or from the middle is a combination of an intransitive path of length 2 and an edge without a bypass of length 2.

$\not\Longleftarrow$: Consider the threshold digraph given in Figure 11b. Vertices i and j cannot be compared w.r.t. $\leqslant^{(\downarrow)}$, and, k and ℓ cannot be compared w.r.t. $\leqslant^{[\downarrow]}$. However, $D_\circ \in [D]_{\sim} \cap \mathcal{F}$ with loops at both vertices i and j corresponds to a hierarchical Ferrers digraph. Therefore $D \notin \mathcal{T}_{\overline{\mathcal{H}}}$.

(iv) $\implies$: Let $D \in \mathcal{T}_{\mathcal{H}}$. We prove the claim by showing that D does not contain any of the forbidden configurations for digraphs which are total on $\leqslant^{(\downarrow)}$ or $\leqslant^{[\downarrow]}$ [15, Lemma 4].

First, we show that D cannot contain an intransitive path of length 2 by contraposition. Assume D contains an intransitive path of length 2 from a vertex $j \in V$ to $k \in V \backslash \{j\}$, on which $i \in V \backslash \{j, k\}$ is in the middle, i.e., $i^+ \longrightarrow^* i^-$. Because $D \in \mathcal{T}_{\mathcal{H}}$, digraph D cannot contain an intransitive path without a return from the middle, such that i and j must be reciprocally adjacent. In general, this means whenever there is an $i \in V$ with $i^+ \longrightarrow^* i^-$, there exist a $j \in V \backslash \{i\}$ with $j <^+ i$ and $i \rightleftarrows j$. For $i \in V$, define

$$\mathcal{C}_i^+ := \Big\{ \{v_1, v_2, \dots, v_t\} \in \mathcal{P}(V) : v_1 <^+ v_2 <^+ \dots <^+ v_t <^+ i =: v_{t+1} \text{ and } \\ v_r \rightleftarrows v_{r+1} \text{ for all r = 1, \dots, t} \Big\}$$

where $\mathcal{P}(V)$ is the power set of V. Let denote

$$M \in \arg\max_{\mathcal{S} \in \mathcal{C}_i^+} f(\mathcal{S}) := |\mathcal{S}|$$

a set of maximal cardinality $m := |M|$ in $\mathcal{C}_i^+$. Since $\{j\} \in \mathcal{C}_i^+$, the set $\mathcal{C}_i^+$ is nonempty such that M exists, but is not necessarily unique. Let $M = \{v_1, \dots, v_m\}$. Then, we know $v_1 <^+ v_2$ with v_1 and v_2 reciprocally adjacent. Then, for any $D_\circ \in [D]_\sim \cap \mathcal{F}$, it applies

$$\Gamma^+(v_1) \subsetneq \Gamma^+(v_2) \text{ and } \Gamma^-(v_1) \supseteq \Gamma^-(v_2).$$

In particular, this implies $v_1 \in \Gamma^-(v_2) \subseteq \Gamma^-(v_1)$, i.e., ${v_1}^+ \longrightarrow^* {v_1}^-$, such that there must be a $v_0 \in V \backslash M$ with $v_0 <^+ v_1$ and $v_0 \rightleftarrows v_1$. However, then $M' := \{v_0, v_1, \dots, v_m\} \in \mathcal{C}_i^+$ with $|M'| > |M|$ which contradicts the maximality assumption of M. Therefore, D cannot contain an intransitive path of length 2. Analogously, it can be shown that D cannot contain an edge without a bypass of length 2.

Next, we show that D neither contains a nonadjacent nor a reciprocally adjacent pair of vertices by contraposition. Assume that D contains a nonadjacent or reciprocally adjacent pair of vertices $i \neq j \in V$. Because, as we have shown above, D does neither contain an intransitive path of length 2, nor an edge without a bypass of length 2, we know

$$N^+(i) \backslash \{j\} \subsetneq N^+(j) \backslash \{i\} \text{ and } N^-(i) \backslash \{j\} \subsetneq N^-(j) \backslash \{i\}$$

(or the other way around) cannot hold. In terms of the strict preorders, by Lemma 13, this implies $i \mathrel{\substack{\not<^{+} \\ {}_{+}\not>}} j$ and $i \mathrel{\substack{\not<^{-} \\ {}_{-}\not>}} j$. Lemma 14 provides us four possibilities each for relations between in- and out-neighborhoods, of which only equality in the open or closed in- and out-neighborhood is feasible due to the adjacency between i and j. Because $D \in \mathcal{T}_{\mathcal{H}}$, by Corollary 26, whenever two vertices i and j have the same open or closed in- or out-neighborhood, they must not or must have loops, respectively. However, a vertex which must or must not have a loop implies D to contain an intransitive path of length 2 or an edge without a bypass of length 2; contradiction. Thus, D does not contain a nonadjacent or reciprocally adjacent pair of vertices.

Finally, D does not contain an isolated path of length 2 due to $D \in \mathcal{T}_{\mathcal{H}}$, and the claim follows.

$\impliedby$: Let D be total on $\leqslant^{(\downarrow)}$ and $\leqslant^{[\downarrow]}$. Then, D does not contain an isolated path of length 2 [15, Lemma 4]. Moreover, every pair of vertices is adjacent by exactly one edge. This adjacency entails that D does not contain a triangle of a reciprocally and nonadjacent pair or a pair of vertices with equal open or closed in- or out-neighborhoods. Because D is threshold, it cannot contain an induced directed 3-cycle; however, since any pair of vertices is adjacent, D must therefore be transitive. Therefore, D cannot contain an intransitive path of length 2 without a return from or to the middle in particular. Then, Corollary 26 provides $D \in \mathcal{T}_{\mathcal{H}}$. □

Lemma 28 provides insight into how different properties capturing a hierarchical (medial) neighborhood inclusion in threshold digraphs relate to each other.

First, for a threshold digraph, not having a hierarchical (medial) corresponding Ferrers digraph is a stronger property than not being total on the hierarchical (medial) preorders $\leqslant^{(\downarrow)}, \leqslant^{[\downarrow]}$ ($\leqslant^{\bowtie}$).

Second, purely hierarchical threshold digraphs are total on both the strict *and* weak hierarchical preorder. This equivalence suggests that having only hierarchical corresponding Ferrers digraphs is a stronger property than being total on $\leqslant^{(\downarrow)}$ *or* $\leqslant^{[\downarrow]}$ and thus captures an extreme among idealized hierarchical threshold digraphs. Furthermore, the proven equivalence provides us directly with the size of the class (unlabelled vertices), namely $|\mathcal{T}_{\mathcal{H}}| = 1$. This single threshold digraph is transitive, acyclic, and between any pair of vertices, there is exactly one edge. In particular, for $n \geq 3$, it can be derived that the threshold digraph cannot have a corresponding Ferrers digraph where all in- and out-neighborhoods are equal, i.e., where the hierarchical and medial neighborhood-nestings coincide, such that $\mathcal{T}_{\mathcal{H}} \cap \mathcal{T}_{\mathcal{M}} = \varnothing$ (cf. Figure 6).

Third, for the medial case, a corresponding result has not been found. This raises the question whether there is an alternative definition of a medial preorder to $\leqslant^{\bowtie}$, such that their intersection equals the threshold digraphs whose corresponding Ferrers digraphs are all medial, akin to the hierarchical situation. Although an equivalent formulation in terms of $\leqslant^{\bowtie}$ is lacking so far, the characterization of $\mathcal{T}_{\mathcal{M}}$ is in accordance with $\leqslant^{\bowtie}$ because both are related to symmetry. For $D \in \mathcal{T}_{\mathcal{M}}$, symmetry between compared vertices is required in their adjacency (either nonadjacent or reciprocally adjacent) whenever a pair-excluding neighborhood is equal. Simultaneously, every symmetric threshold digraph is total on $\leqslant^{\bowtie}$ [15, Lemma 7].

8 Conclusion and Future Work

We have investigated the correspondence between threshold digraphs and Ferrers digraphs from the perspective of neighborhood inclusion. Vertex rankings defined through neighborhood inclusion in a Ferrers digraph must preserve strict preorders defined by a relation similar to neighborhood inclusion in the corresponding threshold digraph. This finding could be used for determining the cardinality of the equivalence class a threshold digraph represents, i.e., to count distinct Ferrers digraphs with the same underlying threshold digraph. The larger this number, the more possibilities there are to rank vertices differently by placing loops. Further, there are threshold digraphs for which none of the proposed neighborhood-inclusion criteria is total [14, 15], such that inspecting their strict preorders could indicate what further suitable criteria might look like. Such criteria allow us to distinguish between centrality indices and are therefore a key to understanding different notions of centrality. In general, they serve as a fundamental tool for investigating the nestedness of a one-mode directed network, a pattern that occurs frequently in empirical networks [13].

In medial (hierarchical) Ferrers digraphs, the position of a vertex in the rankings is unique (up to reversal). The classes of purely medial (hierarchical) threshold digraphs whose corresponding Ferrers digraphs are *all* medial (hierarchical) are non-trivial. These simple digraph classes are interesting because their notion of centrality (medial or hierarchical) is fixed, while there is potentially freedom to rank vertices differently. Although we find the purely hierarchical threshold digraphs to consist of exactly one (unlabelled) digraph, the corresponding medial subclass is yet unknown and warrants further study. The mutually exclusiveness of purely medial and hierarchical threshold digraphs is in line with that of uniquely medially and hierarchically ranked threshold digraphs, further evidencing that, in contrast to undirected networks, there are different types of nestedness for directed networks. Therefore, in practice, the selection of a specific centrality index is even more crucial in the directed case.

In representational measurement [18], hierarchically ranked threshold digraphs, and thus also the subset of purely hierarchical ones, are essential in the theory of preference when indifference is intransitive [11]. It remains to be seen whether the medial equivalents also occur as empirical preference relations.

Several other open questions may benefit from a neighborhood-inclusion perspective. For instance, both undirected threshold graphs and Ferrers digraphs share the property that they can be constructed iteratively by adding isolated or dominating (with respect to their neighborhoods) vertices [12, 5]. While analogous constructions exist for subclasses of threshold digraphs, namely for oriented threshold graphs [1] and directed threshold graphs [14, 10], this is an open problem for threshold digraphs in general. Further, a core/periphery characterization of threshold graphs is that of split graphs [8] with nested neighborhoods in the independent set of vertices in the periphery [12, Theorem 1.2.4]. Threshold digraphs are split digraphs [4, Corollary 6] which were introduced as the simple directed extension to split graphs [7]. However, it is not known whether the vertices of the independent set in the directed case admit a corresponding nesting of their neighborhoods.

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