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Structural Parameterizations of k-Planarity

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Abstract. The concept of k-planarity is extensively studied in the context of Beyond Planarity. A graph is k-planar if it admits a drawing in the plane in which each edge is crossed at most k times. The local crossing number of a graph G is the minimum integer k such that G is k-planar. The problem of determining whether an input graph is 1-planar is known to be NP-complete even for near-planar graphs [Cabello and Mohar, SIAM J. Comput. 2013], that is, the graphs obtained from planar graphs by adding a single edge. Moreover, the local crossing number is hard to approximate within a factor $2 - \varepsilon$ for any $\varepsilon > 0$ [Urschel and Wellens, IPL 2021]. To address this computational intractability, Bannister, Cabello, and Eppstein [JGAA 2018] investigated the parameterized complexity for the case of $k = 1$, particularly focusing on structural parameterizations on input graphs, such as treedepth, vertex cover number, and feedback edge number. In this paper, we extend their approach by considering the general case $k \geq 1$ and give (tight) parameterized upper and lower bound results. In particular, we strengthen the aforementioned lower bound results to subclasses of constant-treewidth graphs: we show that testing 1-planarity is NP-complete even for near-planar graphs with *feedback vertex set number at most 3* and *pathwidth at most 4*, and the local crossing number is hard to approximate within *any constant factor* for graphs with *feedback vertex set number at most 2*.

1 Introduction

Graph drawing is recognized as an important area in the research of graph theory and algorithms due to various real-world applications. In particular, efficient algorithms for drawing graphs on the plane without edge crossings have received considerable attention from various perspectives.

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However, to visualize real-world networks, it is necessary to consider drawing *non-planar* graphs in many cases. This research direction is positioned as Beyond Planarity [11,14,27], and a significant amount of effort has been dedicated to analyzing graph-theoretic properties of graphs with a good drawing and designing algorithms for drawing these graphs. In this context, the problem of drawing an input graph on the plane with the minimum number of crossings is one of the most well-studied problems. In other words, the problem asks for the *crossing number* of an input graph, which is known to be NP-hard [19]. Among various studies on this problem, the (recent) progress on the parameterized complexity of Crossing Number, where the goal is to determine whether an input graph G can be drawn in the plane with at most k crossings, is remarkable [10, 22, 29, 33]. In particular, this problem is *fixed-parameter tractable* (FPT) when parameterized by k, that is, it admits an algorithm with running time $f(k)n^{O(1)}$, where n is the number of vertices in the input graph and f is a computable function.

The *local crossing number* is one of the well-studied variations of the standard crossing number [1, 2, 5, 21, 26, 32, 38, 40], as witnessed by the fact that it is selected as the topic for the live challenge[1] held in conjunction with GD 2025. Let k be an integer. We say that a graph G is *k-planar* if it can be drawn in the plane so that each edge involves at most k crossings. The minimum integer k such that G admits a k-planar drawing is called the *local crossing number* of G, denoted by $\mathrm{lcr}(G)$. The problem of deciding if $\mathrm{lcr}(G) \le k$ for a given graph G is called k-Planarity Testing. Unlike Crossing Number, it is already NP-complete to decide whether $\mathrm{lcr}(G) \le 1$ [2, 5, 21, 32]. More strongly, this problem is NP-complete even when the input is restricted to planar graphs with an extra edge [5] or graphs with constant bandwidth [2]. Urschel and Wellens [40] later extended the NP-hardness by showing that it is NP-hard for any fixed $k \ge 1$ to differentiate graphs with local crossing number at most k from those with at least $2k$. This implies that, unless P = NP, there is no polynomial-time $(2-\varepsilon)$-approximation for local crossing number for any $\varepsilon > 0$.

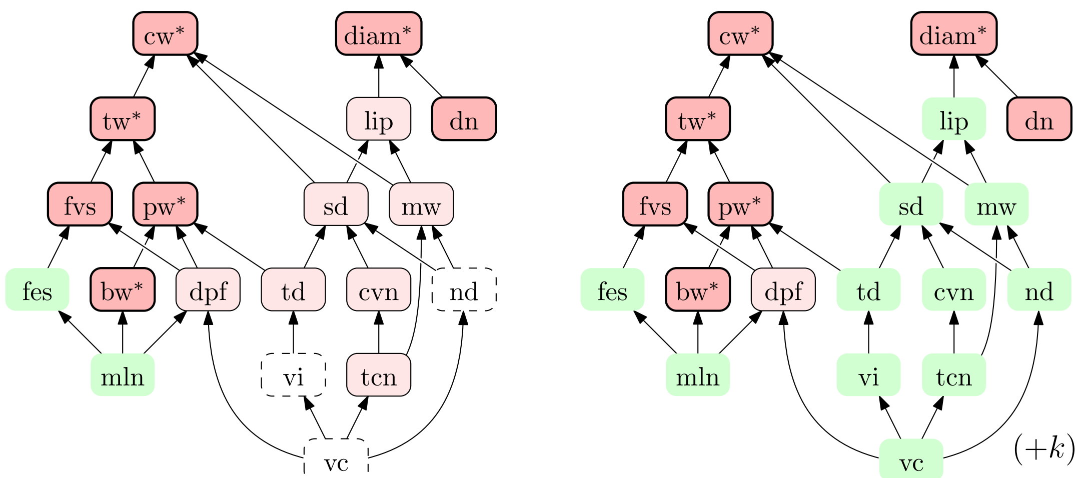
To overcome such an intractability, Bannister, Cabello, and Eppstein [2] pursued the parameterized complexity of the case $k = 1$, namely 1-Planarity Testing, by focusing on *structural parameterizations*. They showed that 1-Planarity Testing is FPT parameterized by the treedepth and by the feedback edge number of the input graph (see Section 2 for definitions). As mentioned above, 1-Planarity Testing is NP-complete even on the classes of graphs with bounded bandwidth. This intractability is indeed inherited by wider classes of graphs, such as those with bounded cliquewidth, treewidth, and pathwidth. In this direction, Münch, Pfister, and Rutter [34] recently considered k-Planarity Testing on special cases of pathwidth-bounded graphs and gave exact and approximation algorithms.

Our results. In this paper, we extend results by Bannister, Cabello, and Eppstein [2] by considering the general case $k \ge 1$ and show a fine parameterized complexity landscape with respect to graph width parameters (see Table 1 and Fig. 1 for a summary of our results). We extend algorithms by Bannister, Cabello, and Eppstein [2] by showing that k-Planarity Testing is FPT parameterized by feedback edge set number (Theorem 6) and by treedepth plus k (Theorem 7). More generally, we show that k-Planarity Testing is FPT on P_t-free graphs when parameterized by $t + k$ (Corollary 4). We also give polynomial kernelizations for k-Planarity Testing with respect to vertex cover number (Theorem 8) and neighborhood diversity (Corollary 6). On the negative side, we show that k-Planarity Testing is W[1]-hard when parameterized solely by treedepth (Theorem 2), which indicates that Theorem 7 is tight in a certain sense, and by twin cover number (Theorem 5). We also show that 1-Planarity Testing is W[1]-hard parameterized by the

[1] https://mozart.diei.unipg.it/gdcontest/2025/live/ (Accessed 6 Jan. 2026)

Table 1: The table summarizes our and known results. The columns "unbounded", "parameter", and "$k = 1$" indicate the results when k is taken as input, as parameter, and $k = 1$, respectively.

	k: unbounded	k: parameter	$k = 1$
feedback edge set number	FPT (Thm. 6)		FPT [2]
feedback vertex set number	NP-complete when fvs $= 2$ (Thm. 1)		
treedepth	W[1]-hard (Thm. 2)	FPT (Thm. 7) (non-uniform FPT [42])	FPT [2]
longest induced path	W[1]-hard (Thm. 5)	FPT (Cor. 4)	
twin cover number			
distance to path forest	W[1]-hard (Thm. 4)		
domination number	NP-complete when dn $= 2$ (Cor. 2)		
diameter	paraNP-complete [21]		



(a) (b)

Figure 1: A visualization of Table 1 of the cases where k is (a) unbounded and (b) a parameter. No borderline, dotted borderline, normal borderline, bold borderline mean that the complexity parameterized by the parameter is FPT, unknown, W[1]-hard, paraNP-complete, respectively. The complexities for parameters with $(*)$ are already known results and the others are our results. The references for known results can be found in Table 1. An arrow indicates that the upper parameter is bounded from above by a function of the lower parameter.

The parameter names are abbreviated as follows: cw is clique width, diam is diameter, tw is treewidth, lip is longest induced path, dn is domination number, fvs is feedback vertex set, pw is pathwidth, sd is shrub-depth, mw is modular-width, fes is feedback edge set, bw is bandwidth, dpf is distance to path forest, td is treedepth, cvn is cluster vertex deletion number, nd is neighborhood diversity, mln is max leaf number, vi is vertex integrity, tcn is twin cover number, and vc is vertex cover number. Refer to the preliminaries for the definitions of the parameters.

distance to path forest of the input graph (Theorem 4). The last result also gives another proof for the NP-hardness of 1-Planarity Testing on planar graphs with a single additional edge [5]. Our reduction further adds a restriction that the input graph has feedback vertex set number at most 3. Using a completely different reduction, we show that 1-Planarity Testing is NP-complete even if the input is restricted to graphs with feedback vertex set number 2 (Theorem 1). This reduction also shows that there is no constant factor approximation for k-Planarity Testing unless P = NP, which significantly strengthens the $(2-\varepsilon)$-inapproximability result of Urschel and Wellens [40].

Related work. Many parameterizations have been considered for problems related to k-Planarity Testing. The authors in [15] showed that extending the given partial (connected) drawing to a 1-planar (full) drawing can be done in XP time with respect to the vertex cover number of the subgraph to be drawn. Hamm and Hliněný [23] gave an FPT algorithm for computing the minimum number of crossings of a k-planar drawing that coincides with the given partial drawing for fixed k, where the parameter is the minimum number of new crossings, namely, crossings that are not in the given partial drawing. Münch and Rutter [35] considered the problem of computing the minimum crossing number of a drawing under a fixed family of (certain) forbidden patterns, including the family of k-planar drawings for any fixed k, and gave an FPT algorithm with respect to the minimum crossing number. This result is further extended to broader families [24] and to surfaces [9]. The authors in [4] considered 1-Planarity Testing where only specified crossing patterns are allowed in a drawing. They gave a dichotomy characterizing the set of crossing patterns such that the corresponding problem is FPT with respect to treewidth.

2 Preliminaries

Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex set $X \subseteq V(G)$ (resp. edge set $X \subseteq E(G)$), $G - X$ denotes the graph obtained from G by deleting all elements in X. The subgraph of G induced by $X \subseteq V(G)$ is denoted by $G[X]$. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v and $N_G[v]$ denotes the closed neighborhood of v, that is, $N_G(v) \cup \{v\}$. We define the *length* of a path $P = (v_1, \dots, v_\ell)$ as the number of edges in P, that is, $\ell - 1$.

A (*topological*) *drawing* Γ of G is a representation in the plane that maps the vertices of G to distinct points in the plane and each edge of G to a non-self-intersecting (Jordan) curve connecting the points corresponding to the endpoints. In the rest of this paper, we may simply refer to these points as vertices and these curves as edges when no confusion is possible. A *crossing* in Γ is an intersection of edges that is not a common endpoint. We assume that, in any drawing, an edge does not pass through any vertex other than its endpoints, and no three (or more) edges cross at a common point. The *crossing number* of G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings over all possible drawings of G. For an integer k, a drawing Γ is said to be *k-planar* if each edge is crossed at most k times. A graph is said to be k-planar if it admits a k-planar drawing. The *local crossing number* of G, denoted by $\mathrm{lcr}(G)$, is the minimum integer k such that G is k-planar. Throughout this paper, we use the following property of the local crossing number.

Lemma 1. *Let G be a graph and k be a positive integer. Let G_k be the graph obtained from G by subdividing each edge $k-1$ times. Then,* $\mathrm{lcr}(G) \le k$ *if and only if* $\mathrm{lcr}(G_k) \le 1$.

Proof: Let Γ be a k-planar drawing of G. For each curve in Γ representing an edge, since there are at most k crossings on it, we can partition it into exactly k subcurves so that each subcurve has at most one crossing on its interior. We obtain a 1-planar drawing of G_k by putting vertices on such $k-1$ division points for each curve.

Conversely, let Γ_k be a 1-planar drawing of G_k. For each $e = \{u, v\} \in E(G)$, let P_e be the corresponding path P_e between u and v in G_k, which consists of k edges. We can assume that no two edges in P_e cross, as otherwise we can remove the crossing without losing 1-planarity (Fig. 9). Let Γ be a drawing of G obtained from Γ_k by, for each $e \in E(G)$, concatenating the curves representing the edges in P_e. Since the vertices of P_e have degree 2 except for the endpoints u and v, concatenating the curves does not yield a new crossing. Hence, every curve representing an edge in Γ has at most k crossings. □

In this paper, we present parameterized algorithms and hardness results with respect to several graph width parameters. In the following, we give definitions and references for the parameters that appear in this paper. For basic concepts in parameterized complexity, we refer the reader to [7].

Let G be a graph. A *vertex cover* of G is a set of vertices S such that $G-S$ is edge-less, and the *vertex cover number* of G is the minimum size of a vertex cover of G, which is denoted by $\mathrm{vc}(G)$. Let $\mathrm{cc}(G)$ denote the set of connected components of G. The *vertex integrity* of G, denoted by $\mathrm{vi}(G)$, is defined as $\min_{S \subseteq V(G)}\{|S| + \max_{H \in \mathrm{cc}(G-S)} |V(H)|\}$. The *feedback vertex set number* (resp. *feedback edge set number*) of G, denoted by $\mathrm{fvs}(G)$ (resp. $\mathrm{fes}(G)$), is the minimum cardinality of a vertex set (resp. edge set) X such that $G - X$ is acyclic. For a linear order $\sigma\colon V(G) \to \{1, \dots, |V(G)|\}$, the *bandwidth* of σ, denoted by $\mathrm{bw}(\sigma)$, is $\max_{\{u,v\} \in E(G)} |\sigma(u) - \sigma(v)|$. The *bandwidth* of G, denoted by $\mathrm{bw}(G)$, is the minimum bandwidth of a linear order of $V(G)$. An *elimination forest* of G is a rooted forest F (i.e., a set of rooted trees) such that $V(F) = V(G)$ and for every edge in G, one of the endpoints is an ancestor of the other in F. The *treedepth* of G, denoted by $\mathrm{td}(G)$, is the minimum integer k such that G has an elimination forest of height k. Two vertices u and v are called *true twins* if they are adjacent and $N_G[u] = N_G[v]$; they are called *false twins* if they are non-adjacent and $N_G(u) = N_G(v)$. The *neighborhood diversity* of G is at most k if the vertex set of G can be partitioned into k sets $V_1, \dots, V_k$ such that either all the pairs in V_i are true twins or they are false twins. Each set V_i is called a *twin class*. Note that each true twin class induces a clique and each false twin class induces an independent set in G. The neighborhood diversity of G is denoted by $\mathrm{nd}(G)$. A vertex set $S \subseteq V(G)$ is called a *twin cover* of G if each connected component of $G - S$ consists of true twins in G. The *twin cover number* of G is the minimum size of a twin cover of G. The *max leaf number* of G is the maximum number of leaves of a spanning tree of G. A vertex set $S \subseteq V(G)$ is called a *dominating set* of G if every $v \in V(G)$ is either in S or adjacent to a vertex in S. The *domination number* of G is the minimum size of a dominating set of G. The *modular-width* of G, denoted by $\mathrm{mw}(G)$, is recursively defined: if $|V(G)| = 1$ then $\mathrm{mw}(G)$ is 1, and otherwise, it is $\min_{\mathcal{V}=\{V_1,\dots,V_k\}} \max\{k, \mathrm{mw}(G[V_1]), \dots, \mathrm{mw}(G[V_k])\}$, where $\mathcal{V}$ is taken over all partitions of $V(G)$ with $k \geq 2$ such that, for every pair V_i and V_j, either G contains all the edges between them or no edge between them. For the definition of *shrub-depth*, as we do not require the precise definition in this paper, we refer the readers to the paper that introduced shrub-depth [18].

3 Hardness

In this section, we give several intractability results for k-Planarity Testing and 1-Planarity Testing. In particular, we show that k-Planarity Testing is hard to approximate within any constant factor in polynomial time, even on graphs with feedback vertex set number at most 2,

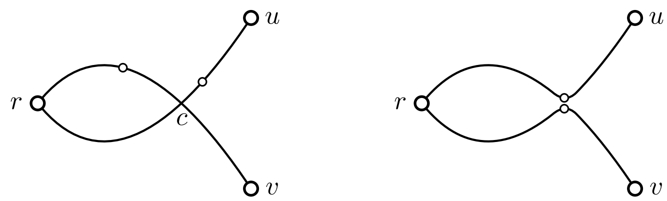


Figure 2: A crossing between spokes P_u and P_v can be removed by exchanging their subcurves.

and 1-Planarity Testing is NP-complete even on near-planar graphs with feedback vertex set number 3, where a graph is *near-planar* if it can be obtained from a planar graph by adding a single edge. These two results improve previous results [5, 40]. To this end, we design two completely different reductions, one is given from Two-Sided k-Planarity and the other is given from Unary Bin Packing, which also yield several consequences other than these two results.

3.1 Inapproximability for Graphs with Feedback Vertex Set Number 2

In this subsection, we show that it is NP-hard to approximate the local crossing number of a given graph within any constant factor, even if the graph has feedback vertex set number 2.

Before describing our reductions, we start with a technical lemma, showing that by adding some vertices and edges to a graph, one can impose a certain form on its k-planar drawings.

Let G be a graph and k be a positive integer. Let $S \subseteq V(G)$ be a nonempty vertex subset. We define a new graph G' by adding a vertex r and $k|E(G)|+1$ paths of length 2 between r and v for each $v \in S$ to G. We refer to these paths of length 2 as *spokes*.

Lemma 2. *Suppose that G' has a k-planar drawing Γ. Then, there is a k-planar drawing Γ' of G' such that each spoke has no crossings and the subdrawings of Γ and Γ' induced by G are identical.*

Proof: Let Γ be a k-planar drawing of G'. Since each edge in G has at most k crossings and there are $k|E(G)|+1$ spokes between r and each $u \in S$, there is at least one spoke connecting r and u that does not cross any edge in G. For $u \in S$, let P_u be such a spoke between r and u. Note that P_u may cross other spoke edges.

Suppose that P_u does not have any crossings with P_v in Γ for all pairs $u, v \in S$. Then, we can redraw spokes between r and u other than P_u along with the curve representing P_u for each $u \in S$. The drawing Γ' obtained in this way is still k-planar, and each spoke between r and u, not limited to P_u, has no crossings at all. Thus, Γ' is a k-planar drawing being claimed.

Suppose otherwise. Let $u, v \in S$ such that P_u and P_v cross in Γ and let c be the crossing point. We can remove the crossing by exchanging the subcurve of P_u between r and c with that of P_v between r and c (Fig. 2). Note that the drawing obtained in this way may not be k-planar. By applying this uncrossing operation between P_u and P_v for all $u, v \in S$, we can obtain a (not necessarily k-planar) drawing of G' such that each P_u has no crossing with P_v for all $v \in S$. Now, we can construct a drawing Γ' of G' by redrawing all spokes between r and u other than P_u along with the curve representing P_u. The drawing Γ' is indeed k-planar as all the spokes have no crossings and the subdrawings of Γ and Γ' induced by G are identical. This completes the proof of the lemma. □

We will use this lemma in the following form.

Corollary 1. *Let $S \subseteq V(G)$. Let G' be a graph obtained from G by adding a vertex s and $k|E(G)|+1$ spokes between s and each $u \in S$. Moreover, let $S' \subseteq V(G')$ and let G'' be a graph obtained from G' by adding a vertex s' and $k|E(G')|+1$ spokes between s' and each $u \in S'$. Suppose that G'' has a k-planar drawing. Then, there is a k-planar drawing of G'' such that all spokes have no crossings.*

Proof: By Lemma 2, G'' has a k-planar drawing Γ' in which all the spokes incident to s' have no crossings. Let Γ be the subdrawing of Γ' induced by G'. As shown in the proof of Lemma 2, for each $u \in S$, there is a spoke P_u between s and u that has no crossing with any edge of G in Γ. Since this spoke P_u does not cross any spoke incident to s', the uncrossing operation and redrawing spokes between s and u used in Lemma 2 never create a new crossing with spokes incident to s'. Thus, by applying Lemma 2 to G' and Γ, we have a k-planar drawing of G'' being claimed. □

We now turn to our reduction from Two-Sided k-Planarity, in which we are given a bipartite graph $G = (X \cup Y, E)$ with two independent sets X and Y, and an integer k. The goal is to determine whether G has a *2-layer k-planar drawing*, which is a special case of a k-planar drawing in which the vertices in X are drawn on a line, the vertices in Y are drawn on another line parallel to the first line, and each edge is drawn as a straight line.

The idea of our reduction is the same as that of Garey and Johnson [19] from Bipartite Crossing Number to Crossing Number. We remark that, however, the proof is more involved due to the difficulty of k-planarity.

Let $\langle G = (X, Y, E), k\rangle$ be an instance of Two-Sided k-Planarity, where $n_X = |X|$, $n_Y = |Y|$, and $m = |E|$. We construct a graph G' from G as follows (see Fig. 3):

1. we introduce two vertices u_X, u_Y;
2. for each $x \in X$, add $\ell_1 := km+1$ spokes between u_X and x;
3. for each $y \in Y \cup \{u_X\}$, add $\ell_2 := k(\ell_1 n_X + m)+1$ spokes between u_Y and y.

Then, $\langle G', k\rangle$ is the instance we construct for k-Planarity Testing. It is easy to verify that the above construction can be done in polynomial time.

Lemma 3. *The graph G admits a 2-layer k-planar drawing if and only if the graph G' admits a k-planar drawing.*

Proof: It is clear that we can obtain a k-planar drawing of G' from a 2-layer k-planar drawing of G by drawing u_X, u_Y, and spokes as in Fig. 3.

Suppose that the graph G' admits a k-planar drawing Γ'. In the following, from this drawing Γ' we extract a 2-layer drawing of G as in Fig. 4b. By Corollary 1, we can assume that all the spokes in G' have no crossings in Γ'. We then replace the set of spokes connecting two vertices with a single edge between them and contract the edge between u_X and u_Y by identifying them into a new vertex u in Γ'. We let G^* denote the graph obtained in this way. As all spokes have no crossings, the obtained drawing of G^*, denoted by Γ^*, is still k-planar. Let us note that G^* is isomorphic to the graph obtained from G by adding a universal vertex u (which is a vertex adjacent to all the vertices in G). We claim that we can draw a closed curve C passing through all the vertices in $V(G^*) \setminus \{u\}$ such that

1. no edge of G^* intersects C (i.e., C is a noose in Γ^*);
2. all the edges incident to u are contained in one of the two regions R with boundary C;

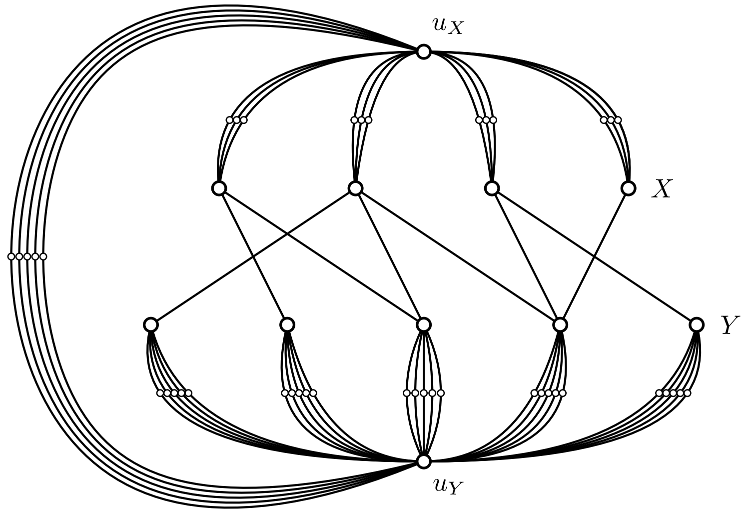


Figure 3: An illustration of the graph G' constructed from an instance $\langle G, k\rangle$ of Two-Sided k-Planarity. Note that the numbers of spokes are different from that of the actual construction.

3. all the edges in $E(G)$ are contained in the other region with boundary C;
4. all the vertices in X appear consecutively on C;
5. all the vertices in Y appear consecutively on C.

To this end, we draw a curve C maintaining the cyclic ordering of the edges incident to u (and hence the neighbors of u) in Γ^*. As these incident edges have no crossings, we can draw such a curve satisfying (1), (2), and (3). To see the properties (4) and (5), suppose for contradiction that there are $x, x' \in X$ and $y, y' \in Y$ such that x, y, x', y' appear in this order on C. Due to the property (2), C can be seen as a closed curve on Γ' such that the region corresponding to R contains all spokes of G'. Let P_x be a path in G' between x and x' via u_X consisting only of spokes. We define P_y similarly. In Γ', those paths are contained in R with their endpoints placed on C. As they are vertex disjoint, they must cross at least once in R, which contradicts the fact that spokes in G' have no crossings in Γ'.

Let us assume that the vertices of $X = \{x_1, \dots, x_{n_X}\}$ and $Y = \{y_1, \dots, y_{n_Y}\}$ appear in this order on C. We now convert the drawing Γ^* into a drawing $\hat{\Gamma}$ such that

- all the vertices in $V(G^*) \setminus \{u\}$ are placed on a line ℓ in the order $x_1, \dots, x_{n_X}, y_1, \dots, y_{n_Y}$;
- all the edges incident to u are drawn as straight segments in a half plane separated by ℓ;
- all the other edges are drawn in the other half plane in such a way that the curve corresponding to an edge forms a semicircular arc between the endpoints.

See Fig. 4a for an illustration. Observe that this drawing $\hat{\Gamma}$ is also k-planar as two edges of G^* cross in $\hat{\Gamma}$ only if they cross in Γ^* as well. Moreover, $\hat{\Gamma}$ is *simple*, meaning that no pair of edges crosses more than once and no two edges incident to a common vertex cross. This in turn can be converted into a k-planar drawing Γ as in Fig. 4b. The subdrawing of Γ induced by $G^* - \{u\}$ is

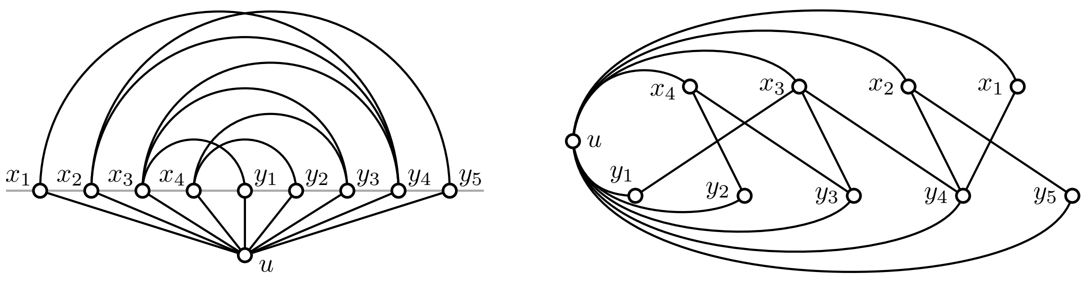


Figure 4: Two topologically equivalent drawings of G^*, where the subdrawing induced by G is (a) an arc diagram, and (b) a 2-layer drawing.

a 2-layer k-planar drawing such that two edges in $G^* - \{u\}$ cross if and only if they cross in $\hat{\Gamma}$. Hence, G $(= G^* - \{u\})$ admits a 2-layer k-planar drawing. □

This reduction immediately gives us the following consequence. Note that Theorem 1 is optimal in the sense that every graph with feedback vertex set number 1 is planar.

Theorem 1. 1-Planarity Testing *is NP-complete even if the given graph has feedback vertex set number* 2*.*

Proof: The NP-membership of 1-Planarity Testing is already known [32]. It is known that Two-Sided k-Planarity is NP-complete even on trees [31, Theorem 11]. If G is a tree, then the graph G' constructed for Lemma 3 has feedback vertex set number at most 2. Hence, by Lemma 3 k-Planarity Testing is NP-complete on such graphs. By Lemma 1 we can transform an instance of k-Planarity Testing into an equivalent instance of 1-Planarity Testing just by taking the k-subdivision of the input graph. Note that subdividing an edge does not increase the feedback vertex set number, and hence the claim follows. □

The above reduction leads to further consequences. To see this, we first sketch the reduction used to show the NP-hardness of Two-Sided k-Planarity on trees [31, Theorem 11]. They performed a polynomial-time reduction from Bandwidth on trees to Two-Sided k-Planarity. The problem Bandwidth asks, given a graph G and a positive integer b, if $\mathrm{bw}(G) \leq b$. Let $\langle T, b\rangle$ be an instance of Bandwidth on trees. Let ℓ be an arbitrary even number with $\ell \geq 2b^2$. Their reduction subdivides each edge $e \in E(T)$ once by introducing a new vertex w_e, and adds ℓ leaves adjacent to each original vertex in T. They then showed that $\mathrm{bw}(T) \leq b$ if and only if the obtained graph T' has a 2-layer k-planar drawing with $k = (\ell + 4)(b - 1)/2$.

Theorem 2. k-Planarity Testing *is W[1]-hard with respect to treedepth, even on graphs with feedback vertex set number 2.*

Proof: It is known that the problem Bandwidth is W[1]-hard with respect to treedepth, even on trees [20]. The above reduction we sketched [31, Theorem 11] also preserves the boundedness of treedepth. To see this, we convert an elimination forest F^* of T to that of T' as follows. Starting from F^*, we add the ℓ leaves attached to each vertex as leaf children of it. For each edge $e \in E(T)$, one of the end vertices, say v, is a descendant of the other in F^*. We then add w_e as a leaf child of v in F^*. It is easy to verify that the rooted forest obtained in this way is an elimination forest of

T' and its height increases by at most 1 compared to the original elimination forest of T, yielding that $\mathrm{td}(T') \leq \mathrm{td}(T) + 1$. Hence, Two-Sided k-Planarity is also W[1]-hard with respect to treedepth, even on trees. Moreover, the reduction used in Lemma 3 increases the treedepth by at most 3, which implies the claim. $\square$

Lemma 4. *Unless P = NP, for any constant $c \geq 1$, there is no polynomial-time c-approximation algorithm for* Two-Sided k-Planarity *on trees.*

Proof: For any constant $c \geq 1$, it is NP-hard to distinguish the following cases: the bandwidth of T is at most b or more than cb for a given tree T and an integer b [13]. We set $\ell = 2t - 4$ for some $t \geq b^2 + 2$ and construct a tree T' according to the above reduction from T. If the bandwidth of T is at most b, then we have $\mathrm{lcr}(T') \leq t(b-1)$. Otherwise, we have $\mathrm{lcr}(T') \geq tcb$. This implies that with a polynomial-time c-approximation algorithm for Two-Sided k-Planarity on trees we can distinguish the above two cases in polynomial time, which is impossible unless P = NP. $\square$

Combining Lemma 3 and Lemma 4, we obtain the following.

Theorem 3. *Unless P = NP, for any constant $c \geq 1$, there is no polynomial-time c-approximation algorithm for k-*Planarity Testing*, even if the given graph has feedback vertex set number 2.*

3.2 Distance to Path Forest

In this subsection, we show that 1-Planarity Testing is W[1]-hard parameterized by distance to path forest, i.e., the minimum number of vertices that have to be removed to make the graph into a disjoint union of paths. The underlying idea of the proof is similar to the one by Grigoriev and Bodlaender [21]. We perform a reduction from Unary Bin Packing. An instance I of Unary Bin Packing consists of a (multi)set of positive integers S and positive integers B and b such that $\sum_{x \in S} x \leq bB$ and $B = O(\mathrm{poly}(|S|))$. Then the goal is to determine if it is possible to partition S into b sets so that the sum of each set is at most B. Unary Bin Packing is NP-complete when b is given as input [19] and W[1]-hard when parameterized by b [28].

Theorem 4. 1-Planarity Testing *is W[1]-hard with respect to distance to path forest.*

Proof: Let $\langle S = \{x_1, \ldots, x_{|S|}\}, B, b\rangle$ be an instance of Unary Bin Packing. In the following, we assume that $b \geq 3$; the problem is trivial when $b = 1$, and when $b = 2$, it can be solved in polynomial time using the well-known pseudo-polynomial-time algorithm for the Subset Sum problem [6]. We construct an instance $\langle G = (V, E)\rangle$ of 1-Planarity Testing as follows. Let G be a graph consisting of vertices $u_1, u_2, v_1, \ldots, v_b$, and paths $P_1, \ldots, P_{|S|}$ with $|V(P_i)| = x_i$ for $1 \leq i \leq |S|$. Let $\mathcal{P} = \{P_1, \ldots, P_{|S|}\}$. We add a path of length B between v_i and v_{i+1} for each $1 \leq i \leq b$, where $v_{b+1} \coloneqq v_1$. These paths form a cycle C of bB vertices, passing through v_i for all i. Next, we add an edge between u_i and each vertex in P_j for $1 \leq i \leq 2$ and $1 \leq j \leq |S|$. Let m be the number of edges in the graph constructed so far (namely, $m = 4bB - |S|$). Finally, for $1 \leq i \leq b$, we add $\ell_1 \coloneqq m + 1$ spokes between u_1 and v_i and add $\ell_2 \coloneqq (\ell_1 b + m) + 1$ spokes between u_2 and v_i. See Fig. 5 for an illustration of the constructed graph G. Observe that, by construction, removing $b + 2$ vertices, say $u_1, u_2, v_1, \ldots, v_b$, yields a path forest. The construction can be done in polynomial time.

From a solution of Unary Bin Packing, we can obtain a 1-planar drawing of G easily, as in Fig. 6. To be more precise, let $\{S_1, \ldots, S_b\}$ be a solution for Unary Bin Packing. We first draw the cycle C as a circle in such a way that the spokes between u_1 and v_i's effectively partition the interior of the circle into b regions $R_1, \ldots, R_b$, where R_i is enclosed by a spoke between u_1 and v_i,

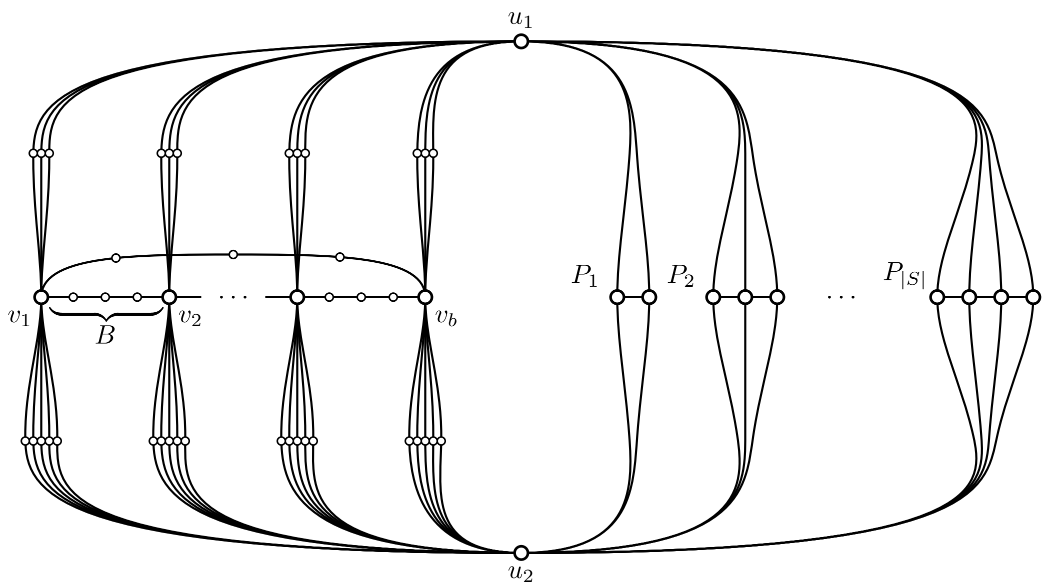


Figure 5: An illustration of the reduction for Theorem 4. Note that the number of depicted spokes between u_i and v_j is not accurate for aesthetic purposes.

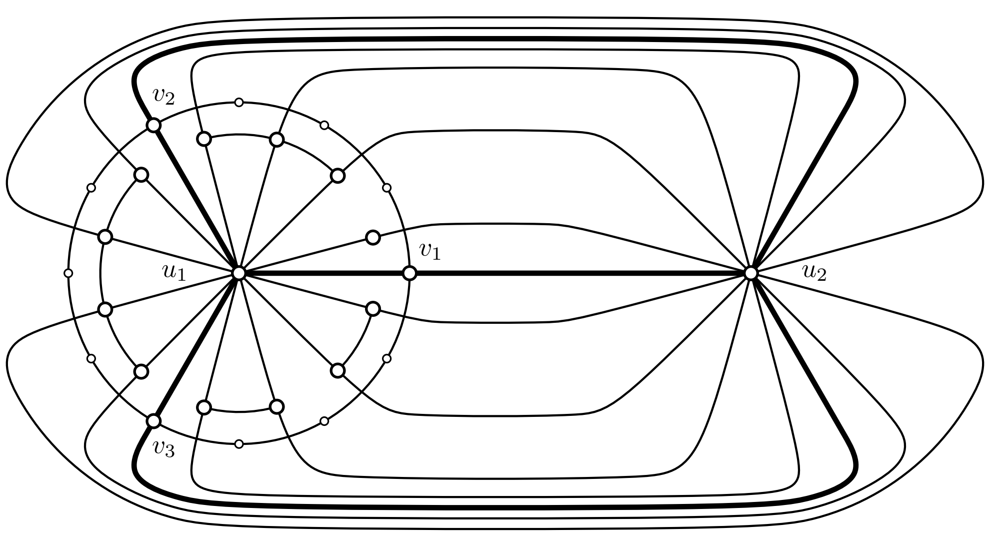


Figure 6: A 1-planar drawing of the graph constructed from an instance $\langle S = \{1, 2, 2, 3, 4\}, B = 4, b = 3\rangle$ of Unary Bin Packing. Each thick edge represents the spokes between its endpoints.

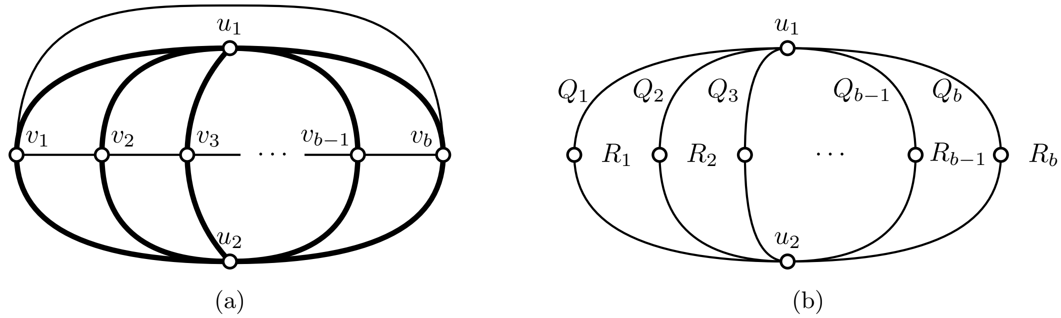


Figure 7: The subgraphs induced by (a) cycle C and spokes (bold lines) and (b) the paths $Q_1, \ldots, Q_b$, which define b regions $R_1, \ldots, R_b$.

a spoke between u_1 and v_{i+1}, and a path between v_i and v_{i+1} on C. For each region R_i, we draw the path $P_j \in \mathcal{P}$ inside R_i if $x_j \in S_i$. Since the sum of the numbers of vertices of the paths drawn in R_j is at most B, we can draw those paths so that each edge in the path between v_i and v_{i+1} gets crossed at most once. The drawing obtained in this manner is a 1-planar drawing of G.

For the other direction, suppose that G has a 1-planar drawing. Note that Corollary 1 can be applied to G; from $G - \{u_1, u_2\}$, we obtain G by connecting $\{v_1, \ldots, v_b\}$ with u_1 (s in Corollary 1), and then with u_2 (s'). Hence, there exists a 1-planar drawing Γ of the graph G such that all the spokes in G have no crossings. Since the spokes have no crossings in Γ, the subdrawing Γ_{spokes} induced by them is planar, and hence there are exactly two vertices v_i and v_j that belong to the outer cycle of Γ_{spokes} other than u_1 and u_2. All the other v_ℓ's are drawn inside the region bounded by this outer cycle. Moreover, the graph G contains a path between v_ℓ and $v_{\ell+1}$ (v_1 when $\ell = b$) for every $1 \leq \ell \leq b$. The subdrawing Γ_{spokes} has only b faces containing multiple vertices from $v_1, \ldots, v_b$, each of which contains in fact exactly two. Hence, the b paths must be distributed to the b faces one by one (in the original drawing Γ). This implies that v_i and v_j are consecutive, namely $j = (i \pm 1) \bmod b$.

By appropriately renaming those vertices, we can assume that $G - \bigcup_{1 \leq i \leq |S|} V(P_i)$ are drawn as in Fig. 7a. Then, there are b internally vertex-disjoint paths $Q_1, \ldots, Q_b$ between u_1 and u_2 such that Q_i is a concatenation of spokes between u_1 and v_i and between v_i and u_2 for each i. These paths separate the plane into b regions, $R_1, \ldots, R_b$ as depicted in Fig. 7b. As each path P_i cannot lie in two different regions, $\mathcal{P}$ can be partitioned into $\{\mathcal{P}_1, \ldots, \mathcal{P}_b\}$, where $\mathcal{P}_i \subseteq \mathcal{P}$ is the collection of paths drawn inside R_i. Observe that, for each path $P_j \in \mathcal{P}_i$, there are $x_j = |V(P_j)|$ edge-disjoint paths between u_1 and u_2 that are drawn inside R_i. Each of them must cross the path between v_i and v_{i+1} on C, implying that there are at most B such paths inside R_i as Γ is 1-planar. Hence, $\sum_{P_j \in \mathcal{P}_i} |V(P_j)| \leq B$ follows and accordingly we can partition S into b sets such that each sum does not exceed B. □

Observe that the above reduction still works even if we add an edge between u_1 and each vertex in C, in which case the vertex set $\{u_1, u_2\}$ forms a dominating set.

Corollary 2. 1-Planarity Testing *is NP-complete even if the domination number is 2.*

As mentioned in Section 1, Cabello and Mohar [5] showed that 1-Planarity Testing is NP-hard even on near-planar graphs. Theorem 4 strengthens their hardness result as the graph G constructed in the reduction is a near-planar graph with the following properties.

Corollary 3. 1-Planarity Testing *is NP-complete on near-planar graphs with feedback vertex set number at most 3 and pathwidth at most 4.*

3.3 Twin Cover Number

In this subsection, we show that k-Planarity Testing is W[1]-hard with respect to twin cover number, which complements the positive result of the "$+k$ setting" to be presented in the subsequent section.

We give a similar reduction from Unary Bin Packing as in Section 3.2. However, in order to substitute cliques for paths, we first show the W[1]-hardness of a restricted version of Unary Bin Packing, where integers in S are sufficiently small that the constructed graph admits a B-planar drawing.

Lemma 5. Unary Bin Packing *is W[1]-hard parameterized by the number b of bins, even if each integer $x \in S$ satisfies $x \leq \sqrt{B} - 1$.*

Proof: Starting from an instance of Unary Bin Packing $I = \langle S, B, b \rangle$, we construct an equivalent instance $I' = \langle S', B', b \rangle$ such that every $x' \in S'$ satisfies $x' \leq \sqrt{B'} - 1$. Without loss of generality, we assume that $x \leq B$ for $x \in S$. We let S' be the multiset obtained from S by adding $7bB$ copies of $B+1$ and let $B' = B + 7B(B+1)$. As B is bounded by a polynomial in $|S|$ from above, I' is indeed an instance of Unary Bin Packing. For every $x' \in S'$, we can verify the restriction as:

$$x' \leq B + 1 = (B+2) - 1 = \sqrt{B^2 + 8B - 4(B-1)} - 1 \leq \sqrt{7B^2 + 8B} - 1 = \sqrt{B'} - 1.$$

Now we show the correctness of the reduction. It is clear that if I is a yes-instance then I' is also a yes-instance. Conversely, let us assume that I' is a yes-instance. Then, each bin must have exactly $7B$ copies of $B+1$, since there are $7bB$ of them and each bin can have at most $7B$ of them by construction. Hence, removing those copies from a solution for I', all bins have integers whose total sum is at most B, which forms a solution for I. □

Let us recall that a twin cover of G is a vertex set $S \subseteq V(G)$ such that each component of $G - S$ consists of true twins, namely, vertices having the same closed neighborhood. Then, the twin cover number of G is the minimum size of a twin cover of G.

Theorem 5. k-Planarity Testing *is W[1]-hard parameterized by twin cover number.*

Proof: The reduction is almost analogous to the one used in Theorem 4. Let $I = \langle S, B, b \rangle$ be an instance of Unary Bin Packing. We can assume that $b \geq 3$ and $x \leq \sqrt{B} - 1$ for all $x \in S$ due to Lemma 5. While the basic construction of the graph G is the same as in Theorem 4, it differs in the following points:

- the cycle C contains exactly b vertices $v_1, \dots, v_b$;
- for each $x_i \in S$, G contains a clique C_i with x_i vertices, instead of a path P_i, such that each vertex in the clique is adjacent to both u_1 and u_2;
- for each v_i, we add $\ell_1 \coloneqq Bm + 1$ spokes between u_1 and v_i and add $\ell_2 \coloneqq B(b\ell_1 + m) + 1$ spokes between u_2 and v_i, where m is the number of edges in G that are not contained in the spokes.

When we draw G as in Fig. 6, edges incident to a vertex in clique C_i may have crossings. Each edge e in C_i may cross other edges in C_i and all the edges between u_j and a vertex in C_i for $1 \leq j \leq 2$. This implies that the number of crossings that e involves is at most

$$|E(C_i)| - 1 + 2|V(C_i)| \leq (\sqrt{B} - 1)(\sqrt{B} - 2)/2 - 1 + 2(\sqrt{B} - 1) < B.$$

Similarly, we can conclude that each edge between u_j and a vertex in C_i has at most $B-1$ crossings. Since we can assume that all the spokes in G have no crossings due to Corollary 1, these spokes define exactly b regions, each of which contains at most B vertices of cliques. Thus, we can show that I is a yes-instance if and only if G is B-planar as well.

As each connected component in $G - (V(C) \cup \{u_1, u_2\})$ consists of true twins, G has a twin cover of size at most $b + 2$. This completes the proof. □

4 FPT Algorithms

In contrast to the previous section, we mainly focus on the algorithmic aspects of k-Planarity Testing in this section and show that k-Planarity Testing is FPT when several well-known graph parameters (and k) are given as parameters.

4.1 Reducing to 1-Planarity Testing

There are several consequences from Lemma 1, combining known FPT algorithms of 1-Planarity Testing [2].

Let G be a graph and let k be an integer. Let G_k be the graph obtained from G by subdividing each edge $k - 1$ times. Clearly, we have $\mathrm{fes}(G) = \mathrm{fes}(G_k)$. By Lemma 1, G is k-planar if and only if G_k is 1-planar. Since 1-Planarity Testing is FPT parameterized by feedback edge set number [2], the following theorem holds.

Theorem 6. k-Planarity Testing *is FPT parameterized by feedback edge set number.*

We next observe that the treedepth of G_k is not much larger than the treedepth of G.

Lemma 6. *For an integer $k \geq 0$, it holds that* $\mathrm{td}(G_k) \leq \mathrm{td}(G) + \lceil \log_2 k \rceil$.

Proof: Let F be an elimination forest of G with height d. Now, we create an elimination forest of G_k with height at most $d + \lceil \log_2 k \rceil$. Let $(u, p_1, \ldots, p_{k-1}, v)$ be the path of G_k corresponding to an edge $\{u, v\} \in E(G)$. Since the treedepth of the path graph P_{n-1} with $n - 1$ vertices is $\lceil \log_2 n \rceil$ [36], there is an elimination tree $T_{u,v}$ of $G_k[\{p_1, \ldots, p_{k-1}\}]$ with height $\lceil \log_2 k \rceil$. For each edge $\{u, v\} \in E(G)$, assume that u is an ancestor of v and attach $T_{u,v}$ as a child subtree to node v in F. Let F_k be the rooted forest obtained as above. It is clear that the height of F_k is at most $d + \lceil \log_2 k \rceil$. Moreover, for each edge of $E(G_k)$, one of the endpoints is an ancestor of the other end in F_k. Thus, F_k is an elimination forest of G_k with height at most $d + \lceil \log_2 k \rceil$. □

Similarly to the above, we have the following theorem.

Theorem 7. k-Planarity Testing *is FPT parameterized by treedepth* $+ k$.

Note that a similar result is already mentioned in a survey by Zehavi [42]. Since k-planarity is closed under vertex and edge deletions, it is known that the class of k-planar graphs with bounded treedepth is well-quasi-ordered by the induced-subgraph relation [37]. This implies a non-uniform version of Theorem 7: k-Planarity Testing can be solved by just checking whether G has an induced subgraph that belongs to a finite set $\mathcal{F}_{k,\mathrm{td}}$ of graphs, where the size of $\mathcal{F}_{k,\mathrm{td}}$ depends only on k and $\mathrm{td}(G)$. However, this is not constructive and does not guarantee the existence of a uniform FPT algorithm. We remark that our algorithm is in contrast uniform and can return a k-planar drawing of the given graph if it exists.

Theorem 7 leads to FPT results for other parameters. To explain this, we introduce the notion of degeneracy. For an integer $d \geq 0$, a graph G is said to be *d-degenerate* if each nonempty subgraph of G contains a vertex of degree at most d. A graph class $\mathcal{C}$ is *degenerate* if there exists d such that every graph in $\mathcal{C}$ is d-degenerate.

Proposition 1 ([36, Proposition 6.4])**.** *A hereditary class of graphs $\mathcal{C}$ has bounded treedepth if and only if $\mathcal{C}$ is degenerate and does not contain a path P_t for some t as an induced subgraph.*

It is known that every k-planar graph with n vertices has at most $3.81\sqrt{k}n$ edges [1]. Since k-planarity is closed under taking a subgraph, the class of k-planar graphs is degenerate for every fixed k. Hence, combined with Proposition 1, k-Planarity Testing on P_t-free graphs parameterized by $k+t$ can be reduced to k-Planarity Testing parameterized by treedepth $+\,k$. It is known that a graph class $\mathcal{C}$ does not contain a path P_t for some t as an induced subgraph if $\mathcal{C}$ has bounded shrub-depth [17] or bounded modular-width [30].

Corollary 4. *Let $\mathcal{C}$ be a hereditary class of graphs that excludes the path P_t for some t as an induced subgraph. Then, k-Planarity Testing over $\mathcal{C}$ is FPT parameterized by $k+t$. In particular, k-Planarity Testing is FPT parameterized by shrub-depth $+\,k$, and modular-width $+\,k$.*

4.2 Polynomial Kernels

In this subsection, we describe a kernelization algorithm for k-Planarity Testing parameterized by k plus the vertex cover number of G. The high-level ideas follow the polynomial kernel for 1-Planarity Testing due to Bannister, Cabello, and Eppstein [2].

Let G be a graph and let S be a vertex cover of G. We can find such a vertex cover S of size at most twice the optimum in linear time by a standard greedy algorithm, which is described as follows. For each edge $\{u, v\}$ in G, we mark both u and v if both of them are unmarked. We then define S as the set of all marked vertices. Clearly, S is a vertex cover of G, and it is well known that the size of S is at most twice the minimum size of a vertex cover of G [41].

First, we observe that the "diversity" of vertices outside of a vertex cover S is sufficiently small when G is k-planar. For a function f, let $\#_f(X)$ denote $|\{f(x) \mid x \in X\}|$, the number of possible images of X through f, where X is a subset of the domain of f.

Lemma 7. *Let $I_{\geq 2} \subseteq V(G) \setminus S$ be the set of vertices with degree at least 2 in $V(G) \setminus S$. Let π be a function that maps each vertex $v \in I_{\geq 2}$ to an unordered pair of its distinct neighbors (that is, $\pi \colon I_{\geq 2} \to \binom{S}{2}$ with $\pi(v) \in \binom{N_G(v)}{2}$). If $\#_\pi(I_{\geq 2}) > |S| \cdot 3.81\sqrt{2k}$, then G is not k-planar.*

Proof: We prove the contrapositive. Assume G is k-planar. Let G_S be the graph with $V(G_S) = S$ and $E(G_S) = \pi(I_{\geq 2})$. Informally, G_S is the simple graph obtained from G by keeping only the edges incident to $I_{\geq 2}$ that are selected by π, "suppressing" all degree-2 vertices in $I_{\geq 2}$, and removing all parallel edges. Here, $E(G_S) = \pi(I_{\geq 2}) \subseteq \binom{S}{2} = \binom{V(G_S)}{2}$ holds since S is a vertex cover.

Let G' be the graph obtained from G_S by subdividing each edge once. Then G' is a subgraph of the k-planar graph G, and hence G' is k-planar. This implies that by Lemma 1, G_S is $2k$-planar.

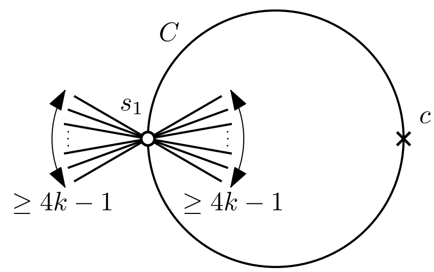


Figure 8: The Jordan curve C, the crossing point c, and the vertex s_1 with edges going out to each of the interior and exterior of C due to the cyclic difference condition.

Hence, $\#_\pi(I_{\geq 2}) = |E(G_S)| \leq |S| \cdot 3.81\sqrt{2k}$, where the last inequality is shown by Ackerman [1]. Therefore, if G is k-planar then $\#_\pi(I_{\geq 2}) \leq |S| \cdot 3.81\sqrt{2k}$. □

It is known that, for a positive integer k, if G contains $K_{7k+1,3}$ as a subgraph then G is not k-planar [8, 39]. Hence, the following also immediately holds.

Corollary 5. *Let $u \in V(G) \setminus S$ be a vertex of degree at least 3. If there are at least $7k+1$ false twins of u (including u itself), then the graph G is not k-planar.*

Next, we consider vertices of degree 2. The following lemma is a key ingredient of our kernelization. It is worth mentioning that, although a similar reduction rule is used in the case $k = 1$ [2], the proof for the general case $k \geq 1$ is considerably more involved.

Lemma 8. *Let $u \in V(G) \setminus S$ be a vertex of degree 2 with $N_G(u) = \{s_1, s_2\} \subseteq S$. If there are more than $16k^2|S|$ vertices in $V(G) \setminus S$ with the same neighborhood $\{s_1, s_2\}$, then G is k-planar if and only if $G - \{u\}$ is k-planar.*

Proof: The ($\Rightarrow$) direction is clear. Thus, we consider the reverse direction. Suppose that $G' \coloneqq G - \{u\}$ is k-planar and let Γ be a k-planar drawing of G'. We can assume that G' has no vertex of degree 1.

Let $T_u = \{v \in V(G') \setminus S \mid N_{G'}(v) = \{s_1, s_2\}\}$. Then, we have $t_u = |T_u| \geq 16k^2|S|$. Let $\mathcal{P}$ be the set of paths between s_1 and s_2 via a vertex in T_u. Let us align the paths in $\mathcal{P}$ according to the cyclic order in Γ around s_1, as $P_0, P_1, \ldots, P_{t_u-1}$. For two paths P_i, P_j, the *cyclic difference* of P_i and P_j is defined as $\min\{|j-i|, t_u - |j-i|\}$. Informally, this is the minimum of the clockwise and counter-clockwise difference according to the cyclic order. We then claim the following.

Claim 1. *Two paths $P_i, P_j \in \mathcal{P}$ with cyclic difference at least $4k$ do not cross in Γ.*

Proof: Suppose that P_i and P_j do cross in Γ. Let c be a crossing point between P_i and P_j, and let C be a Jordan curve consisting of two arcs: the arc of P_i from s_1 to c and the arc of P_j from c to s_1. Such a crossing point c can be chosen so that C forms a closed curve. In both cases where s_2 is in the interior or in the exterior of regions bounded by C, from the condition that the cyclic difference of P_i and P_j is at least $4k$, there are at least $4k-1$ edges that must cross C as shown in Fig. 8. As P_i, P_j can have at most $2k-1$ crossings other than c each, and hence $4k-2$ crossings in total, this leads to a contradiction. □

Let $\mathcal{P}' = \{P_{4ki} \mid 0 \leq i < 4k|S|\}$. As the cyclic difference of any pair of two paths in $\mathcal{P}'$ is at least $4k$, by claim 1, no two paths in $\mathcal{P}'$ cross. We can also assume that each path in $\mathcal{P}'$ does not

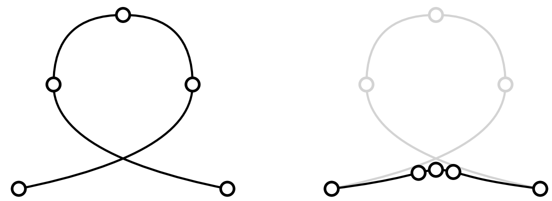


Figure 9: Let P be a path whose internal vertices have degree exactly 2. A “self-crossing” on the path can be removed by suppressing the “loop” formed by the edges in P.

cross itself, as it can be resolved as shown in Fig. 9. Hence, they separate the plane into $4k|S|$ regions similar to Fig. 7b. For each $0 \le i < 4k|S|$, let R_i be the region that is bounded by two paths P_{4ki} and $P_{4k(i+1)}$, where $P_{4k \cdot 4k|S|} \coloneqq P_0$. Observe that, for each vertex $v \in S \setminus \{s_1, s_2\}$, the number of paths in $\mathcal{P}'$ crossed by the edges incident to a vertex in $N_{G'}[v]$ is at most $4k$: when v belongs to region R_i, each edge incident to a vertex in $N_{G'}[v]$ is contained in the regions at most $2k$ apart from R_i, namely, the union of $\{R_{j \bmod 4k|S|} \mid i - 2k \le j \le i + 2k\}$ including its boundary.

Now we pick a path from $\mathcal{P}'$, along which we later redraw the other paths in $\mathcal{P}$ with no crossings. Since S is a vertex cover of G', every edge $e \in E(G')$ is either (1) $\{s_1, s_2\}$, (2) contained in a path in $\mathcal{P}$, or (3) incident to a vertex in $N_{G'}[v]$ for some $v \in S \setminus \{s_1, s_2\}$. Note that this follows from the assumption that G' has no vertex of degree 1. Among the paths in $\mathcal{P}'$, at most k of them can be crossed by $e = \{s_1, s_2\}$ (if it exists). Moreover, at most $4k(|S| - 2)$ paths in $\mathcal{P}'$ can be crossed by the edges of type (3) from the observation in the previous paragraph. Hence, there are at least $4k|S| - 4k(|S| - 2) - k = 7k$ paths in $\mathcal{P}'$ that are crossed only by the edges of type (2), namely, the paths in $\mathcal{P}$. Let $P \in \mathcal{P}'$ be such a path.

We remove all the paths in $\mathcal{P}$ but P from Γ. Observe that now P has no crossings at all. Hence, we can redraw all the removed paths along P with no crossings. We can also add the path (s_1, u, s_2) to Γ along P in the same way and obtain a k-planar drawing of G. $\square$

By summarizing the above lemmas, we can obtain a kernelization algorithm.

Theorem 8. *k-Planarity Testing has a polynomial kernel with $O(\mathrm{vc}(G)^2 k^2 \sqrt{k})$ vertices. Moreover, such a kernel can be computed in linear time.*

Proof: Since removing vertices of degree at most 1 does not change the local crossing number, we can assume that the minimum degree of G is at least 2. Let S be a vertex cover of G. We define a labeling π that maps $v \in V(G) \setminus S$ to an unordered pair $\{x, y\} \in \binom{N(v)}{2}$, and compute $\#_\pi(V(G) \setminus S)$ in polynomial time. From Lemma 7, if $\#_\pi(V(G) \setminus S) > 3.81|S|\sqrt{2k}$ then G is not k-planar.

Suppose that there are more than $7k(|S| - 2)$ vertices of degree at least 3 in $V(G) \setminus S$ having the same labeling on π. Since $|S \setminus \pi(v)| = |S| - 2$, by the pigeonhole principle, G contains $K_{7k+1,3}$ as a subgraph. Hence, by Corollary 5, we can assume that G has at most $7k(|S|-2) \cdot 3.81|S|\sqrt{2k} \in O(|S|^2 k\sqrt{k})$ vertices of degree at least 3 in $V(G) \setminus S$. Applying Lemma 8 exhaustively, the number of degree-2 vertices is at most $16k^2|S| \cdot 3.81|S|\sqrt{2k} \in O(|S|^2 k^2 \sqrt{k})$. Thus, we conclude that $|V(G)| \in O(|S|^2 k^2 \sqrt{k})$.

Moreover, each processing can be done by counting vertices on the labeling based on π and the degree, and removing some of them. Therefore, the entire running time of our kernelization is polynomial in the size of G.

Now, we consider the linear-time implementation. In the following, we identify the vertices of G with $\{1, \ldots, |V(G)|\}$. We first create the adjacency list for each $v \in V(G)$ sorted in ascending order. This can be done in linear time by bucket sort. For each $v \in V(G) \setminus S$, let $\pi(v) = \{x, y\}$, where x and y are the two smallest vertices in $N(v)(\subseteq S)$. Then, create an empty list L and insert tuples (x, y, d, v) to L for each $v \in V(G) \setminus S$ where $\pi(v) = \{x, y\}$ and d is the degree of v. This can be done in linear time by scanning the sorted adjacency list. Now, we can sort L in lexicographic order in linear time by calling bucket sort four times with keys v, d, y, and x in this order. Then, we can check the conditions and mark vertices to be removed by scanning the sorted list L. Hence, the entire running time of our kernelization is linear in the size of G since S with $|S| \leq 2\mathrm{vc}(G)$ can be computed in linear time. □

We can obtain a similar result for the case of neighborhood diversity. Recall that the neighborhood diversity $\mathrm{nd}(G)$ of a graph G is the minimum size of a partition $\mathcal{P}$ of $V(G)$ such that each vertex set in $\mathcal{P}$ is a twin class. Note that there are two types of twin classes: a true twin class is a clique of G consisting of vertices having the same closed neighborhood, and a false twin class is an independent set of G consisting of vertices having the same open neighborhood.

Lemma 9. *The vertex cover number of a k-planar graph G is at most $O(\mathrm{nd}(G)\sqrt{k})$.*

To show Lemma 9, we first show the following lemma.

Lemma 10. *Let G be a graph with neighborhood diversity $\mathrm{nd}(G)$. If G does not contain $K_{t,t}$ as a subgraph for some t, then the vertex cover number of G is at most $2\mathrm{nd}(G) \cdot t$.*

Proof: We construct a vertex cover S with at most $2\mathrm{nd}(G) \cdot t$ vertices of G in the following manner. Suppose that the vertex set of G is partitioned into twin classes $\{V_1, \ldots, V_{\mathrm{nd}(G)}\}$. For each true twin class V_i, we put all the vertices in V_i into S. Since V_i induces a clique in G and G does not have $K_{t,t}$ as a subgraph, we have $|V_i| < 2t$. If S is a vertex cover of G, we are done. Suppose that there is an edge that is not covered by S. As each true twin class is already contained in S, this edge spans two distinct false twin classes V_i and V_j. Since G has no $K_{t,t}$, at least one of V_i and V_j, say V_i, has less than t vertices. We then add all the vertices in V_i to S. As long as there is an edge not covered by S, we continue this process, and then we eventually have a vertex cover S of G with $|S| \leq 2\mathrm{nd}(G) \cdot t$. □

Proof: [Proof of Lemma 9] Let $t = 8\lceil\sqrt{2k}\rceil$. By Lemma 10, it suffices to show that $K_{t,t}$ is not k-planar. Clearly, $K_{t,t}$ has $16\lceil\sqrt{2k}\rceil$ vertices and $4\lceil\sqrt{2k}\rceil \cdot 16\lceil\sqrt{2k}\rceil$ edges. Since every k-planar graph with n vertices has at most $3.81n\sqrt{2k}$ edges [1], $K_{t,t}$ is not k-planar. □

By plugging Lemma 9 to Theorem 8, we have the following.

Corollary 6. k-PLANARITY TESTING *has a kernel with $O(\mathrm{nd}(G)^2 k^3 \sqrt{k})$ vertices.*

Finally, we remark that the results of Theorem 8 and Corollary 6 cannot be generalized to vertex integrity and modular-width parameterizations unless NP $\subseteq$ coNP/poly, respectively. Let $G_1, \ldots, G_t$ be graphs and G be the disjoint union of $G_1, \ldots, G_t$. Then, it holds that (1) G is 1-planar if and only if G_i is 1-planar for all $1 \leq i \leq t$ and (2) $\mathrm{vi}(G) \leq \max_i |V(G_i)|$ and $\mathrm{mw}(G) \leq \max_i |V(G_i)|$. This implies that 1-PLANARITY TESTING parameterized by vertex integrity and by modular-width is AND-cross-compositional [3,12,16], implying that unless NP $\subseteq$ coNP/poly, there is no kernel for 1-PLANARITY TESTING of size $\mathrm{poly}(\mathrm{vi}(G))$ or $\mathrm{poly}(\mathrm{mw}(G))$. As $\mathrm{td}(G) \leq \mathrm{vi}(G)$, this kernelization lower bound also holds for treedepth parameterization.

Proposition 2. k-PLANARITY TESTING *has no polynomial kernel parameterized by vertex integrity, treedepth, or modular-width for any fixed $k \geq 1$ unless* NP $\subseteq$ coNP/poly.

5 Conclusion

In this paper, we study the parameterized complexity of k-Planarity Testing from the perspective of graph structural parameterizations and show several (tight) upper and lower bound results. We leave several interesting open problems relevant to our results.

When k is considered as input, we have only shown that k-Planarity Testing is FPT parameterized by feedback edge set number. It would be interesting to seek similar results using other graph parameters. In this direction, a highly related problem Crossing Number is known to be FPT parameterized by vertex cover number [25]. However, it seems that their approach cannot be directly applied to k-Planarity Testing, requiring new insights for this problem. Towards this, showing intractability with a general parameter of vertex cover number, such as vertex integrity or neighborhood diversity (see Fig. 1a), would also be a nice open problem.

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References

[1] Eyal Ackerman. On topological graphs with at most four crossings per edge. *Comput. Geom.*, 85, 2019. doi:10.1016/J.COMGEO.2019.101574.

[2] Michael J. Bannister, Sergio Cabello, and David Eppstein. Parameterized complexity of 1-planarity. *J. Graph Algorithms Appl.*, 22(1):23–49, 2018. doi:10.7155/JGAA.00457.

[3] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *J. Comput. Syst. Sci.*, 75(8):423–434, 2009. doi:10.1016/J.JCSS.2009.04.001.

[4] Sergio Cabello, Alexander Dobler, Gasper Fijavz, Thekla Hamm, and Mirko H. Wagner. A dichotomy for 1-planarity with restricted crossing types parameterized by treewidth. In Ho-Lin Chen, Wing-Kai Hon, and Meng-Tsung Tsai, editors, *36th International Symposium on Algorithms and Computation, ISAAC 2025, Tainan, Taiwan, December 7-10, 2025*, volume 359 of *LIPIcs*, pages 16:1–16:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2025. URL: https://doi.org/10.4230/LIPIcs.ISAAC.2025.16, doi:10.4230/LIPICS.ISAAC.2025.16.

[5] Sergio Cabello and Bojan Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013. doi:10.1137/120872310.

[6] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, 3rd Edition*. MIT Press, 2009. URL: http://mitpress.mit.edu/books/introduction-algorithms.

[7] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. doi:10.1007/978-3-319-21275-3.

[8] Július Czap and Dávid Hudák. 1-planarity of complete multipartite graphs. *Discrete Applied Mathematics*, 160(4):505–512, 2012. doi:10.1016/j.dam.2011.11.014.

[9] Éric Colin de Verdière and Petr Hlinený. A unified FPT framework for crossing number problems. In Anne Benoit, Haim Kaplan, Sebastian Wild, and Grzegorz Herman, editors, *33rd Annual European Symposium on Algorithms, ESA 2025, Warsaw, Poland, September 15-17, 2025*, volume 351 of *LIPIcs*, pages 21:1–21:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2025. URL: https://doi.org/10.4230/LIPIcs.ESA.2025.21, doi:10.4230/LIPICS.ESA.2025.21.

[10] Éric Colin de Verdière and Thomas Magnard. An FPT algorithm for the embeddability of graphs into two-dimensional simplicial complexes. In *29th Annual European Symposium on Algorithms, ESA 2021*, volume 204 of *LIPIcs*, pages 32:1–32:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPICS.ESA.2021.32.

[11] Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281.

[12] Andrew Drucker. New limits to classical and quantum instance compression. *SIAM J. Comput.*, 44(5):1443–1479, 2015. doi:10.1137/130927115.

[13] Chandan Dubey, Uriel Feige, and Walter Unger. Hardness results for approximating the bandwidth. *Journal of Computer and System Sciences*, 77(1):62–90, 2011. Celebrating Karp's Kyoto Prize. doi:10.1016/j.jcss.2010.06.006.

[14] Vida Dujmovic, Seok-Hee Hong, Michael Kaufmann, János Pach, and Henry Förster. Beyond-planar graphs: Models, structures and geometric representations (dagstuhl seminar 24062). *Dagstuhl Reports*, 14(2):71–94, 2024. doi:10.4230/DAGREP.14.2.71.

[15] Eduard Eiben, Robert Ganian, Thekla Hamm, Fabian Klute, and Martin Nöllenburg. Extending nearly complete 1-planar drawings in polynomial time. In Javier Esparza and Daniel Král', editors, *45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020, Prague, Czech Republic, August 24-28, 2020*, volume 170 of *LIPIcs*, pages 31:1–31:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. URL: https://doi.org/10.4230/LIPIcs.MFCS.2020.31, doi:10.4230/LIPICS.MFCS.2020.31.

[16] Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Meirav Zehavi. *Kernelization: Theory of Parameterized Preprocessing*. Cambridge University Press, 2019. doi:10.1017/9781107415157.

[17] Robert Ganian, Petr Hlinený, Jaroslav Nesetril, Jan Obdrzálek, and Patrice Ossona de Mendez. Shrub-depth: Capturing height of dense graphs. *Log. Methods Comput. Sci.*, 15(1), 2019. doi:10.23638/LMCS-15(1:7)2019.

[18] Robert Ganian, Petr Hlinený, Jaroslav Nesetril, Jan Obdrzálek, Patrice Ossona de Mendez, and Reshma Ramadurai. When trees grow low: Shrubs and fast MSO1. In Branislav Rovan, Vladimiro Sassone, and Peter Widmayer, editors, *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27-31, 2012. Proceedings*, volume 7464 of *Lecture Notes in Computer Science*, pages 419–430. Springer, 2012. doi:10.1007/978-3-642-32589-2_38.

[19] M. R. Garey and D. S. Johnson. Crossing number is NP-complete. *SIAM Journal on Algebraic Discrete Methods*, 4(3):312–316, 1983. doi:10.1137/0604033.

[20] Tatsuya Gima, Tesshu Hanaka, Masashi Kiyomi, Yasuaki Kobayashi, and Yota Otachi. Exploring the gap between treedepth and vertex cover through vertex integrity. *Theor. Comput. Sci.*, 918:60–76, 2022. doi:10.1016/J.TCS.2022.03.021.

[21] Alexander Grigoriev and Hans L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007. doi:10.1007/S00453-007-0010-X.

[22] Martin Grohe. Computing crossing numbers in quadratic time. *J. Comput. Syst. Sci.*, 68(2):285–302, 2004. doi:10.1016/J.JCSS.2003.07.008.

[23] Thekla Hamm and Petr Hlinený. Parameterised partially-predrawn crossing number. In Xavier Goaoc and Michael Kerber, editors, *38th International Symposium on Computational Geometry, SoCG 2022, Berlin, Germany, June 7-10, 2022*, volume 224 of *LIPIcs*, pages 46:1–46:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. URL: https://doi.org/10.4230/LIPIcs.SoCG.2022.46, doi:10.4230/LIPICS.SOCG.2022.46.

[24] Thekla Hamm, Fabian Klute, and Irene Parada. Computing crossing numbers with topological and geometric restrictions. *CoRR*, abs/2412.13092, 2024. URL: https://doi.org/10.48550/arXiv.2412.13092, arXiv:2412.13092, doi:10.48550/ARXIV.2412.13092.

[25] Petr Hliněný and Abhisekh Sankaran. Exact crossing number parameterized by vertex cover. In *Graph Drawing and Network Visualization - 27th International Symposium, GD 2019*, volume 11904 of *Lecture Notes in Computer Science*, pages 307–319. Springer, 2019. doi:10.1007/978-3-030-35802-0_24.

[26] Michael Hoffmann, Chih-Hung Liu, Meghana M. Reddy, and Csaba D. Tóth. Simple topological drawings of k-planar graphs. In David Auber and Pavel Valtr, editors, *Graph Drawing and Network Visualization - 28th International Symposium, GD 2020, Vancouver, BC, Canada, September 16-18, 2020, Revised Selected Papers*, volume 12590 of *Lecture Notes in Computer Science*, pages 390–402. Springer, 2020. doi:10.1007/978-3-030-68766-3_31.

[27] Seok-Hee Hong and Takeshi Tokuyama. *Beyond Planar Graphs, Communications of NII Shonan Meetings*. Springer, 2020. doi:10.1007/978-981-15-6533-5.

[28] Klaus Jansen, Stefan Kratsch, Dániel Marx, and Ildikó Schlotter. Bin packing with fixed number of bins revisited. *J. Comput. Syst. Sci.*, 79(1):39–49, 2013. doi:10.1016/J.JCSS.2012.04.004.

[29] Ken-ichi Kawarabayashi and Bruce A. Reed. Computing crossing number in linear time. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing, STOC 2007*, pages 382–390. ACM, 2007. doi:10.1145/1250790.1250848.

[30] Minki Kim, Bernard Lidický, Tomás Masarík, and Florian Pfender. Notes on complexity of packing coloring. *Inf. Process. Lett.*, 137:6–10, 2018. doi:10.1016/J.IPL.2018.04.012.

[31] Yasuaki Kobayashi, Yuto Okada, and Alexander Wolff. Recognizing 2-Layer and Outer k-Planar Graphs. In Oswin Aichholzer and Haitao Wang, editors, *Proc. 41st Annu. Sympos. Comput. Geom. (SoCG'25)*, volume 332 of *LIPIcs*, pages 65:1–65:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025. doi:10.4230/LIPIcs.SoCG.2025.65.

[32] Vladimir P. Korzhik and Bojan Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *J. Graph Theory*, 72(1):30–71, 2013. doi:10.1002/JGT.21630.

[33] Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Roohani Sharma, Jie Xue, and Meirav Zehavi. Crossing number in slightly superexponential time (extended abstract). In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2025*, pages 1412–1424. SIAM, 2025. doi:10.1137/1.9781611978322.44.

[34] Miriam Münch, Maximilian Pfister, and Ignaz Rutter. Exact and approximate k-planarity testing for maximal graphs of small pathwidth. In *Graph-Theoretic Concepts in Computer Science - 50th International Workshop, WG 2024*, volume 14760 of *Lecture Notes in Computer Science*, pages 430–443. Springer, 2024. doi:10.1007/978-3-031-75409-8_30.

[35] Miriam Münch and Ignaz Rutter. Parameterized algorithms for beyond-planar crossing numbers. In Stefan Felsner and Karsten Klein, editors, *32nd International Symposium on Graph Drawing and Network Visualization, GD 2024, Vienna, Austria, September 18-20, 2024*, volume 320 of *LIPIcs*, pages 25:1–25:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024. URL: https://doi.org/10.4230/LIPIcs.GD.2024.25, doi:10.4230/LIPICS.GD.2024.25.

[36] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-27875-4.

[37] Jaroslav Nešetřil and Patrice Ossona de Mendez. On low tree-depth decompositions. *Graphs Comb.*, 31(6):1941–1963, 2015. URL: https://doi.org/10.1007/s00373-015-1569-7, doi:10.1007/S00373-015-1569-7.

[38] János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Comb.*, 17(3):427–439, 1997. doi:10.1007/BF01215922.

[39] Maximilian Pfister. *Algorithms and Combinatorics for Beyond Planar Graphs*. Phd thesis, Eberhard Karls Universität Tübingen, Tübingen, Germany, 2025. doi:10.15496/publikation-101469.

[40] John C. Urschel and Jake Wellens. Testing gap k-planarity is NP-complete. *Inf. Process. Lett.*, 169:106083, 2021. doi:10.1016/J.IPL.2020.106083.

[41] Vijay V. Vazirani. *Approximation algorithms*. Springer, 2001. URL: http://www.springer.com/computer/theoretical+computer+science/book/978-3-540-65367-7.

[42] Meirav Zehavi. Parameterized analysis and crossing minimization problems. *Comput. Sci. Rev.*, 45:100490, 2022. doi:10.1016/J.COSREV.2022.100490.