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New Reducible Configurations for Graph Multicoloring with Application to the Experimental Resolution of McDiarmid-Reed's Conjecture

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Abstract.A (a, b)-*coloring* of a graph G associates to each vertex a b-subset of a set of a colors in such a way that the color-sets of adjacent vertices are disjoint. We define general handle reduction methods for (a, b)-coloring of graphs for $2 \leq a/b \leq 3$. In particular, using necessary and sufficient conditions for the existence of an (a, b)-coloring of a path with prescribed color-sets on its end-vertices, more complex (a, b)-colorability reduction handles are presented. The utility of these tools is exemplified on finite triangle-free induced subgraphs of the triangular lattice for which McDiarmid-Reed's conjecture asserts that they are all $(9, 4)$-colorable. Computations on millions of such graphs generated randomly show that our tools allow to find a $(9, 4)$-coloring for each of them except for one specific regular shape of graphs (that can be $(9, 4)$-colored by an easy ad-hoc process). We thus obtain computational evidence towards the conjecture of McDiarmid&Reed.

1 Introduction

For two integers a and b, a (a, b)-*coloring* of a graph G is a mapping that associates to each vertex a set of b colors from a set of a colors in such a way that adjacent vertices get disjoints sets of colors. In particular, an $(a, 1)$-coloring is simply a proper coloring. Equivalently, an (a, b)-coloring of G is a homomorphism to the Kneser graph $K_{a,b}$. This type of coloring is also in relation to fractional colorings: the fractional chromatic number of a graph G can be defined as $\chi_f(G) = \min\{a/b, G \text{ is } (a, b)\text{-colorable}\}$.

Since the introductory paper of Stahl [15], multicoloring attracted much research, one of the most recent results being the one of Cranston and Rabern [2] showing that planar graphs are $(9, 2)$-colorable (without using the four color theorem).

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An associated problem is weighted coloring, in which the number of colors to assign can vary from vertex to vertex. Weighted coloring has a natural application for frequency allocation in cellular networks [6]. In particular, since the equidistant placement of transmitters induces a triangular lattice, weighted coloring of triangular lattices has been the subject of many studies [6, 8, 9, 12]. In most of these works, the aim is to bound the weighted chromatic number by a constant times the weighted clique number. Multicoloring is a particular case of weighted coloring; however, the fact that a graph is (a, b)-colorable can be translated into a bound for weighted chromatic number in terms of weighted clique, see [17] for instance.

Any graph with at least one edge is not (a, b)-colorable for $a < 2b$, so it is useful to introduce $e = a - 2b$, e symbolizing the entropy of the (a, b)-coloring (the smaller e is, the harder it is to color the graph). Note also that a graph is $(2b, b)$-colorable if and only if it is bipartite. In this paper we concentrate on the pairs (a, b) such that $2 < \frac{a}{b} \leq 3$, thus $e \in \{1, ..., b\}$. It is easy to observe that if a graph is (a, b)-colorable, it is also (am, bm)-colorable for any $m \geq 1$. Moreover, the following decomposition property also holds: for any integer $y \geq 1$, if G is $(2x+1, x)$-colorable for any $x \in \{1, \dots, y\}$, then G is (a, b)-colorable for any a, b such that $\frac{a}{b} \geq \frac{2y+1}{y}$. Hence $(2b+1, b)$-colorings are the extremal objectives for non-bipartite graphs.

For finite triangle-free induced subgraphs of the triangular lattice, called *hexagonal graphs* in this paper, it is easy to observe that they are $(3, 1)$-colorable and it has been proven by Havet [5] that such graphs are also $(5, 2)$-colorable and $(7, 3)$-colorable. Sudeep and Vishwanathan [16] then presented a simpler $(14, 6)$-coloring algorithm. Later, Sau et al. [11] a simpler $(7, 3)$-coloring, but using the four colors theorem as a subroutine, followed by a paper of Šparl et al. [13] providing a linear time $(7, 3)$-coloring algorithm (hence wihout using the four color theorem). The list version has been considered by Aubry et al. [1]. They showed that hexagonal graphs are $(5m, 2m)$-choosable. Of course, as a hexagonal graph can have an induced 9-cycle, the best we can hope is to find a $(9, 4)$-coloring. In 1999, McDiarmid and Reed proposed the following conjecture (initially stated in terms of weighted coloring):

Conjecture 1 (McDiarmid-Reed [8])**.** *Every hexagonal graph is* $(9, 4)$*-colorable.*

This paper aims to define general reducible configurations for (a, b)-coloring when $2 < a/b \leq 3$, which may allow one to prove that some graphs are (a, b)-colorable. In particular, we will apply our reduction tools on triangle-free induced subgraphs of the triangular lattice, in order to solve Conjecture 1. A part of the results is based on the Ph.D. thesis of Godin [4] that also contains reducibility results for (a, b)-choosability, even when $a/b \geq 3$. Searching for reducible configurations is common when dealing with graph (multi)coloring. It is, for instance, a part of the discharging method, extensively used for proving colorability properties on planar or bounded maximum average degree graphs. In the case $2 < a/b \leq 3$, reducible configurations for (a, b)-coloring take the form of induced paths in which interior vertices have degree 2 in the graph, called *handles*. In order to get sharper results, we also have to define more sophisticated handles by looking at the lengths of the induced paths that start at one or both of the end-vertices of the handle (in which cases we speak about S-handle and H-handle, respectively).

The paper is organized as follows: In Section 2, we define different types of handles we will consider and present the general handle-reducibility results. As the associated proofs are technical and quite long, they are placed in the Appendix section. In Section 3, we apply our reduction tools on finite triangle-free induced subgraphs of the triangular lattice in order to try to solve Conjecture 1. We also present the results of our computations, giving empirical evidence towards the conjecture and listing some possible extensions of this work and a conjecture generalizing Conjecture 1.

2 Handle reductions

The path P_{n+1} of length n is the graph with vertex set $\{v_0, \ldots, v_n\}$ and edge set $\{v_i v_{i+1}, i = 0, 1, \ldots, n-1\}$. For a path $P = v_1, \ldots, v_k$ in a graph G, v_1 and v_k are called its *ends* while each other vertex is called an *interior* vertex. Let $\mathsf{int}(P)$ denote the set of interior vertices of P.

A *handle* $P(n)$ of length n in a graph G is a path of length n that is an induced subgraph of G with vertices of degree two in $P(n)$ having the same degree in G.

A *parity handle (or P-handle)* $PP(n)$ in a graph G is a handle $P(n)$ with the additional property that there exists another path of length $m \leq n$, of the same parity as n in $G - \mathsf{int}(P(n))$, between the two end-vertices of $P(n)$.

An *S-handle* $S(n_1, n_2, n_3)$ in a graph G is a handle of length n_1 such that one of its end-vertices has degree 3 in G and is also the end-vertex of two other induced paths of length n_2 and n_3.

An *H-handle* $H(n_1, n_2, n, n_3, n_4)$ in a graph G is a handle of length n such that its two end-vertices are of degree three in G and one of them is also the end-vertex of two other induced paths of length n_1 and n_2 and the other end-vertex is also the end-vertex of two induced paths of length n_3 and n_4. Examples of handles are illustrated in Figure 1.

Note that for an S-handle $S(n_1, n_2, n_3)$, $\mathsf{int}(H)$ is the set of vertices between the two ends of the path of length n_1; while in an H-handle $H(n_1, n_2, n, n_3, n_4)$, $\mathsf{int}(H)$ is the set of vertices between the two ends of the path of length n (square vertices of Figure 1).

For a handle $H = S(n_1, n_2, n_3)$, the two end-vertices of the paths of length n_2, n_3 of H are called the *ports* of H and similarly for $H = H(n_1, n_2, n, n_3, n_4)$, the four end-vertices of the paths of length n_1, n_2, n_3, n_4 of H are called its ports.

Due to symmetry reasons, we will consider only S-handles with $n_1 \geq n_2 \geq n_3$ and H-handles with $n_1 \leq n_2 \leq n$ and $n_4 \leq n_3 \leq n$. Note that (some of) the ports of a handle may be the same vertices (hence, a handle may induce a cycle in the graph).

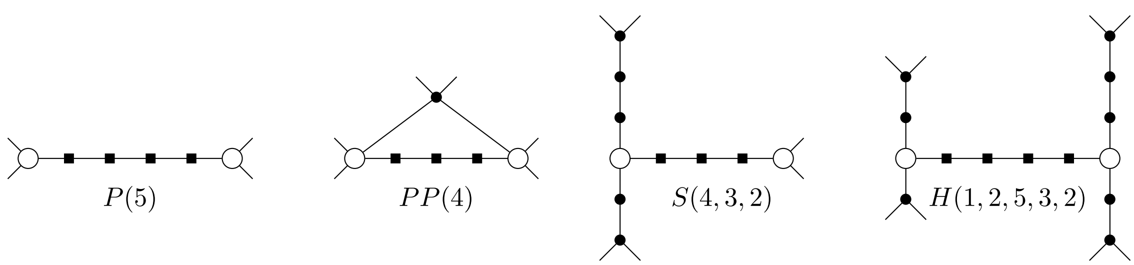

Figure 1: Examples of handles in a graph, from left to right: handle, parity handle, S-handle and H-handle (square vertices: vertices of $\mathsf{int}(H)$; white vertices: end-vertices of the handle).

A handle H is (a,b)*-reducible* if for any graph G, any (a,b)-coloring of $G - \mathsf{int}(H)$ can be extended to an (a,b)-coloring of G, possibly, for S- and H-handles, by modifying the color sets of some vertices of degree 2 of H (other than those of $\mathsf{int}(H)$).

In order to characterize graphs of list chromatic number 2, Erdös, Rubin, and Taylor [3] defined the core of a graph as the graph obtained after iteratively removing vertices of degree 1. In the same vein, for a graph G and a family $\mathcal{F}$ of handles in G, we define $\mathsf{core}_{\mathcal{F}}(G)$ as any (induced) subgraph obtained after successively removing vertices of degree 0 and 1 and vertices of $\mathsf{int}(H)$ for each handle $H \in \mathcal{F}$ until no more degree 0 or 1 vertex nor handle of $\mathcal{F}$ remains. By the definition of reducibility, we immediately have the following result:

Theorem 2 (Core theorem)**.** *For any graph G and any family $\mathcal{F}$ of (a,b)-reducible handles,*

$$G\ (a,b)\text{-colorable} \Leftrightarrow \mathsf{core}_{\mathcal{F}}(G)\ (a,b)\text{-colorable.}$$

In particular, if $\mathcal{F}$ allows to completely reduce the graph G, then we have the following corollary:

Corollary 3. *Let G be a graph and $\mathcal{F}$ be a family of (a,b)-reducible handles such that $\mathsf{core}_{\mathcal{F}}(G) = \emptyset$, then G is (a,b)-colorable.*

Figure 2 illustrates the method on an example. Since the reduced graph is empty and since, by the results of Section 2, each handle of $\mathcal{H}$ is $(7,3)$-reducible, then the graph is $(7,3)$-colorable.

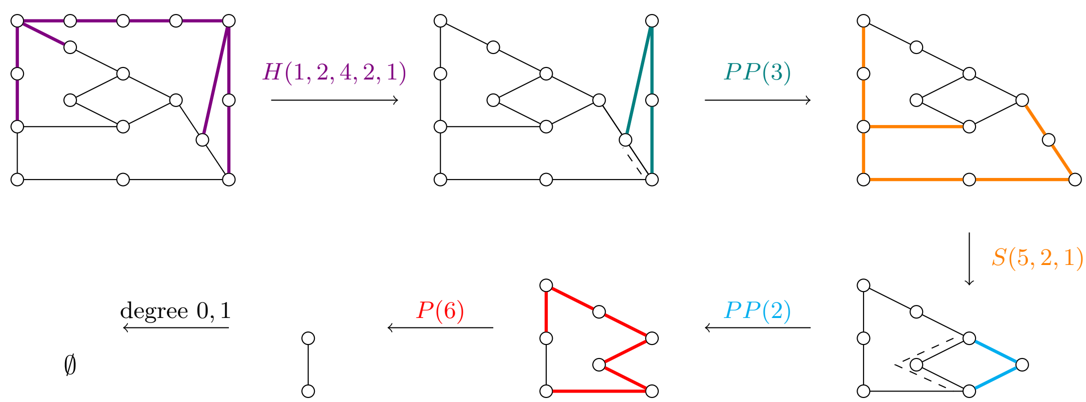

Figure 2: Example of the reduction of a graph with the set of handles $\mathcal{H} = \{H(1,2,4,2,1), S(5,2,1), P(6), PP(3), PP(2)\}$ (Thick edges: handle to reduce ; dashed path: second path of the same parity).

2.1 Path multicoloring

Recall that a, b, e are three integers such that $a = 2b + e$. For any real x, let $\mathsf{Even}(x)$ be the minimum even integer m such that $m \geq x$.

All our reducibility results are based on the following fact about the minimum length of a path in order it can be colored whatever the color-sets on its two end-vertices: It is easy to observe that in any $(2b+e,b)$-coloring of a graph G, the number of shared colors of any two vertices at distance d must be 0 if $d = 1$ (by definition), at most e if $d = 3$, at most $2e$ if $d = 5$, and so on; and for even d, at least $b - e$ if $d = 2$, at least $b - 2e$ if $d = 4$, and so on. This fact is formalized in the following definition and lemma:

Definition 4. *Let X, Y be b-subsets of $\{1, ..., a\}$ and k be an integer. The sets X and Y are said to be* k-compatible *if :*

$$|X \cap Y| \geq b - e\frac{k}{2}\ ,\ \textit{if } k \textit{ is even;}$$

$$|X \cap Y| \leq e\frac{k-1}{2}\ ,\ \textit{if } k \textit{ is odd.}$$

Moreover, if the above two inequalities are equalities, we say that X and Y are k-exactly-compatible.

Lemma 1 ([6, 10]). *For any integer $n \geq 0$, if φ is an (a,b)-coloring of the path P_{n+1}, then $\varphi(0)$ and $\varphi(n)$ are n-compatible.*

The following result, which is the central point of the paper, shows that the converse is also true. Despite the result was already known before (see [6, 10]), we give a constructive new proof.

Theorem 5. *If C_0, C_n are n-compatible sets, then an (a,b)-coloring φ of the path P_{n+1} such that $\varphi(0) = C_0$ and $\varphi(n) = C_n$ can be found in linear time.*

Proof: Let $P = v_0, \dots, v_n$ be a path on $n+1$ vertices and $a = 2b + e$. We first treat the case when C_0, C_n are n-exactly-compatible in the following claim:

Claim 6. *If C_0, C_n are n-exactly-compatible sets, then an (a,b)-coloring φ of the path P_{n+1} such that $\varphi(0) = C_0$ and $\varphi(n) = C_n$ can be found in linear time.*

Proof: If $n = 2p$ is even, then by hypothesis, $|C_0 \cap C_n| = b - ep$. By color symetry, we assume that $C_0 = \{1, \dots, b\}$ and $C_n = \{ep+1, \dots, ep+b\}$. Similarly, if $n = 2p+1$ is odd, then $|C_0 \cap C_n| = ep$ and we assume that $C_0 = \{1, \dots, b\}$ and $C_n = \{1, \dots, ep, ep+b+e+1, \dots, 2b+e\}$.

We define an (a,b)-coloring φ of P such that $\varphi(v_0) = C_0$ and $\varphi(v_n) = C_n$. For $0 \leq i \leq \lfloor (n-1)/2 \rfloor$, set

$$\begin{cases} \varphi(v_{2i}) &= \{ei+1, \dots, b+ei\}, \\ \varphi(v_{2i+1}) &= \{ei+a+1-b, \dots, a\} \cup (\{1, \dots, ei\}, \text{ if } i > 0). \end{cases}$$

Note that the color sets of consecutive vertices are disjoint and that $\varphi(v_0) = C_0$ and $\varphi(v_n) = C_n$. □

Now, assume C_0, C_n are n-compatible. Let $Q_t = v_0, \dots, v_t$ be the subpath of P induced by the first $t+1$ vertices. If n is even, let m be an integer such that $b - em \leq |C_0 \cap C_n| < b - em + e$ and let $k = 2m$. If n is odd, let m be an integer such that $em \leq |C_0 \cap C_n| < em + e$ and let $k = 2m+1$. We first define a coloring of Q_k and then show how to extend it to the whole path P:

- If n is even, let $X \subset C_0 \cap C_n$ such that $|X| = b - em$; and $Y = C_n - C_0$, then $em \geq |Y| > em - e$. Let $Z \subset \{1, \dots, a\} - C_0 - C_n$ such that $|Z| = em - |Y|$. We set $C_k = X \cup Y \cup Z$, then $|C_k| = b$ and $|C_0 \cap C_k| = b - em$.
- If n is odd, let $X \subset C_0 \cap C_n$ such that $|X| = em$; and $Y \subset C_n - C_0$ such that $|Y| = b - em - e$. Let $Z \subset \{1, \dots, a\} - C_0 - C_n$ such that $|Z| = e$. We choose $C_k = X \cup Y \cup Z$, then $|C_k| = b$ and $|C_0 \cap C_k| = em$.

Therefore in both cases, C_0 and C_k are k-exactly-compatible and thus by Claim 6, an (a,b)-coloring φ of the path Q_k such that $\varphi(v_0) = C_0$ and $\varphi(v_k) = C_k$ can be found in linear time. Moreover, by construction, $|C_k \cap C_n| \geq b - e$ and $n - k$ is even. Hence $|C_k \cup C_n| \leq 2b - (b-e) = b + e$ and thus there exists a set of b colors $T \subset \{1, \dots, a\} - C_0 - C_n$. Then we can complete the coloring for the rest of the path P by alternatively assigning the color sets T and C_n on the vertices $v_{k+1}, \dots, v_n$. □

2.2 Handle reductions

The results of Section 2.1 immediately imply the following:

Theorem 7. *For any integers b, e such that $b \geq 2$ and $e < b$, any handle $P(n)$ with $n \geq \mathsf{Even}(2b/e)$ is $(2b+e, b)$-reducible and any parity handle $PP(n)$ with $n \geq 2$ is $(2b+e, b)$-reducible in G.*

In order to present our reducibility results on S-handles and H-handles, we first need to define an ordering among them. For two integer vectors $\boldsymbol{v} = (v_1, v_2, \dots, v_k), \boldsymbol{v'} = (v'_1, v'_2, \dots, v'_k)$ of $\mathbb{N}^k$ we consider the natural order: $\boldsymbol{v} \leq \boldsymbol{v'}$ if and only if $\forall i \in \{1, \dots, k\}$, $v_i \leq v'_i$. Therefore, we will say that an S-handle $S(n_1, n_2, n_3)$ is *smaller* than (or equal to) an S-handle $S(n'_1, n'_2, n'_3)$ if $(n'_1, n'_2, n'_3) \geq (n_1, n_2, n_3)$ and similarly for H-handles. Finally, an (a, b)-reducible S- or H-handle H is *minimal* if no smaller handle is (a, b)-reducible.

We now present the following reducibility results for which the proofs, that consist in long case analysis, are postponed to the appendix section.

Theorem 8. *For any integers b, e, k such that $b \geq \max\{2, e+1, k\}$, $S(\mathsf{Even}(2b/e) - 1, 2, 1)$ is a minimal $(2b+e, b)$-reducible S-handle and $S(2b-k, k, k)$ is a minimal $(2b+1, b)$-reducible S-handle.*

Theorem 9. *For any integers b, e such that $b \geq 2$ and $e < b$, $H(1, 2, \mathsf{Even}(2b/e) - 2, 2, 1)$ is a minimal $(2b+e, b)$-reducible H-handle and the following H-handles are minimal $(2b+1, b)$-reducible H-handles:*

- $H(2, 2, 2b-3, 2, 2)$, $H(1, 2, 2b-3, 3, 2)$, $H(1, 4, 2b-3, 2, 2)$;
- $H(1, 2, 2b-4, 4, 3)$, $H(1, 4, 2b-4, 3, 3)$.

Remark that the above Theorem may be completed with the handle $H(2, 2, 2b-4, 3, 2)$ that seems to be a minimal $(2b+1, b)$-reducible H-handle with $n_1 = 2$ and $n = 2b-4$. Its reducibility was checked by computer for $b = 4$ but a general proof for any b seems to necessitate a very long case analysis and, as the case $b = 4$ is sufficient for Conjecture 1, we decided to skip this proof. Remark also that we do not try to go below $2b-4$ for the third parameter of a H-handle since these results will be mainly used for the case $(a, b) = (9, 4)$ in Section 3, and for these values, we have $2b - 4 = 4$. Hence, going below four will result in a H-handle with $n \leq n_3$ (i.e., the handle is not 'centered' on the longest path). Nevertheless, similar results can be obtained for lower values of n (and greater values of b). For instance, we have found by computer these complementary reducible H-handles for $(a, b) = (9, 4)$:

Proposition 10. *The H-handles $H(2, 2, 4, 3, 2)$, $H(2, 3, 3, 4, 3)$ and $H(2, 4, 3, 3, 3)$ are all three $(9, 4)$-reducible.*

Proof: (Sketch of) By computer or straightforward (but long) case analysis. We just give an idea of the proof for the handle $H(2, 2, 4, 3, 2)$.

Let v_i, $1 \leq i \leq 4$ and x, y be vertices of $H = H(2, 2, 4, 3, 2)$ such that there is a path of length 2 between v_1 and x, (v_2 and x; v_4 and y, respectively) and a path of length 3 between v_3 and y and a path of length 4 between x and y.

Let C_i be the color set of v_i, $1 \leq i \leq 4$. By Lemma 1, we have

$$i_1 = |C_1 \cap C_2| \geq 2, \text{ and } i_2 = |C_3 \cap C_4| \leq 2.$$

By Theorem 5, the color set X (Y, respectively) assigned to x (y, respectively) must satisfy

$$|X \cap Y| \geq 2, |C_1 \cap X| \geq 3, |C_2 \cap X| \geq 3, |C_3 \cap Y| \leq 1, |C_4 \cap Y| \geq 3.$$

It remains to show that for any $2 \leq i_1 \leq 4$ and $0 \leq i_2 \leq 2$, it is possible to find such sets X and Y. Hence there are 9 cases to consider. We only present one of them: Suppose $i_1 = 4$ and $i_2 = 0$. Then $C_1 = C_2$ and $C_3 \cap C_4 = \emptyset$. Let $\alpha = |C_1 \cap C_3|$ and $\beta = |C_1 \cap C_4|$. By the constraints, we have

$3 \leq \alpha + \beta \leq 4$. Suppose that $\alpha = 0$ and thus $\beta = 3$ or 4. Then we can set $X = Y = C_1$. Since $\alpha = 0$, then $|Y \cap C_3| = \alpha = 0 \leq 1$ and $|Y \cap C_4| = \beta \geq 3$ and the other inequations are staified as well. Other values of α, β can be treated similarly. □

It can be shown (but the proof is tedious) that for the above handles, the core is in fact unique, i.e., whatever the order of the reductions made, the process will end with the same graph.

Remark 11 (Unicity of the core)**.** *For the family $\mathcal{F}$ of all handles from Theorems 7, 8, 9 for fixed $b \geq 2$ and $e < b$, $\mathsf{core}_{\mathcal{F}}(G)$ is unique.*

3 Multicoloring triangle-free induced subgraphs of the triangular lattice

Recall that a finite triangle-free induced subgraph of the triangular grid is called a *hexagonal graph*.

In a similar way to the method used by Havet [5], i.e., starting from a degree 3 vertex in a 'corner' of the graph and exploring the configurations around it to prove that a handle from $\mathcal{F}$ is present, we can prove the following:

Theorem 12. *Let G be a hexagonal graph. For each of the following three families of handles, we have $\mathsf{core}_{\mathcal{F}}(G) = \emptyset$:*

1. $\mathcal{F} = \{P(2)\}$;
2. $\mathcal{F} = \{P(4), PP(3)\}$;
3. $\mathcal{F} = \{P(6), PP(3), PP(4), PP(5), S(5,2,1), H(1,2,4,2,1)\}$.

Proof: (Sketch of) Let G be a hexagonal graph.

Since G is finite, it is easy to observe that G contains a handle $P(2)$ and hence $\mathsf{core}_{\{P(2)\}}(G) = \emptyset$.

We now prove that $\mathsf{core}_{\{P(4),PP(3)\}}(G) = \emptyset$. Let us call a vertex of degree 3 in G a *node*. If G has no nodes, then G is a union of cycles of length at least 6, hence it contains a handle $P(4)$. Now, assume that G has at least one node. As explained in [5, 1], G can be embedded in the plane in such a way that edges leaving each vertex are only in at most three of the six following directions: left (L), right (R), up-left (UL), up-right (UR), down-left (DL), down-right (DR). According to this, there are two kinds of nodes: left nodes (with L-edge, UR-edge and DR-edge) and right nodes with (R-, UL- and DL-edges). The idea is to look at a node x the most on the left among those the most on the top. We can assume w.l.o.g. that x is a left node (see [5, 1] for details). For a node v and a direction D among L,R,UL,UR,DL,DR, let us call the *D-path from v* the induced path ending on a node y and with vertices of degree two starting by the D-edge from v. Consider the UR-path P from x. If P has length at least 4 then we are done. Otherwise, since the first edge of P is a UR-edge and x is the node the most on the top, the only remaining possibility is that of P having length 3. In this case, we have an induced hexagon and hence a parity handle $PP(3)$ of length 3.

For the case $\mathcal{F} = \{P(6), PP(3), PP(4), PP(5), S(5,2,1), H(1,2,4,2,1)\}$, we proceed similarly as for the previous case, with x, y and P being defined accordingly and y being the end-vertex of P other than x. If P has length at least 6 then we have a handle $P(6)$. Otherwise if P has length 5, then we have either a handle $S(5,2,1)$ if one of the neighbors of x or y outside P has degree two or a parity handle (there are seven possible configurations, see [4]). If P has length 4 then either we have a handle $H(1,2,4,2,1)$ or the two neighbors of y outside P are also nodes and we have a parity handle $PP(4)$ (see Figure 3 for the three possible configurations). If P has length 3, as for the previous case, we have a parity handle $PP(3)$. □

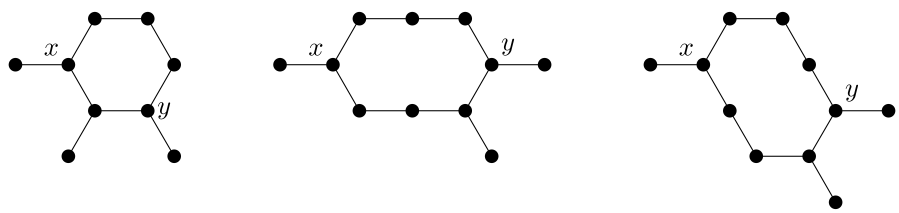

Figure 3: Three types of parity handles $PP(4)$ between nodes x and y in a hexagonal graph.

This theorem along with Theorems 7, 8, 9 and Corollary 3 allow us to prove the known results [5] for (a,b)-colorability of hexagonal graphs for $b=1,2$ and 3 in a unified way:

Corollary 13. *Any hexagonal graph is* $(3,1)$*-colorable,* $(5,2)$*-colorable and* $(7,3)$*-colorable.*

Now, we turn our attention to $(9,4)$-colorings. Consider the set of handles
$\mathcal{F}_{9,4} = \mathcal{P} \cup \mathcal{S} \cup \mathcal{H}$, with
$\mathcal{P} = \{P(8), PP(3), PP(4), PP(5), PP(6), PP(7)\}$,
$\mathcal{S} = \{S(7,2,1), S(6,2,2), S(5,3,3), S(4,4,4)\}$, and
$\mathcal{H} = \{H(1,2,6,2,1), H(1,2,5,3,2), H(1,4,5,2,2), H(2,2,5,2,2), H(2,2,4,3,2),$
$H(1,2,4,4,3), H(1,4,4,3,3), H(2,3,3,4,3), H(2,4,3,3,3)\}$.

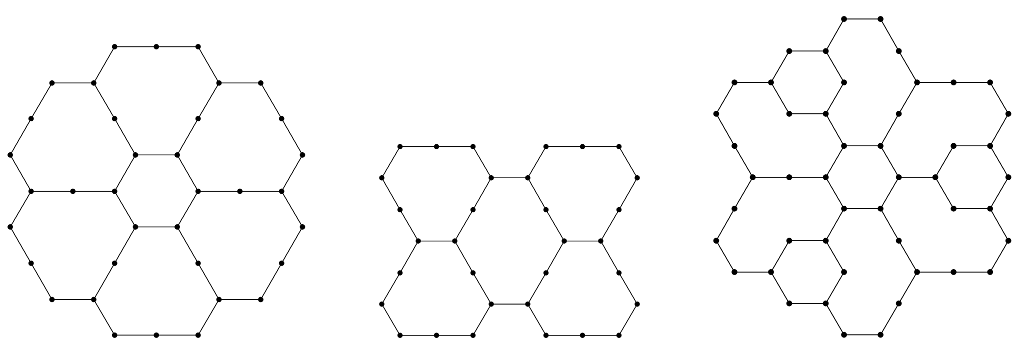

Figure 4: Three hexagonal graphs that contain no handle from $\mathcal{P} \cup \mathcal{S}$.

Note that, by Theorems 7, 8, 9 and Proposition 10, the handles of $\mathcal{F}_{9,4}$ are all $(9,4)$-reducible. Moreover, all the handles of $\mathcal{F}_{9,4}$ are necessary to reduce hexagonal graphs since we have examples of hexagonal graphs for which the core is not empty if we remove one of the handles from $\mathcal{F}_{9,4}$. Figure 4 shows three examples of hexagonal graphs that only possess handles from $\mathcal{H}$ (the graph on the left contains only handles $H(2,4,4,4,2)$ (hence also $H(2,2,4,3,2)$), the one on the center only handles $H(1,2,6,2,1)$ and the one on the right only handles $H(2,2,4,4,2)$.

However, the argument used in proof of Theorem 12 is not sufficient in this case to show that $\mathsf{core}_{\mathcal{F}_{9,4}}(G) = \emptyset$ for any hexagonal graph, i.e., to show that Conjecture 1 is true.

3.1 More $(9, 4)$-reducible handles

In order to (try to) prove Conjecture 1, the computational experiments we made lead us to find a set $\mathcal{F}'_{9,4}$ of more sophisticated $(9, 4)$-reducible configurations that we call *Cyclic H-handles*. The 25 configurations of $\mathcal{F}'_{9,4}$ are presented in Figure 5. These configurations are in fact H-handles with additional constraints (other paths of prescribed length between some ports). Note that each of these configurations is reduced by removing the interior of the central path (between big white vertices). Note also that sometimes the distance between two ports along this central path may be lower than 8, meaning that if the central path is removed, there will be no more constraints on the 4-color sets of the two ports (like for the top left configuration of Figure 5 where the bottom left and bottom right ports are at distance 6 along the central path). But in this case, it can be observed that there is always a second path between the two ports of the same parity and with lower than or equal length than the one going through the central path.

Proposition 14. *Every handle from $\mathcal{F}'_{9,4}$ from Figure 5 is $(9, 4)$-reducible.*

Proof: (Sketch of) We only prove the result for one of the handles of $\mathcal{F}'_{9,4}$: the one depicted on the fifth raw and second column of Figure 5. The result for the other configurations can be proved similarly, whereas some configurations require a deeper (and tedious) case analysis. They have all been tested by computer (see Section 3.2 for the method and link to the code).

Let G be a graph containing this configuration, i.e., a handle $H = H(2, 2, 4, 2, 2)$ with the additional constraints $d_G(v_1, v_2) = 2$ and $d_G(v_3, v_4) = 2$, where h_0 and h_4 are the end-vertices of the handle and v_1, v_2 are the extremities of the paths of length 2 from h_0 and v_3, v_4 are the extremities of the paths of length 2 from h_4, see Section 6. Let φ be a $(9, 4)$-coloring of $G - \mathsf{int}(H)$. By the additional conditions, we have $|\varphi(v_1) \cap \varphi(v_2)| \geq 3$ and $|\varphi(v_3) \cap \varphi(v_4)| \geq 3$. Hence there exist X and Y such that $|X \cap \varphi(v_1) \cap \varphi(v_2)| = 3$ and $|Y \cap \varphi(v_3) \cap \varphi(v_4)| = 3$. If $|X \cap Y| < 2$, then choose any $x \in Y - X$ and any $y \in X - Y$ else choose any $x \in \{1, \ldots, a\} - X$ and any $y \in \{1, \ldots, a\} - Y$. Then, we have $|(X \cup \{x\}) \cap (Y \cup \{y\})| \geq 2$, hence, by Theorem 5, the coloring φ can be extended to G by setting $\varphi(h_0) = X \cup \{x\}$ and $\varphi(h_4) = Y \cup \{y\}$. □

Note that all the cyclic handles of $\mathcal{F}'_{9,4}$ are necessary to reduce hexagonal graphs, i.e., we found examples of hexagonal graphs that cannot be reduced completely if any handle is removed from this set. One such example is illustrated in Figure 6. As it has many vertices, we only draw the nodes (degree 3 vertices) along with the path lengths of the induced paths connecting them, indicated as an edge weight between corresponding vertices. We can observe that this graph contains cyclic handles from $\mathcal{F}'_{9,4}$, like the one depicted in red (in fact, present many times around the graph), which is the leftmost handle on the top of figure 5.

As we will see in Section 3.3, we experimentally observed that the handles from $\mathcal{F}_{9,4} \cup \mathcal{F}'_{9,4}$ are sufficient to obtain an empty core starting from any hexagonal graph, i.e. to completely $(9, 4)$-color it, except if the graph contains the graph of Figure 7 as a subgraph.

However, we further present in Figure 9 of Appendix B a new set $\mathcal{F}''_{9,4}$ of (more complex) $(9, 4)$-reducible configurations obtained by extending S-handles and H-handles on one or two ports. The $(9, 4)$-reducibility of each of these configurations has been tested with the computer.

3.2 Testing configurations with the computer

In order to test that the configurations of $\mathcal{F}'_{9,4}$ and $\mathcal{F}''_{9,4}$ are each $(9, 4)$-reducible, we used the computer. The general process was to test that by assigning any 4-color set on each of the ports of the configuration, it is always possible to complete the $(9, 4)$-coloring of the other vertices of the

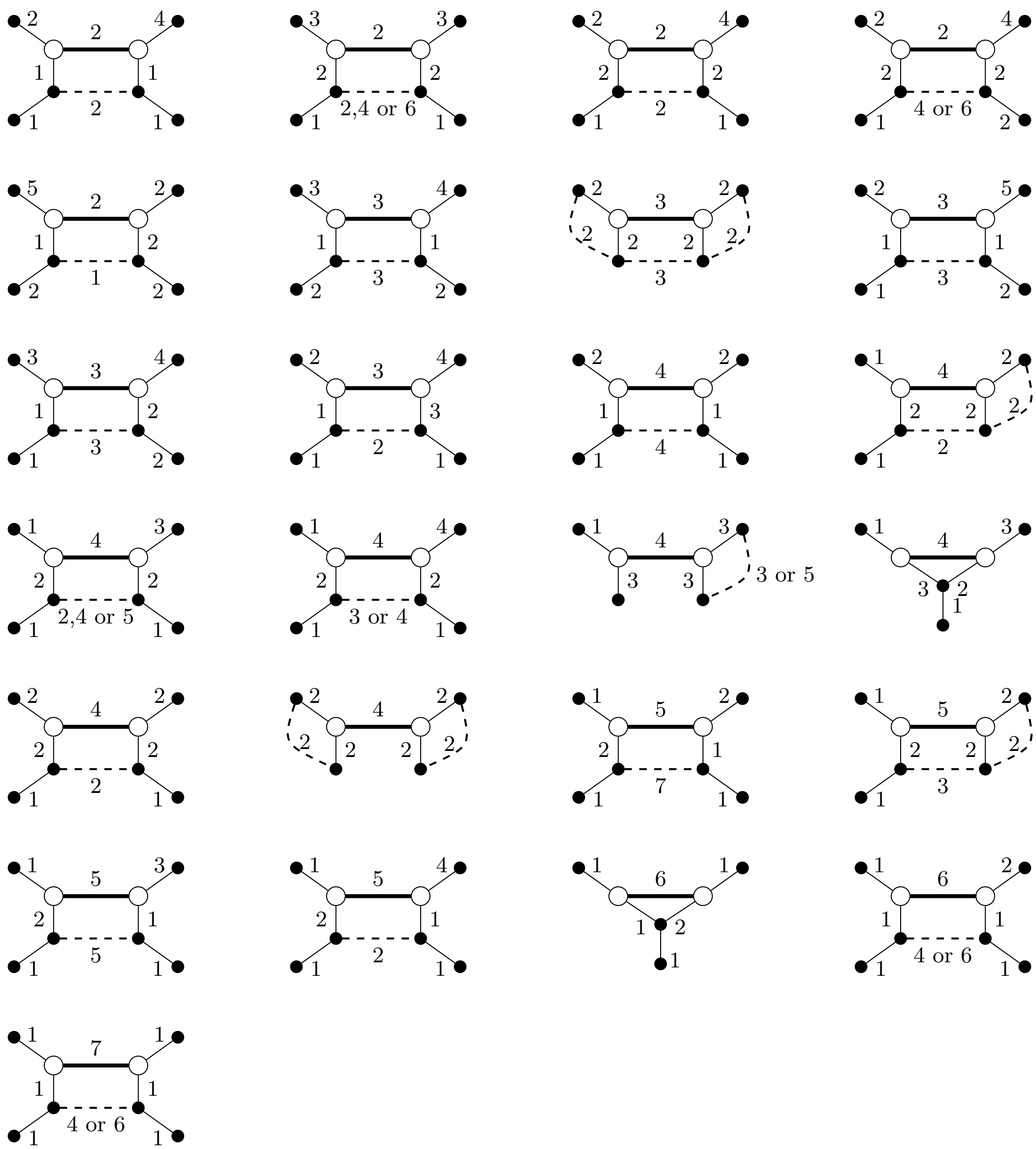

Figure 5: $\mathcal{F}'_{9,4}$: Cyclic $(9,4)$-reducible H-handles (numbers: path lengths; plain lines: induced paths with degree 2 interior vertices; dashed lines: induced paths with possibly vertices of degree greater than two; thick line: path to be removed).

configuration. To reduce the complexity (and running time), we take into account the symmetry in colors. Let us illustrate on an example of a H-handle (with 4 ports and possibly with cycles, like the ones of Figure 5). There are $\binom{9}{4} = 126$ 4-color sets and 4 ports, hence $126^4 = 252047376$

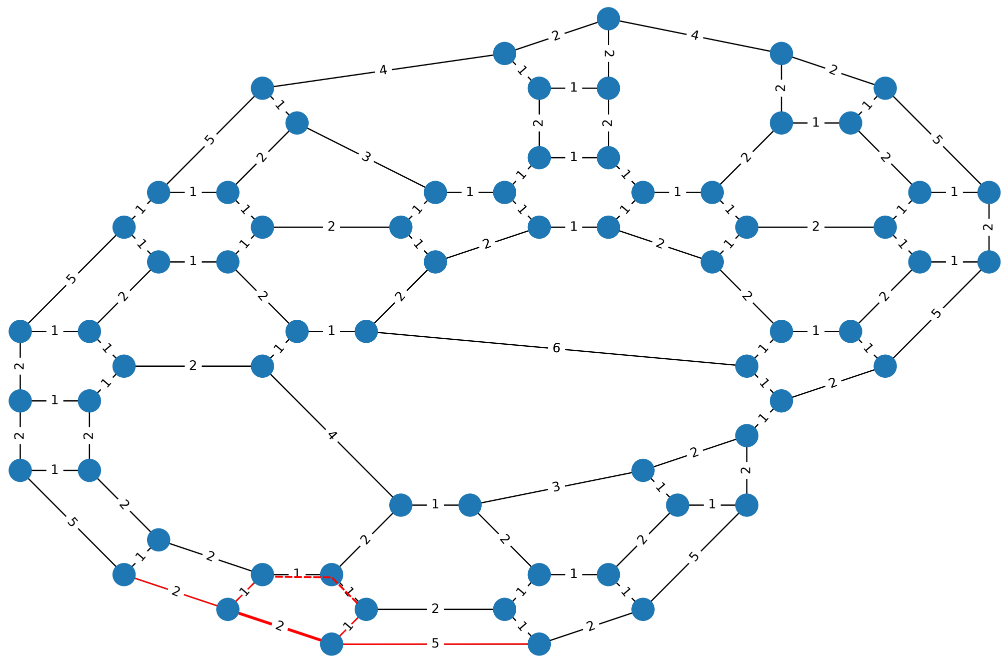

Figure 6: An example of hexagonal graph that cannot be reduced by the handles from $\mathcal{F}_{9,4}$, but that contains handles from $\mathcal{F}'_{9,4}$, like the one in red (numbers are path-lengths connecting degree 3 vertices).

assignments to test. However, by permuting the colors, we can always consider that the first port is given the color set $C_1 = \{1, 2, 3, 4\}$, which we write as $C_1 = 1234$ for short. For the second port, we can restrict to five 4-color sets depending on the number of colors shared with C_1: By color symmetries, Port 2 can be assigned 1234 if 4 colors are shared, 1235 if 3 colors are shared, 1256 if two colors are shared, 1567 if one color is shared, and 5678 if no color is shared. Hence, we can assign to Port 2 the color-sets from $C_2 = \{1234, 1235, 1256, 1567, 5678\}$.

Then we can continue the same reasoning for the third port. Depending on the number of shared colors between the color sets of Port 1 and Port 2, there can be 5, 9, 19, 14, or 22 4-color-sets for Port 3:

$$C_3 = \begin{cases} C_2, & if c(v_2) = 1234, \\ \{1234, 1235, 1236, 1245, 1246, 1256, 1267, 1456, 1467, 1567, 1678, & \\ 4567, 4678, 5678, 6789\} & if c(v_2) = 1235, \\ \{1234, 1235, 1237, 1256, 1257, 1278, 1345, 1347, 1356, 1357, 1378, & \\ 1567, 1578, 1789, 3456, 3457, 3478, 3567, 3578, 3789, 5678, 5789\} & if c(v_2) = 1256, \\ \{1234, 1235, 1238, 1256, 1258, 1289, 1567, 1568, 1589, 2345, 2348, & \\ 2356, 2358, 2389, 2567, 2568, 2589, 5678, 5689\} & if c(v_2) = 1567, \\ \{1234, 1235, 1239, 1256, 1259, 1567, 1569, 5678, 5679\} & if c(v_2) = 5678. \end{cases}$$

For the fourth port, we then test the 126 color sets (this could also be optimized by looking at the sets chosen in C_2 and C_3 but we did not implement this optimization).

Thus, in total, we have reduced the total number of assignments to test from 126^4 to $(5+15+22+19+9)126 = 8820$.

The Python program `TestReducibility.py` developed for checking reducibility of a configuration of type S, H, S', S'', H', H'' and those of Figure 5 can be accessed at https://github.com/otogni/MCD-coloring.git.

3.3 Computational reducibility experiments

We have performed computational experiments for testing the reduction tools defined in the previous section on hexagonal graphs for finding a $(9,4)$-coloring. The algorithms were coded in C++ and ran on an Intel Xeon CPU at 2.67 GHz and 24 GB of memory. The source code can be accessed at https://github.com/otogni/MCD-coloring.git.

Experiment 1: Random generation

Graph generation: The graphs are generated randomly on a grid of size $\ell \times h$ (the triangular lattice is considered as a square lattice with diagonals) by choosing randomly the coordinates of a point and testing if the corresponding vertex can be added to the graph without creating a triangle (repeated $5\ell h$ times). In order to obtain the 'harder' instances of random hexagonal graphs, we then do a final pass in which we consider the points of the grid in sequence and test if they can be added to the graph. Hence the graphs obtained are maximal triangle-free subgraphs of the triangular lattice, i.e., no point in the area can be added without creating a triangle.
Reduction algorithm: it consists in testing, for each vertex x in sequence, if x is the end-vertex of a handle from $\mathcal{F}_{9,4}$ or $\mathcal{F}'_{9,4}$.

Table 1 presents some measures of Experiment 1: the mean number of nodes (degree-3 vertices) and time needed to generate and completely reduce the graph depending on the side-length of the grid n (the square root of the grid' size). One can observe that even for a hexagonal graph with hundreds of thousands degree-3 vertices (and around 2M vertices in total), our reduction algorithm can still construct a $(9,4)$-coloring in a reasonable time, even if it is far from being optimized.

n (grid size $n \times n$)	10	25	50	100	200	500	1000	2000
# nodes	8	92	492	1763	7450	46722	188452	754485
time (s)	0.0022	0.0026	0.0046	0.026	0.31	12.3	185	3618

Table 1: Mean number of nodes (degree 3 vertices) and mean total time needed for generation and reduction depending on the grid side size n.

Since the graphs generated in Experiment 1 have all been reduced, we tried to build other hexagonal graphs that might possibly not be completely reduced by our reductions. For this, our intuition and some mathematical arguments make us believe that hexagonal graphs with many 9-cycles will be harder to reduce. Experiments in this direction lead us to generate semi-random hexagonal graphs with many 9-cycles by using the flower graph depicted on the left of Figure 4, as explained in the following experiment.

Experiment 2: Semi-random generation

Graph generation: The graphs $G_{p,d}$ are generated by starting from an empty grid of size $\ell \times h$ and then first putting flower graphs isomorphic to the graph depicted on the left of Figure 4 in a

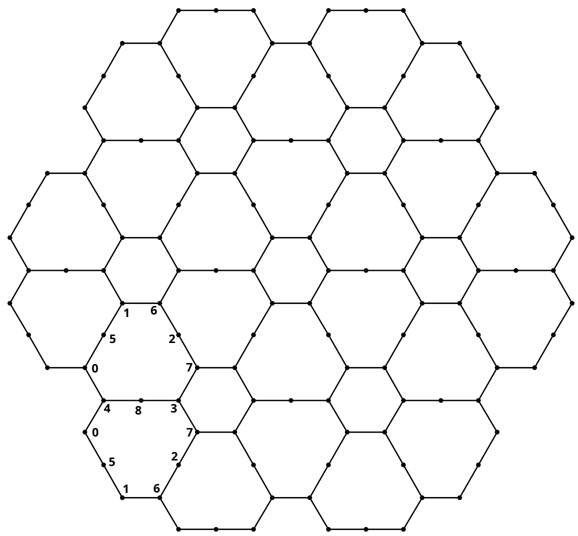

Figure 7: A hexagonal graph that contains none of our $(9,4)$-reducible configurations but can be $(9,4)$-colored by extending the coloring of the two 9-cycles (where number i represents the 4-colors subset $[i, i+4 \bmod 9]$) to the other 9-cycles.

quasi-regular pavement, i.e. a flower is positioned in coordinates $(pi + r^1_{ij} + 5, pj + r^2_{ij} + 5)$, for $0 \leq i < h/15$ and $0 \leq j < \ell/15$, with $p \geq 10$ being a parameter for controlling the distance between flowers (observing that $p = 11$ produces a maximum number of hard instances) and r^1_{ij}, r^2_{ij} being random integers between 0 and d. Note that d has to be small enough compared to p in order to avoid triangles. The obtained graph is then completed randomly in order to obtain a maximal induced triangle-free subgraph of the triangular grid, as described in Experiment 1.

Reduction algorithm: it consists in testing, for each vertex x in sequence, if x is the end-vertex of a handle from $\mathcal{F}_{9,4}$ or $\mathcal{F}'_{9,4}$.

Again, the Reduction algorithm has been tested on hundreds of millions of semi-random graphs (mainly for grid size 50×50) and it allows to reduce completely all the graphs except for $G_{9,0}$ which is always a subgraph of the regular lattice containing faces of length 6 and 9 obtained by translating the graph of Figure 7. For such a graph, the reduction process always falls into a non-reducible configuration. However, this tilling can be easily $(9,4)$-colored using only 9 4-subsets of $\{0, \ldots, 8\}$ by extending the precoloring of Figure 7.

Note that our reduction algorithm is very basic and could be improved in many ways, for instance by searching first for reducible configurations on its boundary instead of looking at each node in sequence.

3.4 Possible Extensions

Even if we were not able to completely solve Conjecture 1 by hand, we made some progress by providing many new reducible configurations that might possibly be used to obtain a proof of the conjecture in a similar way as for proving such graphs are $(7,3)$-colorable, i.e., showing that in any hexagonal graph, there always exists one of these $(9,4)$-reducible configurations somewhere in (the periphery of) the graph. Maybe these new configurations are not enough for such a proof, and one will have to find other reducible configurations. In particular, we believe that there are many $(9,4)$-reducible configurations with a cycle and 3 or 4 paths of prescribed lengths going outside the cycle, other than the 25 configurations of Figure 5, but we did not try to find them exhaustively.

Another possible direction is to restrict to subclasses of hexagonal graphs; for instance, we can ask: is any hexagonal graph with odd-girth 11 $(9,4)$-colorable? or even $(11,5)$-colorable? We thus propose the following conjecture that generalizes Conjecture 1 (since a triangle-free induced subgraph of the triangular lattice has odd girth at least 9):

Conjecture 15. *For any $k \geq 1$, any hexagonal graph G of odd-girth at least $2k+1$ is $(2k+1,k)$-colorable.*

The conjecture is true for $k \leq 3$ and for $k = 4$ for hexagonal graphs G such that $\mathsf{core}_{\mathcal{F}}(G) = \emptyset$, with $\mathcal{F} = \mathcal{F}_{9,4} \cup \mathcal{F}'_{9,4} \cup \mathcal{F}''_{9,4}$. A more general (but weaker) result in this direction is the one of Klostermeyer and Zhang [7] proving that any planar graph with odd girth at least $10k-7$ (and $k \geq 2$) is $(2k+1,k)$-colorable. The three-dimensional generalization (called cannonball graph [14]) also seems interesting. One question could be: Is any triangle-free subgraph of the 3D triangular grid $(5,2)$-colorable?

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A Reducibility proofs

The purpose of this section is to prove Theorems 8 and 9 of Section 2. For an S-handle $S(n, n_1, n_2)$, let $P = h_0, h_1, \cdots, h_n$ be the central path and let v_1 and v_2 be the end-vertices of the paths of lengths n_1 and n_2, respectively. Similarly, for a H-handle $H(n_1, n_2, n, n_3, n_4)$, $P = h_0, h_1, \cdots, h_n$ is the central path and v_1 and v_2, v_3 and v_4 are the end-vertices of the paths of lengths n_1, n_2, n_3 and n_4, respectively. See Figure 8 for an illustration.

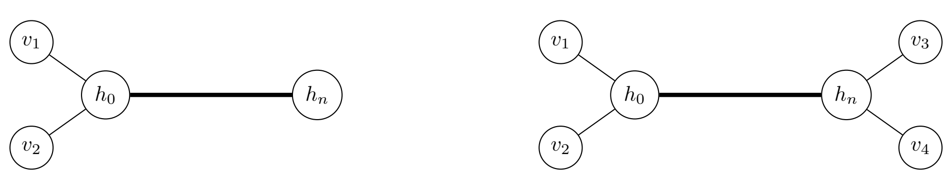


Figure 8: Notation for the vertices of S-handles (on the left) and H-handles (on the right).

A.1 Preliminary lemmas

We first present a series of lemmas giving properties on the color sets of some interior vertices of a path with a given $(2b+1,b)$-coloring. These lemmas will be used to reduce the number of cases to consider when proving the reducibility of some S- and H-handles.

Remember that the vertices of a path P_{n+1} are denoted by $v_0, v_1, \dots, v_n$. For any two good sets C_0, C_n, let $C(1) = C_0 \cap C_n$, $C(2) = C_0 - C_n$, $C(3) = C_n - C_0$, $C(4) = \{1, \dots, a\} - C_0 - C_n$.

Lemma 2. *For any $(2b+1,b)$-coloring φ of P_5, there exist four distinct good sets X_1, X_2, X_3 and X_4, such that*

(i) $|X_1 \cap X_2| = |X_1 \cap X_3| = b-1$, $|X_1 \cap X_4| = b-2$, *and*

(ii) *for each $i \in \{1,2,3,4\}$, there exists a $(2b+1,b)$-coloring φ' of P_5 such that $\varphi'(v_0) = \varphi(v_0)$, $\varphi'(v_4) = \varphi(v_4)$ and $\varphi'(v_2) = X_i$.*

Proof: Let φ be a $(2b+1,b)$-coloring of P_5 and let $C_0 = \varphi(v_0)$ and $C_4 = \varphi(v_4)$. By Theorem 1, $|C(1)| = |C_0 \cap C_4| \geq b-2$. Thus we have three cases to consider:

Case 1: $|C(1)| = b$. We choose any $Y \subset C(1)$, any $Y_1 \subset C_0 - Y$, $Y_2 = C_0 - Y - Y_1$, any $Z, Z' \subset \{1, \dots, a\} - C_0$, such that $|Y| = b-2$, and $|Z| = |Z'| = |Y_1| = |Y_2| = 1$. Then we set $X_1 = Y \cup Y_1 \cup Z$, $X_2 = C_0$, $X_3 = Y \cup Y_1 \cup Z'$, and $X_4 = Y \cup Y_2 \cup Z'$.

Case 2: $|C(1)| = b-1$. We choose any $Y \subset C(1)$, and any $Z, Z' \subset \{1, \dots, a\} - C_0 - C_4$, such that $|Y| = b-2$, and $|Z| = |Z'| = 1$, and we set $X_1 = C(1) \cup Z$, $X_2 = C_0$, $X_3 = C(1) \cup Z'$ and $X_4 = Y \cup (C_0 - C(1)) \cup (C_4 - C(1))$.

Case 3: $|C(1)| = b-2$. Then we note $C(2) = Y_2 \cup Y_2'$ and $C(3) = Y_3 \cup Y_3'$, with $|Y_i|Y_i'| = | = 1$, for $i = 1,2$, and we set $X_1 = D_1 \cup Y_2 \cup Y_3$, $X_2 = C(1) \cup Y_2 \cup Y_3'$, $X_3 = C(1) \cup Y_2' \cup Y_3$, and $X_4 = C(1) \cup Y_2' \cup Y_3'$.

For each case and each $i \in \{1,2,3,4\}$, the pairs (C_0, X_i) and (X_i, C_4) are both 2-compatible and thus Theorem 5 allows to conclude. □

Lemma 3. *For any $(2b+1,b)$-coloring φ of P_6, there exist two distinct sets X and Y such that $|X| = |Y| = 2$ and for any $x \in X$ and $y \in Y$ there exists a $(2b+1,b)$-coloring φ' of P_6 such that $\varphi'(v_0) = \varphi(v_0)$, $\varphi'(v_5) = \varphi(v_5)$ and $\{x, y\} \subset \varphi'(v_2)$.*

Proof: Let φ be a $(2b+1,b)$-coloring of P_6 and let $C_0 = \varphi(v_0)$ and $C_5 = \varphi(v_5)$. By Theorem 1, $|C(1)| \leq 2$ and thus $|C(2)| \geq b-2$.

Case 1: $|C(1)| = 0$. We choose any $I \subset C_0$, $X = C_0 - I$ and any $Y \subset \{1, \dots, a\} - C_0$, such that $|I| = b-2$, and $|Y| = 2$.

Case 2: $|C(1)| \geq 1$. We choose any $I \subset C(2)$, $X = C_0 - I$ and $Y \subset C(4)$ such that $|I| = b-2$ and $|Y| = 2$.

For each case and each $x \in X, y \in Y$, by construction, $C_0, I \cup \{x, y\}$ are 2-compatible and $I \cup \{x, y\}, C_5$ are 3-compatible, hence Theorem 5 allows to conclude. □

Lemma 4. *For any $(2b+1,b)$-coloring φ of P_{2n+2} with $n \leq b-1$, there exist two distinct sets X and Y, such that $|X| = b-n$ and $|Y| = n+1$, and for any $Y' \subset Y$ such that $|Y'| = n$, there exists a $(2b+1,b)$-coloring φ' of P_{2n+2} such that $\varphi'(v_0) = \varphi(v_0)$, $\varphi'(v_{2n+1}) = \varphi(v_{2n+1})$ and $\varphi'(v_1) = X \cup Y'$.*

Proof: Let $n \leq b-1$ and φ be a $(2b+1,b)$-coloring of P_{2n+2} and let $C_0 = \varphi(v_0)$ and $C_{2n+1} = \varphi(v_{2n+1})$. We have $|C(1)| \leq n$ and thus $|C(3)| \geq b-n$. We choose any $X \subset C(3)$ with $|X| = b-n$ and $Y = \{1,\ldots,a\} - X - C_0$. For any $Y' \subset Y$ such that $|Y'| = n$, we let $C_1 = X \cup Y'$. Then $|C_0 \cap C_1| = 0$ and $|C_{2n+1} \cap C_1| \geq b-n$, therefore Theorem 5 allows to conclude. □

Lemma 5. *For any $(2b+1,b)$-coloring φ of P_{4n+1} with $2n \leq b$, there exists a set $X \subset \{1,\ldots,a\}$ such that $|X| = b+2n$, and for any $X' \subset X$ with $|X'| = n$, there exists a $(2b+1,b)$-coloring φ' of P_{4n+1} such that $\varphi'(v_0) = \varphi(v_0)$, $\varphi'(v_{4n}) = \varphi(v_{4n})$ and $X' \subset \varphi'(v_{2n})$.*

Proof: Let $2n \leq b$ and φ be a $(2b+1,b)$-coloring of P_{4n+1}. Let $C_0 = \varphi(0)$ and $C_{4n} = \varphi(4n)$.

Case 1: $|C(1)| \geq b-n$. We choose any set $X \subset \{1,\ldots,a\}$ with $|X| = b+2n$, and for any $X' \subset X$ such that $|X'| = n$, we choose any $I \subset C(1)$ such that $|I| = b-n$, and we choose any $Y \subset C(4)$ such that $|Y| = |I \cap X'|$. We then set $C_{2n} = I \cup X' \cup Y$.

Case 2: $|C(1)| \leq b-n-1$. Let $i = b-n-|C(1)|$. We choose any set X with $C_0 \cup C_{4n} \subset X \subset \{1,\ldots,a\}$ and $|X| = b+2n$, and for any $X' \subset X$ such that $|X'| = n$, we choose any Y_2 with $X' \cap C(2) \subset Y_2 \subset C(2)$ and $|Y_2| = \max(i, |C(2) \cap X'|)$, and we choose any Y_3 with $X' \cap C(3) \subset Y_3 \subset C(3)$ and $|Y_3| = \max(i, |C(3) \cap X'|)$, and we choose any Z with $X' \cap C(4) \subset Z \subset X - C(1) - Y_2 - Y_3$ and $|Z| = b - |C(1)| - |Y_2| - |Y_3|$. Such a choice is always possible since $|X' \cap C(4)| \leq |X - C_0 - C_{4n}| = b+2n-(b+n+i) = n-i$. We then set $C_{2n} = C(1) \cup Y_2 \cup Y_3 \cup Z$.

In both cases, the pairs (C_0, C_{2n}) and (C_{2n}, C_{4n}) are both $2n$-compatible and thus Theorem 5 allows to conclude. □

Lemma 6. *For any $(2b+1,b)$-coloring φ of P_{4n+3} with $3 \leq 2n+1 \leq b$, there exists a set $X \subset \{1,\ldots,a\}$ with $|X| = b+2n+1$, and for any $X' \subset X$ such that $|X'| = n+1$, there exists a $(2b+1,b)$-coloring φ' of P_{4n+3} such that $\varphi'(v_0) = \varphi(v_0)$, $\varphi'(v_{4n+2}) = \varphi(v_{4n+2})$ and $X' \cap \varphi'(v_{2n+1}) = \emptyset$.*

Proof: Let φ be a $(2b+1,b)$-coloring of P_{4n+3} and $3 \leq 2n+1 \leq b$. Let $C_0 = \varphi(v_0)$ and $C_{4n+2} = \varphi(v_{4n+2})$.

Case 1: $|C(1)| \geq b-n$. Let $i = b-|C(1)|$. We choose any set $X \subset \{1,\ldots,a\}$ such that $|X| = b+2n+1$. For any $X' \subset X$ such that $|X'| = n+1$, if $|C(4) - X'| \geq b$ then we choose any good set $I \subset C(4) - X'$ and we set $C_{2n+1} = I$; otherwise we choose $I = C(4) - X'$, so $|I| = b-i+1-|X' \cap C(4)|$. If $|I| \geq b-n$, then we choose any $Y \subset C_0 - X'$ such that $|Y| = b-|I|$ and we set $C_{2n+1} = I \cup Y$. Else ($|I| < b-n$), we let $Y_2 = C(2) - X'$. If $|C(3) - X'| \geq b-|I|-|Y_2|$, then we choose $Y_3 \subset C(3) - X'$ such that $|Y_3| = b-|I|-|Y_2|$ and we set $C_{2n+1} = I \cup Y_2 \cup Y_3$. Otherwise, $|C(3) - X'| < b-|I|-|Y_2|$, and then we choose $Y_3 = C(3) - X'$. By construction we have $|I| + |Y_2| + |Y_3| = |C(4)| + |C(2)| + |C(3)| - (n+1) + |X' \cap C(1)| \geq b+i-n$, then we choose any $Z \subset C(1) - X'$ such that $|Z| = b-|I|-|Y_2|-|Y_3|$ and we set $C_{2n+1} = I \cup Y_2 \cup Y_3 \cup Z$.

Case 2 : $|C(1)| \leq b-n-1$. Let $i = b-n-|C(1)|$. We choose any set $C_0 \cup C_{4n} \subset X \subset \{1,\ldots,a\}$ such that $|X| = b+2n+1$, and for any $X' \subset X$ such that $|X'| = n+1$, we choose $I = C(4) - X'$. If $|C(2) - X'| \geq n$, then we choose $Y_2 \subset C(2) - X'$ such that $|Y_2| = n$. If $|C(3) - X'| \geq b-|I|-|Y_2|$ then we choose $Y_3 \subset C(3) - X'$ such that $|Y_3| = b-|I|-|Y_2|$ and we set $C_{2n+1} = I \cup Y_2 \cup Y_3$. Otherwise, $|C(3) - X'| < b-|I|-|Y_2|$, so $b-n = |C(3)| + |C(4)| - |X'| < b-|Y_2| = b-n$, a contradiction. We do the same with $|C(3) - X'| \geq n$. Else $|C(2) - X'| < n$

and $|C(3) - X'| < n$, then we choose $Y_2 = C(2) - X'$. If $|C(3) - X'| \geq b - |I| - |Y_2|$, then we choose $Y_3 \subset C(3) - X'$ such that $|Y_3| = b - |I| - |Y_2|$ and we set $C_{2n+1} = I \cup Y_2 \cup Y_3$. Otherwise, $|C(3) - X'| < b - |I| - |Y_2|$, so $b + 1 \leq b + |C(3)| - n = |C(3)| + |C(4)| + |C(2)| - |X'| < b$, a contradiction.

Therefore by Theorem 5, the lemma is proved. □

Now we present two lemmas used to shorten some proofs for H-reducibility.

Lemma 7. *Let $n \geq 1$ and G be a graph. If $H(n_1, n_2, n, n_3, n_4)$ is (a,b)-reducible in G and $H(n_1, n_2, n-1, n_3+1, n_4+1)$ is (a,b)-reducible in G when the color-sets of v_3, v_4 are (n_3+n_4+2)-exactly-compatible, then $H(n_1, n_2, n-1, n_3+1, n_4+1)$ is (a,b)-reducible in G.*

Proof: Let $n \geq 1$. Let H be a handle $H(n_1, n_2, n-1, n_3+1, n_4+1)$ in G and φ be an (a,b)-coloring of $G - \mathsf{int}(H)$. Assume $H(n_1, n_2, n, n_3, n_4)$ is (a,b)-reducible in G, and $H(n_1, n_2, n-1, n_3+1, n_4+1)$ is (a,b)-reducible in G for $\varphi(v_3), \varphi(v_4)$ being $(n_3 + n_4 + 2)$-exactly-compatible. If $\varphi(v_3), \varphi(v_4)$ are (n_3+n_4+2)-exactly-compatible then it is true by hypothesis. Otherwise $\varphi(v_3), \varphi(v_4)$ are (n_3+n_4)-compatible. Then, by hypothesis, there exists an (a,b)-coloring φ' of $H' = H(n_1, n_2, n, n_3, n_4)$ with $\varphi'(v'_i) = \varphi(v_i)$, where v'_i is the end-vertex of the path of length n_i of H', $i = 1, \ldots, 4$. Now we complete the coloring of G: to each vertex of the paths of H of length n_1, n_2, and $n-1$ starting at h_0 we associate the color of the corresponding vertex in H'; for the two neighbors of h_{n-1} along the paths towards v_3 and v_4 we assign the color $\varphi'(h'_n)$; for the remaining subpaths of lengths n_3 and n_4 of H, we use the colors of the corresponding paths of the same lengths in H'. We obtain an (a,b)-coloring of H that completes the (a,b)-coloring φ of G, hence H is (a,b)-reducible in G. □

By symmetry and adding the new condition $n_1 < n_4$ in order to fulfill our convention, we have this lemma:

Lemma 8. *Let $n \geq 1$ and $n_1 < n_4$. If $H(n_1, n_2, n, n_3, n_4)$ is (a,b)-reducible in G and $H(n_1+1, n_2+1, n-1, n_3, n_4)$ is (a,b)-reducible in G when the color-sets of v_1, v_2 are (n_1+n_2+2)-exactly-compatible, then $H(n_1+1, n_2+1, n-1, n_3, n_4)$ is (a,b)-reducible in G.*

A.2 proof of Theorem 8

Proof: We prove that for any b, e with $b \geq e$, $S(\mathsf{Even}(\frac{2b}{e}) - 1, 2, 1)$ is a minimal $(2b+e, b)$-reducible S-handle.

Let $n = \mathsf{Even}(\frac{2b}{e}) - 1$. For the minimality, we present a counter-example showing that $S(\mathsf{Even}(\frac{2b}{e}) - 1, 1, 1)$ is not reducible: if $C(v_1) = \{1, \ldots, b\}$, $C(v_2) = \{e+1, \ldots, b+e\}$, $C(h_n) = \{b+e+1, \ldots, 2b+e\}$, then the color set of h_0 must be $\{b+e+1, \ldots, 2b+e\}$ but then the color sets of h_0 and h_n are not n-compatible, hence a contradiction. Also, the handle $S(\mathsf{Even}(\frac{2b}{e}) - 2, 2, 1)$ is not reducible since in this case there is a path between v_2 and h_n of length $\frac{2b}{e} - 2 + 1 < \frac{2b}{e}$, hence the coloring cannot be extended to the interior vertices of the handle if these two vertices have the same color set.

In order to prove reducibility, let G be a graph containing a handle $H = S(n, 2, 1)$ and φ be a $(2b+e, b)$-coloring of $G - \mathsf{int}(H)$. Let $X = \varphi(h_n)$. We have $|\varphi(v_1) \cap \varphi(v_2)| \leq e$, thus there exists $I \subset \varphi(v_2) - \varphi(v_1)$ such that $|I| = b - e$. If $|X \cap (\{1, \ldots, a\} - I - \varphi(v_1))| \leq e$, then there exists $Y \subset \{1, \ldots, a\} - I - \varphi(v_1) - X$ such that $|Y| = e$. Otherwise, $|X \cap (\{1, \ldots, a\} - I - \varphi(v_1))| > e$, and there exists $Y' \subset (\{1, \ldots, a\} - I - \varphi(v_1)) \cap X$ such that $|Y'| = e - |\{1, \ldots, a\} - I - \varphi(v_1) - X|$.

We then choose $Y = (\{1, \dots, a\} - I - \varphi(v_1) - X) \cup Y'$. By construction, $|X \cap (I \cup Y)| \leq b - e \leq e\frac{\mathsf{Even}(\frac{2b}{e})-2}{2}$, hence by Theorem 5, there exists a $(2b+e, b)$-coloring φ' of H such that $\varphi'(v_1) = \varphi(v_1)$, $\varphi'(v_2) = \varphi(v_2)$, $\varphi'(h_n) = X$ and $\varphi'(h_0) = I \cup Y$.

We now prove that for any integer k with $2 \leq k \leq b$, $S(2b-k, k, k)$ is a minimal $(2b+1, b)$-reducible S-handle in G. Let $n = 2b - k$.

The minimality follows from the fact that the handles $S(2b-k, k, k-1)$ and $S(2b-k-1, k, k)$ are both not reducible since in both of them there is a path of length $2b-1$ between v_2 and h_n, hence the color sets of these two vertices must share at least $b-1$ common colors.

Let G be a graph containing a handle $H = S(2b-k, k, k)$, with k an integer such that $2 \leq k \leq b$, and let φ be a $(2b+1, b)$-coloring of $G - \mathsf{int}(H)$. Let $\varphi(v_1) = C_0$ and $\varphi(v_2) = C_{2k}$. Then C_0, C_{2k} are $2k$-compatible. We are going to construct a good set C_k such that C_k, C_0 and C_k, C_{2k} are both k-compatible and $C_k, \varphi(h_n)$ are $(2b-k)$-compatible. Then, Theorem 5 will assert that there exists a $(2b+1, b)$-coloring φ' of H, such that $\varphi'(h_n) = \varphi(h_n)$, $\varphi'(v_1) = \varphi(v_1) = C_0$ and $\varphi'(v_2) = \varphi(v_2) = C_{2k}$.

Case 1 : $k = 2m$ is even. By Lemma 5 there exists X such that $|X| = b+2m$, then $|X \cap \varphi(v_n)| \geq b + 2m + b - a = 2m - 1 \geq m$, so there exists $X' \subset X \cap \varphi(h_n)$ such that $|X'| = m$ and $X' \subset C_k$. Therefore $|C_k \cap \varphi(h_n)| \geq m = b - \frac{2b-k}{2}$.

Case 2 : $k = 2m+1$ is odd. By Lemma 6 there exists X such that $|X| = b + 2m + 1$, then $|X \cap \varphi(h_n)| \geq b + 2m + 1 + b - a = 2m \geq m + 1$, so there exists $X' \subset X \cap \varphi(h_n)$ such that $|X'| = m + 1$ and $X' \cap C_k = \emptyset$. Therefore $|C_k \cap \varphi(h_n)| \leq b - (m+1) = \frac{(2b-k)-1}{2}$.

□

A.3 Proof of Theorem 9

Proof: We prove that the H-handle $H(1, 2, \mathsf{Even}(\frac{2b}{e}) - 2, 2, 1)$ is a minimal $(2b+e, b)$-reducible H-handle.

Let $n = \mathsf{Even}(\frac{2b}{e}) - 2$. For the minimality, it can be observed that $H(1, 1, n, 2, 1)$ and $H(1, 2, n-1, 2, 1)$ are both not reducible since the first one is in fact an S-handle $S(n, 2, 1)$ and is not reducible by Theorem 8 and the second one has a path of length $1 + n - 1 + 1 = n + 1 < \mathsf{Even}(\frac{2b}{e})$ between two of its extremities and thus is not reducible. □

Now let G be a graph containing a handle $H = H(1, 2, n, 2, 1)$ and let φ be a $(2b+e, b)$-coloring of $G - \mathsf{int}(H)$. Let v_2' be the common neighbor of h_0 and v_2 and v_3' be the common neighbor of h_n and v_3. By hypothesis, $|\varphi(v_1) \cap \varphi(v_2)| \leq e$ and $|\varphi(v_3) \cap \varphi(v_4)| \leq e$, hence $|\varphi(v_2) - \varphi(v_1)| \geq b - e$ and $|\varphi(v_3) - \varphi(v_4)| \geq b - e$. Let $X \subset \varphi(v_2) - \varphi(v_1)$ and $X' \subset \varphi(v_3) - \varphi(v_4)$ such that $|X| = |X'| = b - e$. We are going to show that sets $Y \subset \{1, \dots, a\} - \varphi(v_1)$ and $Y' \subset \{1, \dots, a\} - \varphi(v_4)$ can be chosen in such a way that $C_0 = X \cup Y$ and $C_n = X' \cup Y'$ are n-compatible, i.e., $|C_0 \cap C_n| \geq e$. Then, Theorem 5 will assert that there exists a $(2b+e, b)$-coloring φ' of H, such that $\varphi'(h_0) = C_0$, $\varphi'(h_n) = C_n$.

Let $L(h_n) = \{1, \dots, a\} - \varphi(v_4)$ and $L(h_0) = \{1, \dots, a\} - \varphi(v_1)$. If $|X \cap X'| \geq e$, then $|C_0 \cap C_n| \geq e$. Hence assume $|X \cap X'| = z < e$. If $|(L(h_n) - X') \cap X| \geq e - z$, then we choose Y' such that $|Y' \cap X| \geq e - z$, therefore $|C_0 \cap C_n| \geq e$. Otherwise $|(L(h_n) - X') \cap X| = t < e - z$. If $|(L(v_0) - X) \cap X'| \geq e - z - t$, then we choose Y such that $|Y \cap X'| \geq e - z - t$, therefore $|C_0 \cap C_n| \geq e$. Otherwise $|(L(v_0) - X) \cap X'| = t' < e - z - t$. We note $z' = |(L(v_0) - X) \cap (L(h_n) - X')|$. Since $a \geq |L(v_0) \cup L(h_n)| = 2b + 2e - z - z' - t - t'$, we have $z + z' + t + t' \geq e$. We choose $Y_0 \subset (L(v_0) - X) \cap (L(h_n) - X')$ such that $|Y_0| = e - z - t - t'$. We choose Y, Y' such that $Y_0 \cup \big((L(h_n) - X') \cap X\big) \subset Y'$ and $Y_0 \cup \big((L(v_0) - X) \cap X'\big) \subset Y$, therefore $|C_0 \cap C_n| \geq e$.

H-handles $H(2,2,2b-3,2,2)$, $H(1,2,2b-3,3,2)$ and $H(1,4,2b-3,2,2)$.

For proving minimality, we have the necessary condition $n_1 + n_4 \geq 2b - (2b-3) = 3$. Moreover, we have $n_4 < 3$. If $n_1 = 1$ then $n_3 \geq n_4 = 2$. Therefore $H(1,2,2b-3,2,2)$ could be the minimal $(2b+1,b)$-reducible handle. However there exists a counter example even for $H(1,3,2b-3,2,2)$: $C(v_1) = \{1,\ldots,b\}$, $C(v_2) = \{1,\ldots,b-2\} \cup \{2b, 2b+1\}$, $C(v_4) = \{b+1,\ldots,2b-1\} \cup \{2b+1\}$, $C(v_3) = \{b+1,\ldots,2b\}$. Therefore, there remain only three configurations to be tested: $H(1,2,2b-3,3,2)$, $H(1,4,2b-3,2,2)$, and $H(2,2,2b-3,2,2)$. We now consider each of these three H-handles in turn and prove that it is $(2b+1,b)$-reducible. Let $n = 2b-3$.

Handles $H = H(2,2,2b-3,2,2)$. Let φ be a $(2b+1,b)$-coloring of $G - \mathsf{int}(H)$. We are going to show that there exist two $(2b-3)$-compatible sets C and C' that can be given to vertices h_0 and h_n. By Lemma 2 there exist four sets $X'_i, i \in \{1,2,3,4\}$ for the vertex h_n with $|X'_1 \cap X'_4| = b-2$. Let $I = \varphi(v_1) \cap \varphi(v_2)$ and $I' = \varphi(v_3) \cap \varphi(v_4)$. By hypothesis, both $\varphi(v_1)$ and $\varphi(v_2)$ and $\varphi(v_3)$ and $\varphi(v_4)$ are 4-compatible. Hence the size of both I and I' is between $b-2$ and b. We then consider three cases depending on the values of $|I|$ and $|I'|$.

Case 1. $|I| = b$ or $|I'| = b$. Assume without loss of generality that $|I| = b$. Then either there exists $i \in \{1,4\}$ such that $|I \cap X'_i| \leq b-2$ and thus we can set $C = I$ and $C' = X'_i$ or we have $|I \cap X'_i| \geq b-1$ for all $i \in \{1,2,3,4\}$. In this case, we can set $C = X'_1$ and $C' = X'_4$.

Case 2. $|I| = b-1$ or $|I'| = b-1$. Assume without loss of generality that $|I| = b-1$. Since $|X_1 \cap X_4| = b-2$, then there exists $i \in \{1,4\}$ such that $|I \cap X'_i| \leq b-2$ and thus we can set $C = I \cup \{x\}$ and $C' = X'_i$, with $x \in \{1,\ldots,a\} - I - X'_i$.

Case 3. $|I| = |I'| = b-2$. If there exist $x \in \varphi(v_1) - I$, $y \in \varphi(v_2) - I$ and $i \in \{1,4\}$ such that $|(I \cup \{x,y\}) \cap X'_i| \leq b-2$ then we can set $C = I \cup \{x,y\}$ and $C' = X'_i$. Otherwise $|(I \cup \{x,y\}) \cap X'_i| = b-1$, then $X'_1 = \varphi(v_1)$ and $X'_4 = \varphi(v_2)$ then $\varphi(v_1) = I \cup \{x,x'\}$, $\varphi(v_2) = I \cup \{y,y'\}$, $\varphi(v_3) = I \cup \{x,y\}$ and $\varphi(v_4) = I \cup \{x',y'\}$. In this case we can choose $C = I \cup \{x,y'\}$ and $C' = I \cup \{x',y\}$.

Handles $H = H(1,2,2b-3,3,2)$. Let φ be a $(2b+1,b)$-coloring of $G - \mathsf{int}(H)$. If $|\varphi(h_0) \cap \varphi(h_n)| \leq b-2$, then $\varphi(h_0)$ and $\varphi(h_n)$ are $(2b-3)$-compatible, therefore by Theorem 5 we conclude. Else $|\varphi(h_0) \cap \varphi(h_n)| > b-2$. By Lemma 3 there exists I, Y and Z such that $|I| = b-2$ and $|Y| = |Z| = 2$. Let $Y = Y_1 \cup Y_2$ and $Z = Z_1 \cup Z_2$, with $|Y_i| = |Z_i| = 1$, $i = 1,2$. If there exists $i,j \in \{1,2\}$ such that $|\varphi(h_0) \cap (I \cup Y_i \cup Z_j)| \leq b-2$, then we conclude. Otherwise for any $i,j \in \{1,2\}$, $|\varphi(h_0) \cap (I \cup Y_i \cup Z_j)| > b-2$, then without loss of generality, $\varphi(h_0) = I \cup Y$. But Lemma 4 allows to construct another coloring φ' with $|\varphi'(h_0) \cap \varphi(h_0)| = b-1$ and thus $I \cup Z \neq \varphi'(h_0) \neq I \cup Y$. Hence there exists $i,j \in \{1,2\}$ such that $|\varphi'(h_0) \cap (I \cup Y_i \cup Z_j)| \leq b-2$, allowing to conclude.

Handles $H = H(1,4,2b-3,2,2)$. Let φ be a $(2b+1,b)$-coloring of $G - \mathsf{int}(H)$. Then $\varphi(v_1)$ and $\varphi(v_2)$ are 5-compatible and $\varphi(v_3)$ and $\varphi(v_4)$ are 4-compatible. If $|\varphi(h_0) \cap \varphi(h_n)| \leq b-2$, then $\varphi(h_0)$ and $\varphi(h_n)$ are $(2b-3)$-compatible, therefore by Theorem 5 we conclude. Else $|\varphi(h_0) \cap \varphi(h_n)| > b-2$. By Lemma 2 we have 4 good sets X'_1, X'_2, X'_3 and X'_4 for the vertex h_n, and by Lemma 4, there exists a set X such that $|X| = b-2$ and a set $Y = \{y_1, y_2, y_3\}$ such that the three good sets $C_1 = X \cup \{y_1, y_2\}$, $C_2 = X \cup \{y_1, y_3\}$ and $C_3 = X \cup \{y_2, y_3\}$ can be given to vertex h_0. If there exists $i,j \in \{1,2,3,4\}$ with $i \neq 4$, such that $|X'_j \cap C_i| \leq b-2$, then we conclude. Else $\forall i,j \in \{1,2,3,4\}$ with $i \neq 4$, we have $b \geq |X'_j \cap C_i| > b-2$, and since $|X'_1 \cap X'_4| = b-2$, then for all $j \in \{1,4\}$ and for all $i \in \{1,2,3\}$ we have the formula $|X'_j \cap C_i| = b-1$, so $X'_1 \cap X'_4 \subset C_i \subset X'_1 \cup X'_4$. Without loss of generality, $y_1 \in X'_1$, thus by the formula with $i=1, j=1$ we have $y_2 \notin X'_1$. Therefore by the formula with $i=3, j=1$ we have $y_3 \in X'_1$, a contradiction with the formula with $i=2, j=1$.

H-handles $H(1,2,2b-4,4,3)$, and $H(1,4,2b-4,3,3)$. We finally prove that $H(1,2,2b-4,4,3)$, and $H(1,4,2b-4,3,3)$ are the minimal $(2b+1,b)$-reducible H-handles $H(n_1,n_2,n,n_3,n_4)$ with $n_1 = 1$ and $n = 2b-4$ in G.

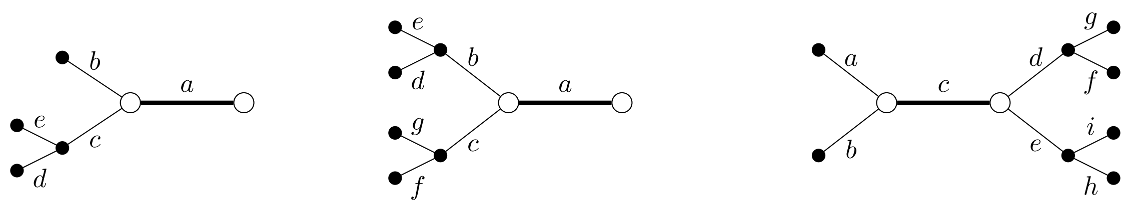
For proving minimality, we have the necessary condition $n_1 + n_4 \geq 2b - (2b-4) = 4$. Hence, as $n_1 = 1$, then $n_3 \geq n_4 = 3$. If $n_2 \leq 3$ then we can construct a simple counter example for $H(1,3,2b-4,3,3)$: $C(v_1) = \{1,\dots,b\}$, $C(v_2) = \{3,\dots,b+2\}$, $C(v_4) = \{b+3,\dots,2b+1\} \cup \{b+1\}$, $C(v_3) = \{b+2,\dots,2b+1\}$. Therefore there remain two possibilities in total: $H(1,2,2b-4,4,3)$ and $H(1,4,2b-4,3,3)$. We consider each of these two H-handles in turn and prove that it is $(2b+1,b)$-reducible in G.

Handles $H = H(1,2,2b-4,4,3)$. Let φ be a $(2b+1,b)$-coloring of $G - \mathsf{int}(H)$. We note $C_0 = \varphi(v_1)$, $C_3 = \varphi(v_2)$, $C'_0 = \varphi(v_4)$ and $C'_7 = \varphi(v_3)$, then C_0, C_3 are 3-compatible, and C'_0, C'_7 are 7-compatible. By the above, $H(1,2,2b-3,3,2)$ is $(2b+1,b)$-reducible, and by Lemma 7, there only remains to prove that H is reducible in the case C'_0, C'_7 are 7-exactly-compatible. We are going to show that there exist two sets of b colors C_1 and C'_3 that can be given to vertices h_0 and h_n, respectively. We note $C'_0 \cap C'_7 = Y'_1 \cup Y'_2 \cup Y'_3$, $I' = C'_7 - C'_0$, $I'' = \{1,\dots,2b+1\} - C'_7 - C'_0 = Z'_1 \cup Z'_2 \cup Z'_3 \cup Z'_4$, $D'_1 = I' \cup Y'_1 \cup Z'_1 \cup Z'_2$, $D'_2 = I' \cup Y'_2 \cup Z'_3 \cup Z'_4$, $D'_3 = I' \cup Y'_3 \cup Z'_1 \cup Z'_4$ and $D'_4 = I' \cup Y'_2 \cup Z'_2 \cup Z'_3$. By Lemma 4 there exist two (distinct) good sets D_1 and D_2, for C_1. If there exists $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$ such that $|D'_j \cap D_i| \geq 2$, then we set $C_1 = D_i$ and $C'_3 = D'_j$. Else for any $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$ we have $|D'_j \cap D_i| \leq 1$. However $|(D'_1 \cup D'_2 \cup D'_3) \cap D_i| \geq b+4+b-a = 3$ and $|(D'_1 \cup D'_4 \cup D'_3) \cap D_i| \geq 3$. Thus $|D'_j \cap D_i| = 1$ and $|(D'_1 \cup D'_2 \cup D'_3) \cap D_i| = |(D'_1 \cup D'_4 \cup D'_3) \cap D_i| = 3$. Then for any $i = 1,2$, $C'_0 \cap C'_7 \subset D_i$ and $D_i \cap (I' \cup I'') = \emptyset$, and therefore $D_i = C'_0$, a contradiction with $D_1 \neq D_2$.

Handles $H = H(1,4,2b-4,3,3)$. Let φ be a $(2b+1,b)$-coloring of $G - \mathsf{int}(H)$, we note $C_0 = \varphi(v_1)$, $C_5 = \varphi(v_2)$, $C'_0 = \varphi(v_4)$ and $C'_6 = \varphi(v_n)$, then C_0, C_5 are 5-compatible and C'_0, C'_6 are 6-compatible. By the above, $H(1,4,2b-3,2,2)$ is $(2b+1,b)$-reducible and, by Lemma 7, there only remains to prove the reduction for the case C'_0, C'_6 are 6-exactly-compatible. We are going to show that there exist two sets of b colors C_1 and C'_3 that can be given to vertices h_0 and h_n, respectively. We note $C'_0 - C'_6 = Y'_1 \cup Y'_2 \cup Y'_3$, $C'_6 - C'_0 = Z'_1 \cup Z'_2 \cup Z'_3$, $I' = \{1,\dots,2b+1\} - C'_6 - C'_0$, $D'_1 = I' \cup Y'_1 \cup Z'_1$, $D'_2 = I' \cup Y'_2 \cup Z'_2$ and $D'_3 = I' \cup Y'_3 \cup Z'_3$. By Lemma 4 there exist I such that $|I| = b-2$, and $X = X_1 \cup X_2 \cup X_3$ such that $D_1 = I \cup X_1 \cup X_2$, $D_2 = I \cup X_1 \cup X_3$ and $D_3 = I \cup X_2 \cup X_3$. For any $i,j \in \{1,2,3\}$, if $|D'_j \cap D_i| \geq 2$, then we set $C_1 = D_i$ and $C'_3 = D'_j$. Else for any $i,j \in \{1,2,3\}$ we have $|D'_j \cap D_i| \leq 1$. However $|(D'_1 \cup D'_2 \cup D'_3) \cap D_i| \geq b+3+b-a = 2$. Thus $I \cap I' = \emptyset$ and if $X_1 \subset I'$ then $(I \cup X_2 \cup X_3) \cap (D'_1 \cup D'_2 \cup D'_3) = \emptyset$, a contradiction. Proceeding similarly for X_2, X_3, we are in the case that $X \cap I' = \emptyset$ and thus $|(I \cup X) \cap ((C'_0 - C'_6) \cup (C'_6 - C'_0))| \geq b+1+6+b-2-a = 4$. Hence, without loss of generality, $(C'_0 - C'_6) \cup Z'_1 \subset I \cup X$, a contradiction with $|D'_1 \cap D_i| \leq 1$.

B Other reducible $(9,4)$-configurations

We present in Figure 9 new $(9,4)$-reducible handles (their reducibility has been checked by computer).



$S'(a,b,c,d,e)$ $S''(a,b,c,d,e,f,g)$ $H''(a,b,c,d,e,f,g,h,i)$

a	b	c	d	e
7	1	1	3	6
7	1	1	4	5
6	3	1	3	3
6	3	1	2	4
6	2	1	3	4
6	2	1	2	5
5	4	2	2	2
5	4	1	3	3
5	4	1	2	4
5	3	2	3	2
5	3	1	4	4
4	4	3	2	2
4	4	3	1	4
4	4	2	2	3
4	4	1	3	4

a	b	c	d	e	f	g
7	1	1	3	4	3	4
7	1	1	3	3	4	4
7	1	1	3	3	3	5
6	2	1	2	2	3	3
6	2	1	3	3	2	3
6	1	1	2	4	3	4
6	1	1	3	3	4	4
6	1	1	2	5	2	5
5	3	2	2	2	2	2
5	2	2	2	3	2	3
5	2	2	3	3	2	2
5	2	2	2	4	2	2
5	2	1	2	3	3	4
5	2	1	3	3	3	3
5	1	1	3	3	4	4
5	1	1	3	4	3	4
4	4	2	1	2	2	2
4	4	1	2	2	3	3
4	4	1	1	4	3	3
4	3	3	1	2	2	2
4	3	2	1	2	3	3
4	3	1	2	2	3	4
4	2	2	2	3	2	3
3	3	3	2	3	2	3

a	b	c	d	e	f	g	h	i
1	2	6	1	1	1	1	5	4
1	2	6	1	1	2	3	4	4
2	2	5	1	1	2	3	4	4
1	2	5	2	2	1	1	2	1
1	2	5	4	1	2	2	2	2
1	2	5	3	1	2	3	2	3
1	2	5	2	1	1	3	4	4
1	2	4	4	2	1	1	2	2
1	2	4	4	2	2	2	2	1
1	2	4	2	2	2	3	2	3
2	2	4	4	1	2	2	3	2
2	4	4	3	1	2	2	3	2
2	4	4	2	1	2	2	3	3
2	2	4	2	2	2	2	3	2
3	3	2	3	3	1	2	2	2

Figure 9: $\mathcal{F}''_{9,4}$: a new set of (9, 4)-reducible configurations of type $S'(a,b,c,d,e)$ on the left, $S''(a,b,c,d,e,f,g)$ (middle), and $H''(a,b,c,d,e,f,g,h,i)$ (right).