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Improving the Crossing Lemma by Characterizing Dense 2-Planar and 3-Planar Graphs

Aaron Büngener *Michael Kaufmann*

University of Tübingen, Tübingen, Germany

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Abstract. The classical Crossing Lemma by Ajtai, Chvátal, Newborn and Szemerédi [4] and Leighton [13] from 1982 gave an important lower bound of $c\frac{m^3}{n^2}$ for the number of crossings in any drawing of a given graph of n vertices and m edges. The original value was $c = 1/100$, which then has gradually been improved. Here, the bounds for the density of k-planar graphs played a central role. Our new insight is that for $k = 2, 3$, k-planar graphs have substantially fewer edges if specific local configurations that occur in drawings of k-planar graphs of maximum density are forbidden. Therefore, we are able to derive better bounds for the crossing number $\mathrm{cr}(G)$ of a given graph G. In particular, we achieve a bound of $\mathrm{cr}(G) \geq \frac{37}{9}m - \frac{155}{9}(n-2)$ for the range of $5n < m \leq 6n$, while our second bound $\mathrm{cr}(G) \geq 5m - \frac{203}{9}(n-2)$ is even stronger for larger $m > 6n$.

For $m > 6.77n$, we finally apply the standard probabilistic proof from the BOOK and obtain an improved constant of $c > 1/27.48$ in the Crossing Lemma. Note that the previous best constant was $1/29$. Although this improvement is not too impressive, we consider our technique as an important new tool, which might be helpful in various other applications.

1 Introduction

We consider simple graphs G (without loops or multi-edges)[1] with vertices V and edges E that are drawn in the plane such that vertices are pairwise distinct points and edges are represented

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E-mail addresses: aaron.buengener@uni-tuebingen.de (Aaron Büngener) michael.kaufmann@uni-tuebingen.de (Michael Kaufmann)

[1] For technical reasons and for the sake of greater generality, most of our results in the main body are proved for multi-graphs.

as Jordan arcs connecting the endpoints. We assume that edges do not overlap other vertices in the interior. Two edges might cross, but we do not allow more than two edges to cross at a single point. We also assume that two edges have only a finite number of common interior points and no two edges meet tangentially.

Such a drawing D of G may have crossings, and often crossings are unavoidable. Our main result establishes a new lower bound for the crossing number $\mathrm{cr}(G)$, defined as the minimum of $\mathrm{cr}(D)$ over all drawings D of G, where $\mathrm{cr}(D)$ denotes the number of crossings in D. The lower bound depends only on $|V|$ and $|E|$, independent from the structure of the graph G.

Theorem 1. *Let G be a simple graph with n vertices and m edges. Then $\mathrm{cr}(G) \geq \frac{1500}{41209}\frac{m^3}{n^2} - \frac{54791}{41209}n > \frac{1}{27.48}\frac{m^3}{n^2} - 1.33n$. If $m \geq 6.77n > \frac{203}{30}n$, then $\mathrm{cr}(G) \geq \frac{1500}{41209}\frac{m^3}{n^2} > \frac{1}{27.48}\frac{m^3}{n^2}$.*

This theorem improves the classical Crossing Lemma by Ajtai, Chvátal, Newborn and Szemerédi [4] and Leighton [13], which has been a subject of extensive research. Starting from its first refinement to $\frac{1}{64}$ [3], the leading constant has been improved in subsequent works [1, 14, 16] to $\frac{1}{29}$. The theorem is tight up to the leading constant, which is at least 0.09 [14]. Variants of the Crossing Lemma have been formulated for many graph classes [19], such as bipartite graphs [5], graphs of bounded girth [15], multi-graphs [12, 17], etc. Székely [20] gave an impressive collection of further applications of the Crossing Lemma in discrete geometry.

The gradual improvement of the above-mentioned constant has mainly been achieved using the linear bounds for the number of edges for planar, 1-planar, 2-planar, etc. graphs. k-planar graphs are graphs that have a drawing where each edge is crossed at most k times. Density bounds for k-planar n-vertex graphs have been an attractive research topic in the past. While planar graphs have at most $3n-6$ edges, the best known upper bounds for 1-planar, 2-planar and 3-planar graphs are $4n-8$ [21], $5n-10$ [16] and $5.5n-11.5$ [9] respectively. They have been directly applied to improve the crossing lemma. The proof for the current best constant of $\frac{1}{29}$ uses even the bound for 4-planar graphs [1], which is $6n-12$.

We will present a more refined analysis by considering drawings that can be viewed as intermediate between k-planar and $k+1$-planar drawings for $k = 1, 2$. In their paper from 2006 [14], Pach, Radoičić, Tardos and Tóth used a similar approach to improve the corresponding constant of the Crossing Lemma. They considered the density of 1-planar drawings with a fixed number of crossing-free triangles, which form a class of drawings lying between planar and general 1-planar drawings. We extend this idea and focus on local configurations that have to occur in optimal 2- and 3-planar drawings, i.e. drawings of k-planar graphs whose densities match the corresponding upper bound. Such local configurations will be excluded or at least restricted: It turns out that either we can refine the bounds for the edge density of the corresponding k-planar classes (which immediately yields a better Crossing Lemma), or we can characterize the drawing in a very good way, which simplifies the way of counting the crossings.

Our approach is motivated by several results in the literature. (Non-simple) optimal 2-planar and 3-planar graphs have been characterized [9], and there is little flexibility in the structure of such graphs. Moreover, even under substantially weaker drawing restrictions, the bounds on the maximum density for some superclasses of 1-planar and 2-planar graphs are still roughly the same. Examples include min-1-planar and min-2-planar graphs [10] as superclasses of 1-planar and 2-planar graphs, as well as gap-planar graphs as a superclass of 2-planar graphs [7]. Min-k-planar graphs have been defined as graphs with drawings such that for each crossing, at least one of two crossing edges has at most k crossings, while in gap-planar drawings crossings are assigned to one of the two crossing edges (gap), and each edge is assigned to at most one crossing.

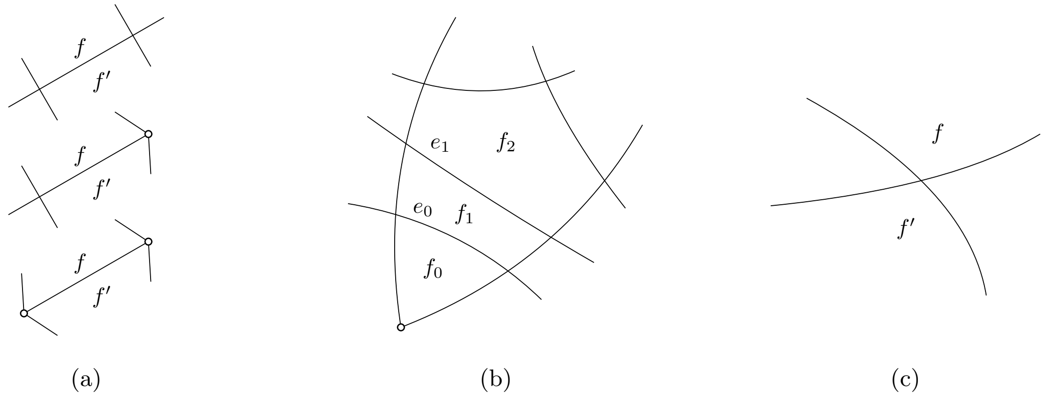


Figure 1: Illustrations of the defined neighborhood-relations. (a) From top to bottom: The faces f and f' are 0-neighbors, 1-neighbors, 2-neighbors resp. (b) The 0-pentagon f_2 is the wedge-neighbor of the 1-triangle f_0 at its edge e_0. (c) The faces f and f' are vertex-neighbors.

Besides Theorem 1, our work also contributes in another direction. There is some literature about edge densities under the exclusion of specific cell types, i.e., closed regions bounded by segments of intersecting edges. For example, Ackerman and Tardos [2] considered drawings without triangular cells that have no vertex on the boundary. But to the best of our knowledge, our Theorems 2 and 3 (see Section 3) are the first results about larger forbidden configurations that consist of multiple cells.

This paper is a refinement and extension of the conference version [11]. Compared to that version, we strengthened a crucial observation, i.e., Proposition 15, yielding a further improvement of the Crossing Lemma constant. Furthermore, we added two simple combinatorial proofs that K_8 is not 2-planar (Corollary 4) and K_9 is not 3-planar (Corollary 5).

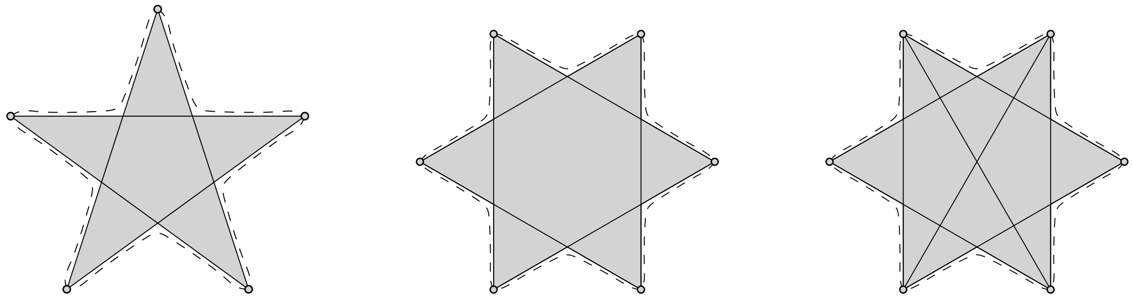
2 Definitions and Notation

Let $G = (V, E)$ be a graph and D be a drawing of G. While our main result Theorem 1 holds only for simple graphs, we consider throughout most of the paper (unless otherwise stated) multi-graphs without loops in E. We require that any two parallel edges are drawn non-homotopically, i.e., they form a closed curve with at least one vertex inside and one outside in D[2]. Note that adjacent edges might cross, but since we mostly assume that the number of crossings will be minimal, there will be no *empty lenses*, i.e., regions that are empty of vertices and have a boundary that is being defined by two edges; cf. Proposition 3.

Next, we introduce some definitions that help to describe a drawing more in detail. We basically adopted the notation by Ackerman [1].

We interpret a drawing D as a plane map $M(D) = (V', E')$ whose vertices V' are either vertices $V(D)$ of D or crossing points of D. An edge e in E' connects two vertices of V', i.e., it is a crossing-free segment of an edge of D, which we denote by $\bar{e}$. We call an edge of E' an *r-edge*, if $r \in \{0, 1, 2\}$ of its endpoints are vertices of D. For a vertex $v \in V(D)$, we write $\deg(v)$ for its *degree*. The degree of a crossing is always four.

[2]This definition is non-standard as it requires an additional vertex outside. It has been used in, e.g., [8].



(a) (b) (c)

Figure 2: (a) An F_5^2, (b) an F_6^2 and (c) an F_6^3 configuration with their boundaries (dashed).

Let F' be the set of faces of $M(D)$. For a face $f \in F'$, we write $|f|$ for the number of edges in E' and $|V(f)|$ for the number of vertices in $V(D)$ counted while traversing the boundary of f (so double-counting is possible). A face with $|f| = s$ is called an *s-gon*. In the cases of $s = 3, 4, 5, 6, 7$, we write instead *triangle, quadrilateral, pentagon, hexagon* and *heptagon*. If we want to denote that $|V(f)| = r$ and $|f| = s$, we write *r-s-gon* and use this wording also for 2-triangles, 0-quadrilaterals, etc. for simplicity. If we only want to specify for a face that $|V(f)| = r$, then we call it an r-face. Note that by Propositions 1–3, we will assume that the boundary of every face is a simple cycle. We say that an edge $e \in E$ *ends* at a face $f \in F'$ if one of its endpoints is incident to f.

Further, we need some definitions for relations between faces in F'. Two faces are *r-neighbors* if they share an r-edge. Let $e_0 \in E'$ be a 0-edge of a face $f_0 \in F'$ and let $f_1 \in F'$ be the 0-neighbor of f_0 at e_0. For $i \geq 1$, if $f_i \in F'$ is a 0-quadrilateral, then let $f_{i+1} \in F'$ be the 0-neighbor of f_i at the edge e_i opposite to e_{i-1}. The face f_i, for which i is maximal, is called the *wedge-neighbor* of f_0 at e_0. Notice the (slightly) different definition of a wedge-neighbor by Ackerman [1]. Finally, we define two faces $f, f' \in F'$ to be *vertex-neighbors*, if f and f' share a crossing-vertex c, but not an edge in E' incident to c. See Fig. 1 for an illustration of the defined terms.

Crossings-dense configurations. We now define three *crossings-dense configurations* that play a key role: An F_5^2 (resp., F_6^2) is a connected region of six (resp., seven) faces in $M(D)$ consisting of a central 0-pentagon (0-hexagon) f, where all 0-neighbors of f are 1-triangles (see Figs. 2a and 2b). An F_6^3 is a connected region of 14 faces in $M(D)$, where two 0-quadrilaterals and two 0-pentagons lie alternating around a central crossing and the other 0-neighbors of each of these faces are 1-triangles (see Fig. 2c). The notation F_p^k indicates that these configurations occur, when a maximal number of edges with at most k crossings is inserted in a convex p-gon.

Consider the 1-triangles of a crossings-dense configuration C. Let t, t' be a pair of 1-triangles that is incident to a common crossing x but does not share an edge in $M(D)$. Let $u, u' \in V(D)$ be the vertices to which t, t' are incident. They have incident edges $e, e' \in E$ that cross at x. Let uu' be a crossing-free edge that forms together with x a 2-triangle. If uu' does not exist in $M(D)$, then one can draw it by closely following e and e' until the vicinity of x. We call uu' a *boundary edge* of C. The *boundary* of C is the set of the five resp., six boundary edges of C, see Fig. 2. Note that the term *boundary* is used here in a purely combinatorial sense, referring to a set of edges of D, and should not be confused with the geometric boundary of a connected region. By definition, the boundary of C is not part of C itself. Thus, any two distinct crossings-dense configurations are edge-disjoint, although they may share vertices and boundary edges.

Using the definitions above, we are able to state our main results in the next section.

3 Results

In this section, we present our main results. The proofs of Theorem 2 and Theorem 3 use the discharging method and can be found in Section 4.

Theorem 2. *Any graph G with $n \geq 3$ vertices that admits a 2-planar F_5^2-free drawing has at most $4.5(n-2)$ edges. If the drawing is also F_6^2-free, then G has at most $\frac{13}{3}(n-2)$ edges. These bounds are tight.*

Tight lower bound constructions come from a plane graph whose faces are hexagons with an F_6^2 inserted into each hexagon, resp., from a plane graph with pentagonal faces, in which four arbitrary edges are added in each pentagon. More details and illustrations are provided in Section 4.1.

Corollary 1. *For every 2-planar drawing of any graph G with $n \geq 3$ vertices and $\frac{13}{3}(n-2)+x$ edges for $x \in [0, \frac{2}{3}(n-2)]$, the number of F_5^2 and F_6^2 configurations is at least x. This is tight for every x.*

Note that G cannot be 2-planar for $x > \frac{2}{3}(n-2)$ by the corresponding density bound.

Proof: Let D be a 2-planar drawing of a graph G with n vertices and $\frac{13}{3}(n-2)+x$ edges such that the number of F_5^2 or F_6^2 configurations is $y < x$. We can destroy those configurations by removing one edge from each F_5^2 and F_6^2. Hence, we still have more than $\frac{13}{3}(n-2)$ edges, which is a contradiction to Theorem 2.

The bound is achieved by a plane graph whose faces are pentagons, s.t. x pentagons are extended by five edges each, i.e. F_5^2's, and the other $\frac{2}{3}(n-2)-x$ pentagons by four edges each. □

This implies that drawings of optimal 2-planar graphs consist of $\frac{2}{3}(n-2)$ F_5^2 configurations, a fact that is already known [9]. Analogous results hold for 3-planar drawings.

Theorem 3. *Any graph G with $n \geq 3$ vertices that admits a 3-planar F_6^3-free drawing has at most $5(n-2)$ edges. This bound is tight.*

The next corollary allows us to characterize drawings of dense 3-planar graphs very well. This extends the characterization of optimal 3-planar graphs, which must have a drawing consisting of $\frac{1}{2}(n-2)$ F_6^3 configurations and their boundaries [9].

Corollary 2. *For every 3-planar drawing of any graph with $n \geq 3$ vertices and $5(n-2)+x$ edges for $x \in [0, 0.5(n-2)]$, the number of F_6^3 configurations is at least x.*

Note that G cannot be 3-planar for $x > 0.5(n-2)$ by the corresponding density bound.

Proof: Analogously to the proof of Corollary 1, we assume that there is a 3-planar drawing D of a graph with n vertices and $5(n-2)+x$ edges such that the number of F_6^3 configurations is $y < x$. Those configurations can be destroyed by removing one edge from each F_6^3, hence we still have more than $5(n-2)$ edges, which is a contradiction to Theorem 3. □

A consequence of this is a new upper bound for the edge density of simple 3-planar graphs, i.e., the case where multi-edges are forbidden. Note that the best known bound before was $5.5n-11.5$ edges [9] and there exist examples with $5.5n-15$ edges [14]. In contrast, there exist non-simple 3-planar graphs with $5.5n-11$ edges [8].

Corollary 3. *There are no 3-planar graphs on $n \geq 3$ vertices with exactly $5.5n-11.5$ edges. Therefore, any simple 3-planar graph on $n \geq 3$ vertices has at most $5.5n-12$ edges.*

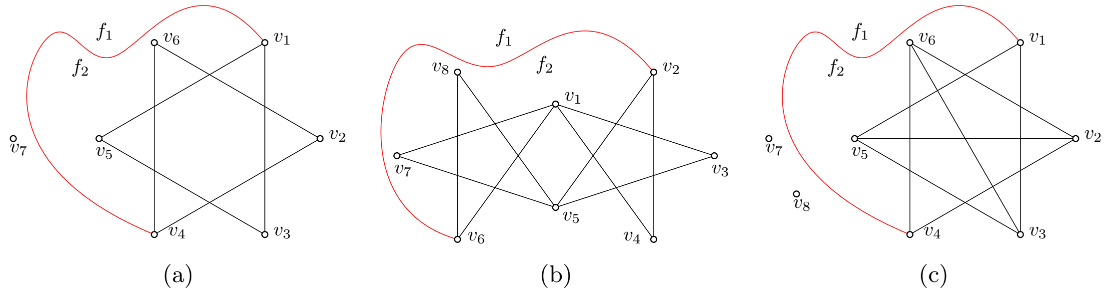


Figure 3: Constructing (a)-(b) a 2-planar drawing of K_8 or (c) a 3-planar drawing of K_9 leads to contradictions.

Proof: Assume that there exists a (not necessarily simple) 3-planar graph G with exactly $5.5n - 11.5$ edges. By Corollary 2, there are at least $0.5(n-2) - 0.5$ many F_6^3 configurations.

Let $\mathcal{H}$ be any triangulation on the set of vertices that includes the boundary of each F_6^3 configuration in D. Since the boundary of each F_6^3 configuration is partitioned into four triangles in $\mathcal{H}$, there are at most $2(n-2)/4 = 0.5(n-2)$ such configurations. As n must be odd, the number of F_6^3 configurations is exactly $0.5(n-2) - 0.5$ leaving exactly two triangles in $\mathcal{H}$ that do not belong to any F_6^3 configuration.

We now count the edges of G. Starting with the edges of $\mathcal{H}$, each F_6^3 consists of five additional edges. The other two triangles may contain one additional edge, which gives in total at most $3(n-2) + 2.5(n-2) - 2.5 + 1 = 5.5n - 12.5$ edges, contradicting the assumed density.

Combining this result with the upper bound of $5.5n - 11.5$ edges for simple graphs established in [9] proves the second part of the corollary. □

In the literature about k-planar drawings of complete graphs, the best results have been found through exhaustive search by checking all embeddings [6]. In particular, K_7 is known to be 2-planar and K_8 is 3-planar, but not 2-planar, while K_9 is the smallest complete graph which is not 3-planar. Our new observations imply short combinatorial proofs.

Corollary 4 ([6], Characterization 2)**.** *The complete graph K_8 is not 2-planar.*

Proof: K_8 has $n = 8$ vertices $\{v_1, ..., v_8\}$ and $m = 28 = \frac{13}{3}(n-2) + 2$ edges. Assume that there exists a 2-planar crossing-minimal drawing D of K_8. By Corollary 1, D has at least two F_5^2 or F_6^2 configurations C_1, C_2. For the case that w.l.o.g. C_1 is an F_6^2 configuration, let $v_1, ..., v_6$ be its vertices, let D' be the subdrawing of C_1 in D and let f be the face in D' such that $V(f)$ contains $v_1, ..., v_6$. Then v_1v_4 must be drawn in f and subdivides f into f_1, f_2, where f_1 contains w.l.o.g. the vertices v_2, v_3, and f_2 contains v_5, v_6 (see Fig. 3a). W.l.o.g., the vertex v_7 lies in f_1. But this implies that the four edges v_2v_5, v_3v_6, v_5v_7 and v_6v_7 all cross v_1v_4 contradicting that D is 2-planar.

Thus, assume that C_1 and C_2 are both F_5^2 configurations. Because $n = 8$, C_1 and C_2 have at least two common vertices a, b. If a, b do not lie consecutively on the boundary of w.l.o.g. C_1, then either ab exists also in C_2, which would be a multi-edge, or ab is a boundary edge of C_2 that could be drawn planar, which would contradict that D is crossing-minimal.

So C_1 and C_2 share exactly two vertices a, b which lie consecutively on their boundary. Let D' be the subdrawing of $C_1 \cup C_2$ in D and let f be the face in D' containing the vertices $v_1 = a, .., v_5 = b, ..., v_8$ in clockwise order. Then, in D the edge v_2v_6 must be drawn in f and subdivides f into f_1, f_2, where f_1 contains w.l.o.g. the vertices v_3, v_4, v_5 and f_2 contains v_1, v_7, v_8 (see Fig. 3b). Thus, the four edges v_3v_7, v_3v_8, v_4v_7 and v_4v_8 all cross v_2v_6 contradicting that D is 2-planar. □

Corollary 5 ([6], Characterization 2)**.** *The complete graph K_9 is not 3-planar.*

Proof: K_9 has $n = 9$ vertices $\{v_1, ..., v_9\}$ and $m = 36 = 5(n-2)+1$ edges. Assume that there exists a 3-planar drawing D of K_9. Then, by Corollary 2, D has at least one F_6^3 configuration C with vertices $v_1, ..., v_6$ and the missing edge v_1v_4. Let D' be the subdrawing of C in D and let f be the face in D' such that $V(f)$ contains $v_1, ..., v_6$. Then v_1v_4 must be drawn in f and subdivides f into f_1, f_2, where f_1 contains w.l.o.g. the vertices v_2, v_3, and f_2 contains v_5, v_6 (see Fig. 3c). W.l.o.g., the vertices v_7, v_8 lie in f_1. But this implies that the four edges v_5v_7, v_6v_7, v_5v_8 and v_6v_8 all cross v_1v_4 contradicting that D is 3-planar. □

From Theorem 2 and Theorem 3 we can also derive new lower bounds for the crossing number of a graph. The proof can be found in Section 5.

Theorem 4. *Let G be a graph with $n > 2$ vertices and m edges. Then*

(a) $\mathrm{cr}(G) \geq \frac{37}{9}m - \frac{155}{9}(n-2)$,

(b) $\mathrm{cr}(G) \geq 5m - \frac{203}{9}(n-2)$.

A slightly weaker bound than in (a) of $\mathrm{cr}(G) \geq 4m - \frac{50}{3}(n-2)$ can be derived with a significantly shorter proof by only applying Theorem 3; we point this out in the proof.

That improves the known lower bounds on $\mathrm{cr}(G)$ for $m > 5(n-2)$, which are $\mathrm{cr}(G) \geq 4m - \frac{103}{6}(n-2)$ [14] respectively $\mathrm{cr}(G) \geq 5m - \frac{139}{6}(n-2)$ [1]. Theorem 4 implies directly a better constant in the Crossing Lemma.

Theorem 1. *Let G be a simple graph with n vertices and m edges. Then $\mathrm{cr}(G) \geq \frac{1500}{41209}\frac{m^3}{n^2} - \frac{54791}{41209}n > \frac{1}{27.48}\frac{m^3}{n^2} - 1.33n$. If $m \geq 6.77n > \frac{203}{30}n$, then $\mathrm{cr}(G) \geq \frac{1500}{41209}\frac{m^3}{n^2} > \frac{1}{27.48}\frac{m^3}{n^2}$.*

Proof: Let G be a graph with n vertices and m edges. For the case $m \geq \frac{203}{30}n$, we construct a random subgraph G' by selecting every vertex of G independently with probability $p = \frac{203}{30}n/m \leq 1$. We denote the number of edges and vertices in G' by m' and n', respectively. By Theorem 4 and linearity of expectation, we obtain $\mathbb{E}[\mathrm{cr}(G')] \geq 5\mathbb{E}[m'] - \frac{203}{9}\mathbb{E}[n']$. We replace $\mathbb{E}[n'] = pn$, $\mathbb{E}[m'] = p^2m$ and $\mathbb{E}[\mathrm{cr}(G')] \leq p^4\,\mathrm{cr}(G)$, and get

$$\mathrm{cr}(G) \geq \frac{5m}{p^2} - \frac{203n}{9p^3} = \frac{1500}{41209}\frac{m^3}{n^2} \geq \frac{1}{27.48}\frac{m^3}{n^2}.$$

For the case $m < \frac{203}{30}n$, recall that the best known linear bounds are $\mathrm{cr}(G) \geq m - 3(n-2)$ for $3(n-2) \leq m < 4(n-2)$, $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$ for $4(n-2) \leq m < 5(n-2)$ [14], $\mathrm{cr}(G) \geq \frac{37}{9}m - \frac{155}{9}(n-2)$ for $5(n-2) \leq m < 6(n-2)$ and $\mathrm{cr}(G) \geq 5m - \frac{203}{9}(n-2)$ for $m \geq 6(n-2)$ by Theorem 4. The claimed bound of $\mathrm{cr}(G) \geq \frac{1500}{41209}\frac{m^3}{n^2} - \frac{54791}{41209}n > \frac{1}{27.48}\frac{m^3}{n^2} - 1.33n$ is in all intervals weaker than these linear bounds except for $m = 4(n-2)$, where they coincide up to a constant. □

One direct application of the improved Crossing Lemma is a new bound on the edge density of k-planar graphs.

Corollary 6. *For $k \geq 2$, any simple k-planar graph with n vertices has at most $3.71\sqrt{k}n$ edges.*

Proof: As in [16], the new bound for k-planar graphs can be derived directly from the new Crossing Lemma and the fact that each edge can be crossed at most k times:

$$\frac{1}{27.48}\frac{m^3}{n^2} \leq \mathrm{cr}(G) \leq km/2,$$

which then leads to $m \leq \sqrt{13.74kn} \leq 3.71\sqrt{k}n$. □

The best previous constant in the bound was 3.81.

4 Proof of Theorems 2 and 3

In this section, we give the proofs of the two central theorems of our paper. As both theorems consider upper bounds for the number of edges for the specific graph classes, we also assume that we consider graphs G that are edge-maximum for the specific class, and for such graphs we assume a corresponding drawing D that is crossing-minimum. These assumptions will enable us to conduct a focused analysis of the bounds for the number of edges.

We prove both theorems by induction. This will allow us to study only drawings that are connected and have no cut vertex that is a crossing in the underlying graph. The approach is similar to [1], but we have to adapt it to the non-simple case, i.e., where multi-edges are allowed.

For $n = 3$, independently from the crossings-dense configurations, there are at most three non-homotopic edges in any drawing and therefore both theorems hold.

Proposition 1. *If D is not connected, then Theorem 2 and Theorem 3 are true.*

Proof: To argue for the different scenarios of Theorem 2 and Theorem 3 at the same time, let $a(n-2)$ for $a \in \{\frac{13}{3}, 4.5, 5\}$ be an upper bound on the number of edges, which we want to prove.

Assume that D is not connected and let D_1 be a connected component of D such that w.l.o.g. the other components of D are within the outerface of D_1 and let D_2 be the union of all other connected components of D. Let $\tilde{D}_1$ be the drawing obtained from D_1 by adding an additional vertex representing the component D_2. Let $\tilde{D}_2$ be the drawing obtained from D_2 by adding an additional vertex representing the component D_1. Note that the additional vertices have degree zero. We denote the corresponding graphs by $\tilde{G}_1, \tilde{G}_2$.

By this procedure, no multi-edges of $\tilde{G}_1$ or $\tilde{G}_2$ are homotopic in $\tilde{D}_1$ or $\tilde{D}_2$, because the added vertex separates the same Jordan arcs in $\tilde{D}_1$ that were separated in D by vertices of D_2 and vice versa.

Let n_i, m_i be the number of vertices and edges of $\tilde{D}_i$. By construction, we have $n_1 + n_2 = n+2 \geq 5$ and w.l.o.g. $n_1 \geq 3$ and $n_2 \geq 2$. For $n_2 = 2$, the component D_2 consists just of one vertex and thus, $m_2 = 0$. By this and in the other cases by induction, $m_i \leq a(n_i - 2)$ holds. Combining this, we get that the number of edges in D is $m \leq a(n_1 - 2) + a(n_2 - 2) = a(n-2)$. □

From now on, we assume that D is connected.

Proposition 2. *If $M(D)$ has a cut vertex that is a crossing in D, then Theorem 2 and Theorem 3 are true.*

Proof: To argue for the different scenarios of Theorem 2 and Theorem 3 at the same time, let $a(n-2)$ for $a \in \{\frac{13}{3}, 4.5, 5\}$ be an upper bound on the number of edges, which we want to prove.

Assume that there is a vertex $x \in V'$ such that $M(D) \setminus \{x\}$ is not connected and assume that x is a crossing in D. Let $\tilde{D}$ be the drawing of the graph $\tilde{G}$ obtained by replacing the crossing x by a vertex. Thus, this adds one vertex and two edges.

Let $\tilde{D}'$ be an arbitrarily connected component of $\tilde{D} \setminus \{x\}$ and let $\tilde{D}''$ be the union of all other connected components. Let $\tilde{G}', \tilde{G}''$ be the corresponding graphs. Let $\tilde{D}_1$ be the drawing induced by the vertices of $\tilde{G}'$ and x, and let $\tilde{D}_2$ be the drawing induced by the vertices of $\tilde{G}''$ and x. Let $\tilde{G}_1, \tilde{G}_2$ be the graphs corresponding to $\tilde{D}_1, \tilde{D}_2$. Note that both include vertex x.

By this procedure, no multi-edges of $\tilde{G}_1$ or $\tilde{G}_2$ are homotopic in $\tilde{D}_1$ or $\tilde{D}_2$ resp., because the vertex x separates the same Jordan arcs in $\tilde{D}_1$ that were separated in D by vertices of $\tilde{G}''$ and vice versa.

Let n_i, m_i be the number of vertices and edges of $\tilde{D}_i$. By construction, we have $n_1 + n_2 = n + 2 \geq 5$ and w.l.o.g., $n_1 \geq 3$ and $n_2 \geq 2$. For $n_2 = 2$, we observe that $n = n_1$ while $\tilde{G}_1$ has one edge more than G. This contradicts that G is maximally dense. For $n_2 \geq 3$, we can apply induction and conclude that G has at most $a(n_1 - 2) + a(n_2 - 2) - 2 = a(n - 2) - 2 < a(n - 2)$ edges. □

Proposition 3. *Let D be a drawing that is either (1) 2-planar F_5^2-free or (2) 2-planar F_5^2-free and F_6^2-free or (3) 3-planar F_6^3-free and maximally-dense-crossing-minimal under this restriction. Then the following properties hold:*

(a) There are no empty lenses.

(b) For all faces $f \in F'$ we have $|f| \geq 3$.

(c) The wedge-neighbor of a 0-triangle or a 1-triangle is a face $f \in F'$ with $|f| \geq 4$ that is not a 0-quadrilateral.

(d) If there are two vertices $u, v \in V(D)$ on the boundary of a face $f \in F'$, then the edge uv is part of the boundary of f. Therefore every face $f \in F'$ with $|V(f)| > 2$ is a 3-triangle.

Proof:

(a) Consider an empty lens that is minimal with respect to containment. Since there are no two homotopic edges, its boundary does not contain two vertices. Hence, the lens can be eliminated by swapping the segments of the edges of D that define the empty lens (without creating one of the crossings-dense configurations). This reduces the number of crossings, contradicting that D is crossing-minimal.

(b) Loops and self-intersecting edges are forbidden, so there is no face $f \in F'$ with $|f| = 1$. Every face $f \in F'$ with $|f| = 2$ is an empty lens, which does not appear in D by (a).

(c) Let f be a wedge-neighbor of an arbitrary 0-triangle or a 1-triangle. By definition, the face f is never a 0-quadrilateral. If $|f| = 3$, then this would imply an empty lens.

(d) For an arbitrary face f, assume that no edge $e = uv$ exists on the boundary of f. Therefore, we may insert e contradicting that D is maximally dense. By this, we cannot create one of the three crossings-dense configurations F_5^2, F_6^2 and F_6^3, since they do not contain planar edges. This does not create homotopic edges as every other edge $e' = uv$ homotopic to e would have been already on the boundary of f or would have formed an empty lens with an edge of the boundary of f contradicting (a).

Assume now that a face f with $|V(f)| > 2$ exists that is not a 3-triangle. Then we find three vertices in $V(D)$ on the boundary of f, which do not all appear next to each other. We introduce a new edge between two of them, contradicting the maximality of D.

□

We conclude that the boundary of every face is a simple cycle: By Proposition 1, each face boundary is connected and by Proposition 2, no crossing is encountered more than once while traversing it. By Proposition 3, if a vertex were encountered more than once on a boundary, then the face would have to be either a 2-face or a 3-triangle. However, in either case this would imply the existence of a loop, contradicting our assumption that loops are not allowed.

In the following, we will use the *discharging method*. See [1, 2, 10, 18] for similar applications of this technique. We define a *charging function* $\mathrm{ch} : F' \to \mathbb{R}$ that assigns an *initial charge* of

$$\mathrm{ch}(f) = |f| + |V(f)| - 4 \tag{1}$$

to every face $f \in F'$. It is known that for the total charge $\sum_{f\in F'} \mathrm{ch}(f) = 4n-8$ holds (refer to [2] for details). The challenge now is to redistribute the charge so that in the end every face $f \in F'$ has a charge of $\mathrm{ch}'(\cdot)$ that satisfies $\mathrm{ch}'(f) \geq \alpha|V(f)|$ for a suitable $\alpha > 0$, while the total charge does not change. From this and the observation that $\sum_{f\in F'} |V(f)| = \sum_{v\in V(D)} \deg(v) = 2m$ holds, we can derive an upper bound of

$$m \leq \frac{2}{\alpha}(n-2) \tag{2}$$

on the number of edges. For a given α and a face f with current charge c, we say that $|c-\alpha|V(f)||$ is the *demand* of f, if $c - \alpha|V(f)|$ is negative, otherwise we call it the *excess* of f. If f has no demand, then we also say that f is satisfied.

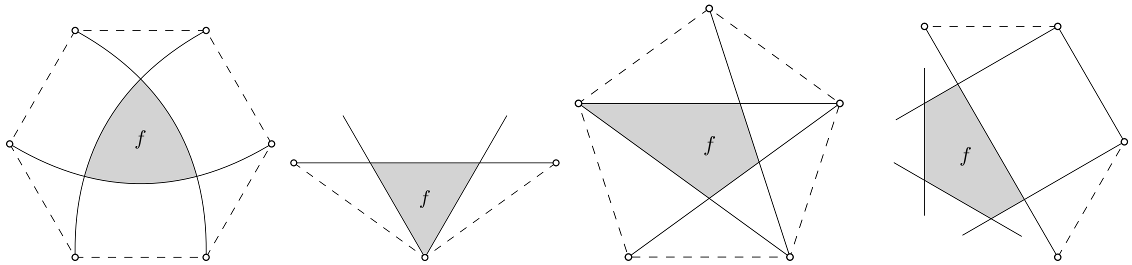
4.1 Proof and Discharging for Theorem 2

Theorem 2. *Any graph G with $n \geq 3$ vertices that admits a 2-planar F_5^2-free drawing has at most $4.5(n-2)$ edges. If the drawing is also F_6^2-free, then G has at most $\frac{13}{3}(n-2)$ edges. These bounds are tight.*

Proof: We start with the bound of $\frac{13}{3}(n-2)$, such that $\alpha = \frac{6}{13}$. Let D be a 2-planar, F_5^2-free and F_6^2-free drawing that is maximally-dense-crossing-minimal. Assign to every face $f \in F'$ the initial charge $\mathrm{ch}(f)$ according to Eq. (1). The initial charges are distributed in the following steps that are executed one after the other:

1. Each 0-triangle receives $\frac{1}{3}$ units of charge from each of its wedge-neighbors.
2. Each 1-triangle receives $\frac{1}{26}$ units of charge from each of its 1-neighbors.
3. Each 1-triangle receives $\frac{5}{13}$ units of charge from its wedge-neighbor.
4. Each 2-quadrilateral contributes its excess to its wedge-neighbor.
5. For each 2-triangle f, let $\mathcal{C}(f)$ be the inclusion-minimal planar cycle of D enclosing f (i.e., the planar cycle that does not contain other planar cycles). Then f distributes its excess equally over those faces that lie inside $\mathcal{C}(f)$ and have a demand.

Denote the charges after the i-th step by $\mathrm{ch}_i(\cdot)$. With this, we have $\mathrm{ch}'(\cdot) = \mathrm{ch}_5(\cdot)$.



(a) (b) (c) (d)

Figure 4: Discharging for Theorem 2. Planar edges that exist by Proposition 3 are dashed.

Proposition 4. *For all faces $f \in F'$, we have* $\mathrm{ch}'(f) \geq \frac{6}{13}|V(f)|$.

Proof: We analyze the final charge $\mathrm{ch}'(\cdot)$ for all faces. Note that a face contributes through each edge of its boundary in Steps 1–3 at most once and the only contributing faces in Step 1 are 2-quadrilaterals (see Fig. 4a) and in Step 2 2-triangles (see Fig. 4b). Also $\mathrm{ch}_3(f) \geq \frac{6}{13}|V(f)|$ already implies $\mathrm{ch}'(f) \geq \frac{6}{13}|V(f)|$. Because of Proposition 3 there are only 3-triangles and faces f with $|f| \geq 3$ and $|V(f)| \leq 2$.

- *f is a 0-triangle.* Then f receives in Step 1 $3 \cdot \frac{1}{3}$ units of charge and never contributes charge. Therefore $\mathrm{ch}_3(f) = -1 + 1 = 0 \geq \frac{6}{13} \cdot 0$.

- *f is a 1-triangle.* Then f receives in Step 2 $2 \cdot \frac{1}{26}$ units of charge, in Step 3 $\frac{5}{13}$ charge and never contributes charge. Therefore $\mathrm{ch}_3(f) = 0 + \frac{6}{13} \geq \frac{6}{13} \cdot 1$.

- *f is a 2-triangle.* Then f starts with 1 unit of charge and contributes in Step 2 at most $2 \cdot \frac{1}{26}$ units of charge. Therefore $\mathrm{ch}_3(f) \geq 1 - \frac{1}{13} = \frac{12}{13} \geq \frac{6}{13} \cdot 2$.

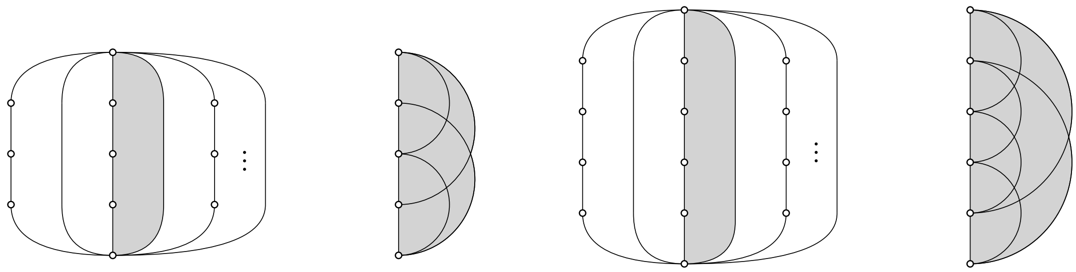
- *f is a 3-triangle.* Then f never receives or contributes charge. Thus $\mathrm{ch}_3(f) = 2 \geq \frac{6}{13} \cdot 3$.

- *f is a 0-quadrilateral.* Then f starts with 0 charge and never receives or contributes charge as it cannot be the wedge-neighbor of another face. Therefore $\mathrm{ch}_3(f) = 0 \geq \frac{6}{13} \cdot 0$.

- *f is a 1-quadrilateral.* Then f starts with 1 unit of charge. If f contributes in Step 3 to less than two 1-triangles, we have $\mathrm{ch}_3(f) \geq 1 - \frac{5}{13} = \frac{8}{13} \geq \frac{6}{13} \cdot 1$. Otherwise, we know that f is bounded by a 5-cycle of planar edges (Fig. 4c). Here, charges do not change in Step 4, but we can find $\frac{3}{13}$ units of charge from the excesses of 2-triangles in this 5-cycle and move that to f in Step 5. Therefore, we have $\mathrm{ch}'(f) = 1 - 2 \cdot \frac{5}{13} + \frac{3}{13} = \frac{6}{13} \geq \frac{6}{13} \cdot 1$.

- *f is a 2-quadrilateral.* Then f has one wedge-neighbor, to which it may contribute $\frac{1}{3}$ units of charge in Step 1 or $\frac{5}{13}$ units of charge in Step 3. So we have $\mathrm{ch}_3(f) \geq 2 - \frac{5}{13} = \frac{21}{13} \geq \frac{6}{13} \cdot 2$.

- *f is a 0-pentagon.* Note that all wedge-neighbors of f are 1-triangles or 2-quadrilaterals, as otherwise there would be an edge with three crossings or a face with two real vertices that are not connected by an edge. If f contributes to five 1-triangles in Step 3, then we would have an F_5^2 configuration, which is forbidden. Otherwise, at least one 2-quadrilateral contributes its excess of $\frac{14}{13}$ to f in Step 4 (see Fig. 4d). Therefore we have $\mathrm{ch}_4(f) \geq 1 + \frac{14}{13} - 4 \cdot \frac{5}{13} = \frac{7}{13} \geq \frac{6}{13} \cdot 0$.



(a) (b) (c) (d)

Figure 5: Lower bound constructions for (a)-(b) F_5^2-free and (c)-(d) additionally F_6^2-free 2-planar graphs.

- *f is a 1-pentagon or a 2-pentagon.* Then f contributes to at most three or two 1-triangles resp. in Step 3. Therefore, we have $\mathrm{ch}_3(f) \geq 2-3\cdot\frac{5}{13} = \frac{11}{13} \geq \frac{6}{13}\cdot 1$ resp. $\mathrm{ch}_3(f) \geq 3-2\cdot\frac{5}{13} = \frac{29}{13} \geq \frac{6}{13}\cdot 2$.
- *f is a 0-hexagon.* If f contributes to six 1-triangles in Step 3, then we would have an F_6^2 configuration, which is forbidden. Otherwise, we have $\mathrm{ch}_3(f) \geq 2-5\cdot\frac{5}{13} = \frac{1}{13} \geq \frac{6}{13}\cdot 0$.
- *f is a 1-hexagon resp. 2-hexagon.* Then f contributes to at most four resp. three 1-triangles in Step 3 and we have $\mathrm{ch}_3(f) \geq 3-4\cdot\frac{5}{13} = \frac{19}{13} \geq \frac{6}{13}\cdot 2$.
- *f is a face with $|f| \geq 7$.* Then f may contribute charge to at most $|f|$ wedge-neighbors in Step 3. Therefore $\mathrm{ch}_3(f) \geq |f| + |V(f)| - 4 - \frac{5}{13}\cdot|f| \geq \frac{8}{13}\cdot 7 + |V(f)| - 4 \geq \frac{6}{13}|V(f)|$.

Therefore, all faces $f \in F'$ are satisfied, which proves the proposition. □

Combining Proposition 4 and Eq. (2), $m \leq 2\cdot\frac{13}{6}(n-2)$ is implied, as claimed.

For a corresponding tight lower bound, consider a plane graph on n vertices whose faces are pentagons (see Fig. 5a). By Euler's Formula, it has $\frac{2}{3}(n-2)$ pentagonal faces. We triangulate each pentagon by inserting two edges and add two more arbitrarily chosen edges each (see Fig. 5b). The total number of edges is $3(n-2) + 2\cdot\frac{2}{3}(n-2) = \frac{13}{3}(n-2)$. Clearly, the drawing is 2-planar and does not contain an F_5^2 or F_6^2 configuration.

For drawings, where F_6^2 configurations are allowed, we can use similar discharging steps to prove the bound of $4.5(n-2)$ on the number of edges. Here we set $\alpha = \frac{4}{9}$, and therefore 1-triangles can receive $\frac{1}{18}$ units of charge from each of its 1-neighbors each in Step 2 without creating a demand for any 2-triangles. Therefore, faces have to contribute in Step 3 only $\frac{1}{3}$ units of charge to satisfy all 1-triangles. Now let f be a 0-hexagon that is the wedge-neighbor of six 1-triangles. Starting with 2 units of charge, it contributes at most $6\cdot\frac{1}{3}$ in Step 3, and therefore ends with $0 \geq \frac{4}{9}\cdot 0$ charge. For all other faces we still have enough charge with the same analysis as above.

Therefore, there exists a function $\mathrm{ch}'(\cdot)$ satisfying $\mathrm{ch}'(f) \geq \frac{4}{9}|V(f)|$ for all $f \in F'$, while the total amount of charge is still $4n-8$. By Eq. (2) we get $m \leq 2\cdot\frac{9}{4}(n-2)$.

A tight lower bound can be constructed starting with a plane graph on n vertices, whose faces are hexagons (see Fig. 5c). It has $\frac{1}{2}(n-2)$ hexagonal faces. Inserting an F_6^2 into each hexagon (see Fig. 5d), $3(n-2) + 3\cdot\frac{1}{2}(n-2) = \frac{9}{2}(n-2)$ can be counted.

□

4.2 Proof and Discharging for Theorem 3

Theorem 3. *Any graph G with $n \geq 3$ vertices that admits a 3-planar F_6^3-free drawing has at most $5(n-2)$ edges. This bound is tight.*

Proof: For the upper bound, let D be a 3-planar F_6^3-free drawing that is maximally-dense-crossing-minimal. As in the proof of Theorem 2, we assign the initial charges $\mathrm{ch}(f)$ to the faces of $M(D)$ and redistribute them to achieve a function $\mathrm{ch}'(\cdot)$. We set $\alpha = \frac{2}{5}$. The discharging takes place in seven subsequent steps:

1. Each 0-triangle receives 1 unit of charge from each 0-neighbor that is a 2-quadrilateral.
2. Each 0-triangle with positive demand receives $\frac{1}{3}$ unit of charge from each of its wedge-neighbors.
3. Each 2-triangle distributes its excess equally over each of its 1-neighbors that are 1-triangles.
4. Each 1-triangle receives its demand from its wedge-neighbor.
5. Each face distributes its excess equally over its wedge-neighbors that are 0-pentagons, but at most 0.3 to each of them, and keeps the rest.
6. Each face distributes its excess equally over each of its vertex-neighbors that are 0-quadrilaterals or 0-pentagons. The 0-quadrilaterals distribute this charge equally over their 0-neighbors that have a demand.
7. For each face f, let $\mathcal{C}(f)$ be the inclusion-minimal planar cycle of D enclosing f (i.e., the planar cycle that does not contain other planar cycles). Then f distributes its excess equally over those faces that lie inside $\mathcal{C}(f)$ and have a demand.

Again, we denote by $\mathrm{ch}_i(\cdot)$ the charges after the i-th step and by $\mathrm{ch}'(\cdot)$ the final charges. Our goal is to show $\mathrm{ch}'(f) \geq 0.4|V(f)|$ for all faces $f \in F'$. Note that this is already implied by $\mathrm{ch}_4(f) \geq 0.4|V(f)|$, as in Steps 5–7 faces contribute only their excesses. We structure the proof into several propositions, collecting statements about the discharging steps.

Proposition 5. *After Step 2, 0-triangles are and remain satisfied.*

Proof: Let f be a 0-triangle. We have $\mathrm{ch}(f) = -1$. If f receives in Step 1 charge, then $\mathrm{ch}_1(f) = 0$. Otherwise, f receives $3 \cdot \frac{1}{3}$ charge in Step 2, so $\mathrm{ch}_2(f) = 0$. 0-triangles do not contribute charge in Steps 3 and 4, since they are not wedge-neighbors of 1-triangles. Therefore, $\mathrm{ch}'(f) \geq 0.4 \cdot 0$ holds. □

Proposition 6. *In Steps 1 and 2, 0-faces contribute no charge.*

Proof: No faces except 2-quadrilaterals contribute charge in Step 1, so we consider only Step 2. Assume that a 0-face f_0 contributes charge to a 0-triangle f in Step 2, and f_0 and f are therefore 0-neighbors at an edge e_0. Let e_1, e_2 be the other edges of f and let f_1, f_2 be the 0-neighbors at these edges (see Fig. 6a). Since f_0 is a 0-face, $\overline{e}_1$ and $\overline{e}_2$ each have two crossings at f_0, and they also cross each other at f. Therefore $\overline{e}_1$ and $\overline{e}_2$ have already three crossings and end at f_1 resp. f_2. The edge $\overline{e}_0$ ends also at one of f_1 or f_2, as otherwise it would have four crossings. W.l.o.g., e_0 ends at f_1 and by Proposition 3 f_1 is a 2-quadrilateral. Hence, $\mathrm{ch}_1(f) \geq 0.4 \cdot |V(f)|$, contradicting that f receives charge later. □

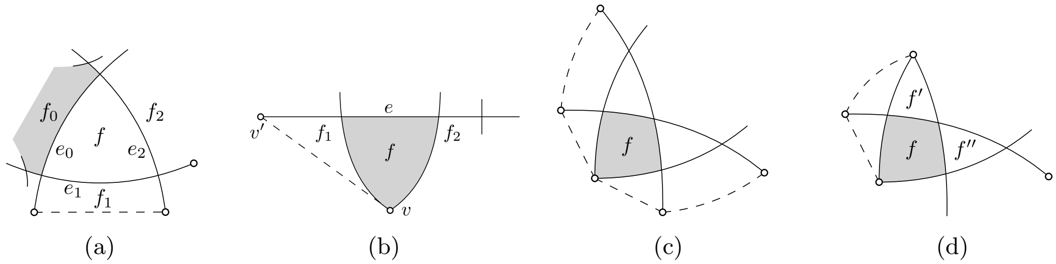


Figure 6: Illustrations for the proofs of Propositions 6–8.

Proposition 7. *After Step 3, 1-triangles have a demand of at most* 0.3 *units of charge.*

Proof: Let f be a 1-triangle with the real vertex v and the 0-edge e. Let further f_1, f_2 be the 1-neighbors of f (see Fig. 6b). Then $\overline{e}$ ends at one of f_1 and f_2, as otherwise it would have more than three crossings. W.l.o.g., let f_1 be that face with the vertex v' to which $\overline{e}$ is incident. Then by Proposition 3 the edge vv' exists and f_1 is a 2-triangle. Therefore, f_1 starts with 1 unit of charge and has an initial excess of 0.2. Thus, f receives at least 0.1 units of charge in Step 3. We have $\mathrm{ch}_3(f) \geq 0.1$, which is equivalent to a demand of at most 0.3. □

Proposition 8. *After Step 4, all 1-quadrilaterals are satisfied.*

Proof: Let f be a 1-quadrilateral. We have $\mathrm{ch}(f) = 1$ and f contributes charge only in Steps 2 and 4. If f contributes to at most one wedge-neighbor or to two 1-triangles, then $\mathrm{ch}_4(f) \geq 1 - 0.6 \geq 1 \cdot 0.4$. Otherwise, f contributes either to two wedge-neighbors that are both 0-triangles or to one 0-triangle and one 1-triangle. In the first case, both 0-triangles are already satisfied after Step 1, as they have wedge-neighbors that are 2-quadrilaterals (see Fig. 6c). For the second case, note that if f and the 1-triangle f' are not 0-neighbors, then f does not contribute charge to the 0-triangle f'', because f'' has a 0-neighbor that is a 2-quadrilateral. So assume f and f' are 0-neighbors. f contributes not more than 0.2 units of charge to the f', because one of its 1-neighbors is a 2-triangle contributing its excess of 0.2 units of charge only to f' in Step 3 (see Fig. 6d). Therefore, we have $\mathrm{ch}_4(f) \geq 1 - \frac{1}{3} - 0.2 \geq 1 \cdot 0.4$. □

Proposition 9. *Let f be any face that is not a 0-pentagon that is the wedge-neighbor of four or five 1-triangles. Then, face f is satisfied after Step 4 and remains satisfied afterwards.*

Proof: Note again that in Steps 5–7, faces only contribute their excess. So for a face f, charge $\mathrm{ch}_4(f) \geq 0.4|V(f)|$ already proves the proposition.

To see that 0-triangles and 1-quadrilaterals are satisfied, we refer to Propositions 5 and 8. 1-triangles are satisfied by definition of Step 4. Remember that only 3-triangles and r-s-gons with $r \leq 2, s \geq 3$ can exist by Proposition 3. Now we discuss the other cases:

- *f is a 2-triangle.* We start with $\mathrm{ch}(f) = 1$. As only wedge-neighbors contribute in Steps 1, 2 and 4 and f cannot be a wedge-neighbor of another face, the only critical step is Step 3. Here, f contributes in total at most its excess of 0.2 units of charge, and therefore $\mathrm{ch}_4(f) \geq 1 - 0.2 \geq 2 \cdot 0.4$.

- *f is a 3-triangle.* We start with $\mathrm{ch}(f) = 2$ and f never contributes charge. It follows that $\mathrm{ch}_4(f) = 2 \geq 3 \cdot 0.4$ holds.

- *f is a 0-quadrilateral.* Again, f never contributes charge, and therefore $\mathrm{ch}(f) = \mathrm{ch}_4(f) = 0 \geq 0 \cdot 0.4$ holds.
- *f is a 2-quadrilateral.* We start with $\mathrm{ch}(f) = 2$. Note that f contributes only once as it has only one wedge-neighbor, and therefore we have $\mathrm{ch}_4(f) \geq 2 - 1 \geq 2 \cdot 0.4$.
- *f is a 0-pentagon with at most three wedge-neighbors that are 1-triangles.* We have $\mathrm{ch}(f) = 1$ and f contributes no charge to 0-triangles by Proposition 6. With Proposition 7, $\mathrm{ch}_4(f) \geq 1 - 3 \cdot 0.3 \geq 0 \cdot 0.4$ follows.
- *f is a 1-pentagon or a 2-pentagon.* f starts with $\mathrm{ch}(f) \geq 2$ and we have $\mathrm{ch}_4(f) \geq 2 - 3 \cdot \frac{1}{3} \geq 2 \cdot 0.4$.
- *f is a face with $|f| \geq 6$.* Then f may contribute to at most $|f|$ wedge-neighbors charge. Therefore, we have $\mathrm{ch}_4(f) \geq |f| + |V(f)| - 4 - \frac{1}{3} \cdot |f| \geq |V(f)|$.

□

It remains to prove that 0-pentagons with four or five wedge-neighbors that are 1-triangles have at least zero charge after Step 7. We show this by the following four propositions, which we only state here; the proofs can be found in Section A.

Proposition 10. *In Step 5, each 0-pentagon receives* 0.3 *units of charge from each of its wedge-neighbors that are not 1-triangles, 0-triangles or 0-pentagons.*

Proposition 11. *In Step 6, each 1-face and 2-face f, in both cases with $|f| \geq 5$, and each 0-face f with $|f| \geq 7$ contributes at least* 0.4 *units of charge to the vertex-neighbors that are 0-quadrilaterals or 0-pentagons.*

Proposition 12. *After Step 7, all 0-pentagons that are the wedge-neighbor of exactly four 1-triangles are satisfied.*

Proposition 13. *After Step 7, all 0-pentagons that are the wedge-neighbor of five 1-triangles are satisfied.*

By Propositions 9, 12 and 13, $\mathrm{ch}'(f) \geq 0.4 \cdot |V(f)|$ holds for all faces $f \in F'$. Since charge is only moved, its total amount is still $4n - 8$ and Eq. (2) implies $m \leq \frac{2}{0.4}(n-2) = 5(n-2)$.

The tightness of the bound can easily be seen by considering 2-planar drawings of optimal 2-planar graphs, which are obviously D_6^3-free [9]. □

5 Proof of Theorem 4

In this section, we present the proof of Theorem 4 that shows how to use the earlier stated observations and theorems and leads to a better bound for the Crossing Lemma.

Theorem 4. *Let G be a graph with $n > 2$ vertices and m edges. Then*

(a) $\mathrm{cr}(G) \geq \frac{37}{9}m - \frac{155}{9}(n-2)$,

(b) $\mathrm{cr}(G) \geq 5m - \frac{203}{9}(n-2)$.

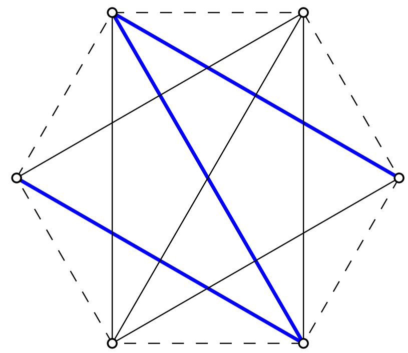


Figure 7: Three independent edges (blue) with three crossings in an F_6^3 configuration. In D_3 one of them is already deleted, in D_{3-} also the other two.

Proof: We begin by proving the bound in (a). If $m \leq 5(n-2)$, then the bound follows from the linear bound $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$ [14]. So assume $m > 5(n-2)$ and let D be a crossing-minimal drawing of G. From D, we iteratively remove the edge with the most crossings until $5(n-2)$ edges are left. In particular, as long as the maximum number of crossings is three, we always remove an edge from an F_6^3 configuration. By Theorem 3, we stop latest, when there are no F_6^3 configurations. By this process, edges are iteratively deleted until we reach $5(n-2)$ edges, as following:

1. m_{5+} edges with five or more crossings – denote the resulting drawing by D_4,
2. m_4 edges with four crossings – denote the resulting drawing by D_3 and the set of edges deleted in this step by E_4,
3. m_3 edges with three crossings from F_6^3 configurations – denote the resulting drawing by D_{3-}.

Note that m_4 or m_3 could be zero in the case that we reached $5(n-2)$ already during step 1 or 2. Afterwards we have m_3 edge-disjoint F_6^3 configurations with a missing edge in D_{3-}. So we are able to find $2m_3$ more independent edges with three crossings and delete them (see Fig. 7). Continue the deletion process by still removing the edge with the most crossings until this edge no longer has three or more crossings; we denote the number of these deleted edges by m_{3-}. Call the achieved drawing D_2. By applying the linear bound from [14] again, we have

$$\begin{aligned} \mathrm{cr}(G) &\geq [5m_{5+} + 4m_4 + 3m_3] + [2 \cdot 3m_3 + 3m_{3-}] + \left[\frac{7}{3}(5(n-2) - 2m_3 - m_{3-}) - \frac{25}{3}(n-2)\right] \\ &= 5m_{5+} + 4m_4 + \frac{13}{3}m_3 + \frac{2}{3}m_{3-} + \frac{10}{3}(n-2). \end{aligned} \tag{3}$$

As all values are non-negative, it is not hard to see that this is at least

$$\geq 4(m_{5+} + m_4 + m_3 + 5(n-2)) - \frac{50}{3}(n-2) = 4m - \frac{50}{3}(n-2).$$

For the better bound of $\mathrm{cr}(G) \geq \frac{37}{9}m - \frac{155}{9}(n-2)$ we have to elaborate on the value m_4, as there was no slack in the last inequality.

As a preparation, we first consider the structure of D_2. Let c_5^2 be the number of F_5^2 configurations and let c_6^2 be the number of F_6^2 configurations in D_2. Let further E_0 be the union of the

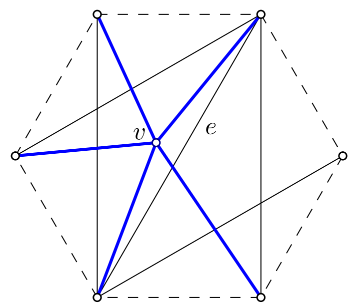


Figure 8: Illustration for the proof of Proposition 14. We augment each F_6^3 configuration after the deletion of the three blue edges in Fig. 7 by one vertex and five edges for the drawing $\tilde{D}$.

(planar) boundary edges of the F_6^3 configurations in D_3 that are not present in D_3, together with the (planar) boundary edges of the F_5^2 and F_6^2 configurations in D_2 that are not present in D_2. We denote $|E_0| = m_0$ and state the following:

Proposition 14. *With the notation above,* $c_5^2 + c_6^2 \geq \frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0$.

Proof: Insert the m_0 missing planar edges to D_2 at the boundaries of the crossings-dense configurations. Further add a vertex v and five edges in every F_6^3 configuration from D_3 as shown in Fig. 8. More precisely, notice that in D_2 three edges have been deleted from each F_6^3 configuration. Those three edges form a path consisting of a 2-hop edge, a 3-hop edge and a second 2-hop edge, where an x-hop edge is an edge whose endpoints are at distance x along the boundary cycle of the F_6^3 configuration. Only one 3-hop edge e still exists and it is crossing-free in D_2. We arbitrarily choose a side of e and place the new vertex v close to e at this side. We realize the five new edges by connecting v to the two vertices of the configuration that are on same side of e, further to the two endpoints of e, and to one of the two endpoints on the opposite side of e. We do not create new crossings-dense configurations by this operation, thus the number of F_5^2 and F_6^2 configurations in D_2 does not change.

As a next step, we remove one edge from each F_5^2 and F_6^2 configuration in D_2 and call this drawing $\tilde{D}$. Note that $\tilde{D}$ is 2-planar, F_5^2-free, F_6^2-free and has $5(n-2) - 2m_3 - m_{3-} + m_0 + 5m_3 - (c_5^2 + c_6^2)$ edges on $n + m_3$ vertices.

Assume we have fewer F_5^2 and F_6^2 configurations in D_2 than stated in the proposition. Then $\tilde{D}$ would have more than

$$5(n-2) - 2m_3 - m_{3-} + m_0 + 5m_3 - \left(\frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0\right) = \frac{13}{3}(n - 2 + m_3)$$

edges, which contradicts the statement of Theorem 2 for $\tilde{D}$. □

Next, we show how to bound the number of the edges in E_4, i.e., the deleted edges that were accounted with four crossings in D_4. For that, we introduce some new definitions.

Let uu' be a boundary edge of a crossings-dense configuration C that exists in D_2 resp., D_3 and is crossed by at least one edge $e \in E_4$ in D_4. Let $\tilde{e}_1$, $\tilde{e}_2$ be the edge-segments enclosing the 2-triangle next to uu' in the crossings-dense configuration so that $\tilde{e}_1$ is crossed more often by edges in E_4 or choose $\tilde{e}_i$ arbitrarily if they have the same number of crossings with edges in E_4. Let uu' be so that u is incident to $\tilde{e}_1$ and u' is incident to $\tilde{e}_2$. We consider the crossing edge, say $e = ab$

that has the crossing c with uu' closest to u. Consider the two segments (a, c) and (c, b) of e, such that (a, c) is completely outside of the crossings-dense configuration C. Then let t be the triangle that is adjacent to uu' and consists of the edge uu', as the second edge, we define the edge that closely follows the two segments (u, c) and (c, a), and as the third edge of t, we take the edge that closely follows the two segments (a, c) and (c, u') (see Fig. 9a). We call t the *charging triangle*[3] of uu' and (a, c) the *pillar* of t.

For the case that $\tilde{e}_1$ is crossed by a second edge $e' \in E_4$, let c' be the crossing of the edges au' and e' and let a' be the endpoint of e' so that ac and $a'c'$ lie on the same side of uu'. If $a \neq a'$, then let t' be the triangle defined by the edges $a'u'$, aa' and au', where aa' closely follows the edge segments (ac') and $(c'a')$ of au' and e' and $a'u'$ closely follows the edge segments $(a'c')$ and $(c'u')$ (see Fig. 9b). We call t' the *charging by-triangle* of uu' and (a', c') the *pillar* of t'.

Additionally, whenever the edges of a charging triangle or a charging by-triangle form an empty lens with edges of G, then we swap the involved edge-segments such that the empty lens is being removed.

Now, we can introduce a triangulation $\mathcal{H}$ on the vertices of G that has the following properties. Edges with property (i) are contained in subset S_i for $1 \leq i \leq 5$:

(1) $\mathcal{H}$ contains the boundary of every F_6^3 configuration in D_3.

(2) $\mathcal{H}$ contains the boundary of every F_5^2 and F_6^2 configuration in D_2.

(3) $\mathcal{H}$ contains every edge in E_4 that lies completely outside of these crossings-dense configurations.

(4) $\mathcal{H}$ contains the edges of all charging triangles.

(5) $\mathcal{H}$ contains the edges of all charging by-triangles.

Note that pillars do not cross a boundary edge, as by this, the corresponding edge $e \in E_4$ would enter a second crossings-dense configuration. Thus, e would have at least two crossings each in the crossings-dense configurations and one with the boundary edge, which contradicts that D_4 is 4-planar.

Claim 1. *A triangulation $\mathcal{H}$ with the properties (1)-(5) always exists.*

Proof: We need to show that the edges of $S_1, ..., S_5$ do not cross each other. The boundary edges of F_6^3 configurations in D_3 and of F_5^2 and F_6^2 configurations in D_2 are planar and therefore edges of S_1 and S_2 do not cross each other. As edges of E_4 do not cross each other, edges of S_3 do not cross other edges of S_3 and by construction they do not cross edges of S_1, S_2.

The edges of S_4 consist of segments (x, y) closely following pillars and segments of boundary edges. Note that every edge crossing (x, y) also crosses the edge segment $(x, y)^*$ closely followed. Therefore, (x, y) crossed by an edge of S_1 or S_2 would imply that either boundary edges cross each other or a pillar crosses a boundary edge, both contradictions. If (x, y) is crossed by an edge of S_3, that would imply that edges of E_4 cross each other or an edge of S_3 crosses a boundary edge, both contradictions. For the case that two edges $(x_1, y_1), (x_2, y_2)$ of S_4 cross each other, it is sufficient to observe that then $(x_1, y_1)^*, (x_2, y_2)^*$ cross and again boundary edges and pillars cannot cross.

For the edges of S_5 we argue as for the edges of S_4. Here, the edges are following pillars or edge segments of S_4. Again, an edge segment (x, y) crossed by an edge e would imply that $(x, y)^*$

[3]The term reflects the role of t in the discharging argument used later to bound the size of E_4.

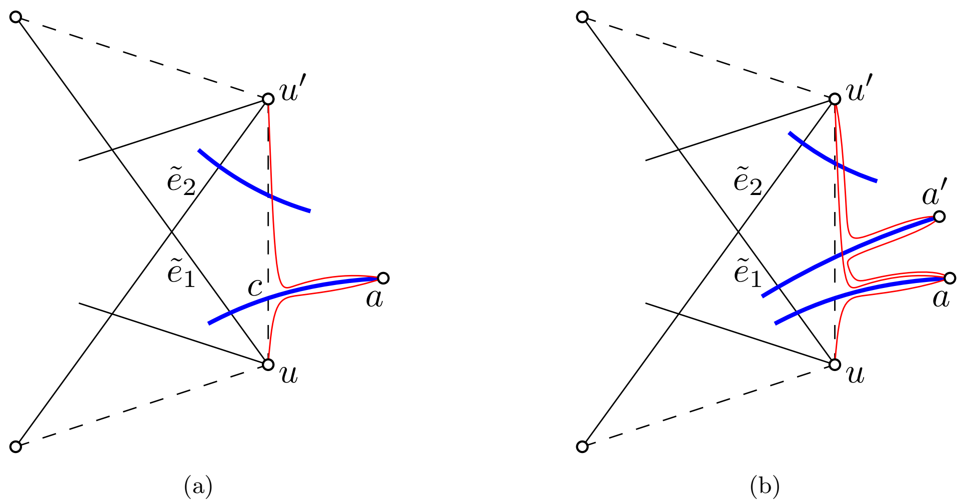


Figure 9: The construction of a charging triangle (a) and a charging by-triangle (b) of uu'.

is crossed by e. Thus, an edge of S_5 crossed by an edge of $S_1, ..., S_5$ would imply that edges of $S_1, ..., S_4$ cross each other or a pillar, which is a contradiction. □

Proposition 15. *Let $\mathcal{H}'$ be the set of triangles induced by the triangulation $\mathcal{H}$ so that the edges of the triangles are edge-disjoint from any crossings-dense configuration. Let $c_\triangle = |\mathcal{H}'|$. Then $m_4 \leq m_3 + c_6^2 + 4m_0 + 2c_\triangle$.*

The proof can be found in Section B. Combining the results of the propositions, we can finish the first part of the proof of Theorem 4. Proposition 14 implies

$$c_\triangle \leq 2(n-2) - 4m_3 - \left(\frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0\right) \cdot 3 - c_6^2 = 3m_{3-} - 3m_0 - c_6^2,$$

because the total number of triangles is $2(n-2)$ and a pentagon resp. hexagon contains three resp. four triangles. Together with Proposition 15, this gives

$$m_4 \leq m_3 + c_6^2 + 4m_0 + 2(3m_{3-} - 3m_0 - c_6^2) \leq 2m_3 + 6m_{3-}.$$

Multiplying this term by $\frac{1}{9}$ and adding it to Eq. (3), we get as desired

$$\begin{aligned} \mathrm{cr}(G) &\geq 5m_{5+} + 4m_4 + \frac{13}{3}m_3 + \frac{2}{3}m_{3-} + \frac{10}{3}(n-2) + \frac{m_4 - (2m_3 + 6m_{3-})}{9} \\ &\geq \frac{37}{9}(m_{5+} + m_4 + m_3) + \frac{10}{3}(n-2) \\ &= \frac{37}{9}(m_{5+} + m_4 + m_3 + 5(n-2)) - \frac{155}{9}(n-2) \end{aligned}$$

and therefore (a) is proved.

For the bound in (b), see the following: If $m \leq 6(n-2)$, then we can apply the bound of (a). So let $m > 6(n-2)$. Iteratively delete the edge with the most crossings in a crossing-minimal drawing D until $6(n-2)$ edges are left; these edges have at least five crossings, as the density of 4-planar graphs is $\leq 6(n-2)$ [1]. With the bound in (a), this implies

$$\mathrm{cr}(G) \geq 5(m-6(n-2)) + \frac{37}{9} \cdot 6(n-2) - \frac{155}{9}(n-2) = 5m - \frac{203}{9}(n-2).$$

□

6 Discussion

We have improved the leading constant of the lower bound for the crossing number of a given graph G. Although this improvement does not seem to be too impressive at first sight, we worked out some interesting observations for drawings with a limited number of crossings per edge. This leads to further improvements, conjectures and suggestions for future research.

In particular, we have improved for $m > 5(n-2)$ the lower bound of the crossing number, unfortunately we did not reach tightness. We still believe that the conjecture $\mathrm{cr}(G) \geq \frac{25}{6}m - \frac{35}{2}(n-2)$ by [14] is true. The corresponding upper bound can be obtained by a construction where the plane subgraph consists only of pentagonal and hexagonal faces [14].

Our improvement compared to the previous version [11] comes from Proposition 15, where we were able to replace $m_4 \leq m_3 + c_6^2 + 4m_0 + 4c_\triangle$ by $m_4 \leq m_3 + c_6^2 + 4m_0 + 2c_\triangle$. Although it seems impossible to get the bound $m_4 \leq m_3 + c_6^2 + 4m_0 + \frac{4}{3}c_\triangle$, from which the conjecture would follow, we think that our general approach in the proof is of interest.

Applying our technique to 4-planar drawings might show that these drawings have density $\leq 5.5(n-2)$, if we forbid the 4-planar crossings-dense configuration created by nine edges inside a planar hexagon. This would provide a characterization of optimal 4-planar graphs, which is a well-known open problem. Further, we can look at 5-planar graphs, a class that has been considered as too complex for actual research. Just applying Corollary 6 improves the current known density bound from $8.52n$ to $8.3n$.

It seems to be worthwhile to apply the idea to bipartite graphs to obtain improvements of the Crossing Lemma for this class of graphs. Here, the corresponding linear bound $\mathrm{cr}(G) \geq 3m - \frac{17}{2}n + 19$ used in the current proof in [5] is not tight.

Furthermore, we have indicated a way how to obtain the exact density bound of optimal simple 3-planar graphs. Note that we only did one step in this direction.

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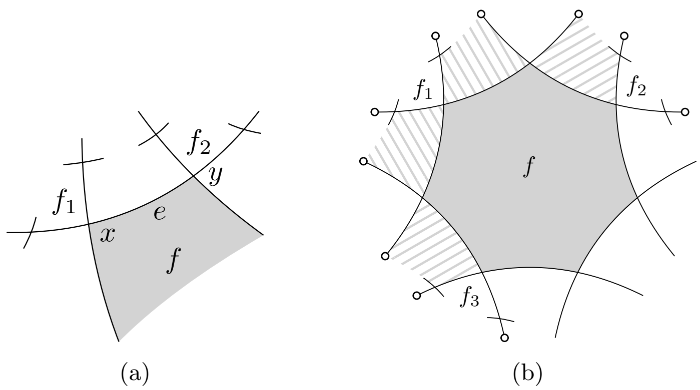


Figure 10: (a) No face contributes to two consecutive vertex-neighbors in Step 6. (b) If a 0-heptagon f contributes to three vertex-neighbors f_1, f_2, f_3 in Step 6, then it contributes not to all its seven wedge-neighbors in Steps 1–5. (c) A 1-pentagon f contributing to each of its three wedge-neighbors in Steps 1–5 and to two vertex-neighbors in Step 6 leads to a contradiction.

A Details for Section 4

Proposition 10. *In Step 5, each 0-pentagon receives* 0.3 *units of charge from each of its wedge-neighbors that are not 1-triangles, 0-triangles or 0-pentagons.*

Proof: Note that in the calculations of Proposition 9, we assumed for all faces that are possibly a wedge-neighbor of a 0-pentagon except 0-triangles, 1-triangles, 1-quadrilaterals and 0-pentagons that they give at least 0.3 charge to all wedge-neighbors. If such a face f has a wedge-neighbor that is a 0-pentagon, then it did not contribute charge to it in Steps 1–4, and therefore has 0.3 charge left for it in Step 5. The only critical case is a 1-quadrilateral f with a 0-pentagon and a 0-triangle f' as wedge-neighbors. Observe that in this case $\mathrm{ch}_1(f') = 0$ already, because there is a 2-quadrilateral next to f' as in Fig. 6c, and therefore $\mathrm{ch}_4(f) = 1$. Thus, f can contribute 0.3 units of charge to the 0-pentagon. □

Proposition 11. *In Step 6, each 1-face and 2-face f, in both cases with $|f| \geq 5$, and each 0-face f with $|f| \geq 7$ contributes at least* 0.4 *units of charge to the vertex-neighbors that are 0-quadrilaterals or 0-pentagons.*

Proof: Let e be a 0-edge of a face f incident to the crossings x and y. Let f_1 be the vertex-neighbor of f at x and let f_2 be the vertex-neighbor at y. If f_1 and f_2 are 0-faces, then $\overline{e}$ has more than three crossings (see Fig. 10a), a contradiction. Therefore, no face can contribute charge through two consecutive crossings on its boundary in Step 6. For a face f, this implies that it can contribute to at most $\left\lfloor \frac{|f|}{2} \right\rfloor$ vertex-neighbors in this step.

Now we distinguish different cases for the face f that might contribute to vertex-neighbors. We start with the case that f is a 0-face. Here, after Step 5, f has an excess of

$$\mathrm{ch}_5(f) \geq |f| - 4 - |f| \cdot 0.3 = 0.7|f| - 4,$$

which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ for $|f| \geq 8$ and therefore enough. If f is a 0-heptagon, then, by the inequality above, there is enough charge if f contributes to at most two vertex-neighbors in Step 6.

So assume that it contributes to three vertex-neighbors. It is not hard to see that there are wedge-neighbors of f with $|f| \geq 4$ and $|V(f)| \geq 1$ (see Fig. 10b), so f contributes in Steps 1–5 to at most six faces. Now we have $\mathrm{ch}_5(f) \geq 3 - 6 \cdot 0.3 = 1.2$, which is sufficient to give three vertex-neighbors 0.4 charge each in Step 6.

Let f now be a 1-face. Then f has an excess of

$$\mathrm{ch}_5(f) - 0.4 \geq |f| + 1 - 4 - (|f| - 2) \cdot \frac{1}{3} - 0.4 = \frac{2}{3}|f| - 3.4 + \frac{2}{3},$$

which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ if $|f| \geq 6$. If f is a 1-pentagon, then, by the inequality above, there is enough charge if f contributes to at most one vertex-neighbor in Step 6. So assume the opposite, i.e., f contributes to two vertex-neighbors f_1, f_2 in Step 6. If f contributes in Steps 1–5 to only one or two wedge-neighbors, then its excess after Step 5 is at least $1.6 - \frac{2}{3}$ charge and therefore sufficient, so assume this is not the case either.

Walking along the boundary of f, let e_0 be a 1-edge of f, let e_1, e_2, e_3 be the 0-edges and let e_4 be the other 1-edge of f. Let further t_i be the wedge-neighbor of f at e_i for $i \in \{1, 2, 3\}$. W.l.o.g., f_1 is adjacent to the crossing of $\overline{e}_0$ and $\overline{e}_1$, and therefore the face t_2 is a 1-triangle, as otherwise f would not contribute charge to t_2 in Steps 1–5 (see Fig. 10c). Therefore, f_2 lies at the crossing of $\overline{e}_3$ and $\overline{e}_4$. Note that $\overline{e}_2$ ends at t_1 or t_3, say w.l.o.g. at t_1. But then $|t_1| \geq 4$ and $|V(t_1)| \geq 1$, so f does not contribute charge to t_1 in Steps 1–5, a contradiction to our assumption. Therefore, 1-pentagons can contribute 0.4 charge to the desired vertex-neighbors.

The last case is that f is a 2-face. Then f has an excess of

$$\mathrm{ch}_5(f) - 0.8 \geq |f| + 2 - 4 - (|f| - 3) \cdot \frac{1}{3} - 0.8 \geq \frac{2}{3}|f| - 1.8,$$

which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ for $|f| \geq 5$. □

Proposition 12. *After Step 7, all 0-pentagons that are the wedge-neighbor of exactly four 1-triangles are satisfied.*

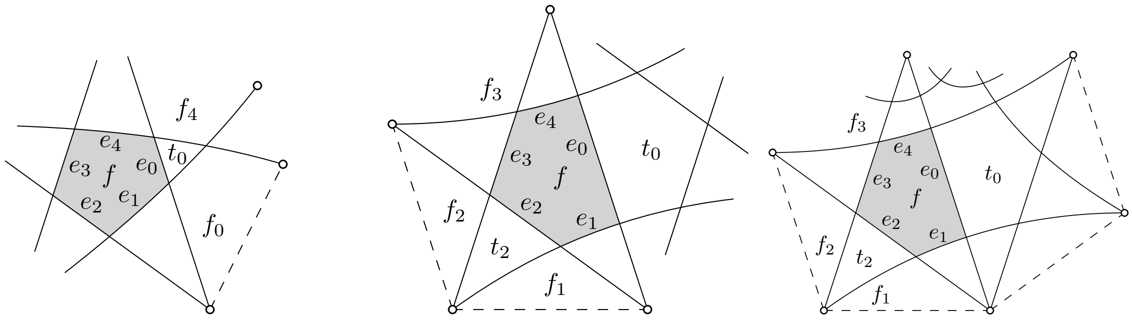
Proof: We introduce a notation for the edges and faces on the boundary of a 0-pentagon f. Let $e_i, i \in \{0, ..., 4\}$ be the edges forming the boundary of f, so that $\overline{e}_i$ and $\overline{e}_{(i+1 \bmod 5)}$ have a crossing at f. Further we denote by t_i the wedge-neighbor of f at e_i and by f_i the vertex-neighbor of f at the crossing of $\overline{e}_i$ and $\overline{e}_{(i+1 \bmod 5)}$.

Let f be a 0-pentagon with four wedge-neighbors that are 1-triangles. So we have $\mathrm{ch}_4(f) \geq 1 - 4 \cdot 0.3 = -0.2$. W.l.o.g., let t_0 be the wedge-neighbor of f that is not a 1-triangle. If t_0 is not a 0-triangle or 0-pentagon, then it contributes, by Proposition 10, 0.3 units of charge to f in Step 5 and f is satisfied. Otherwise, distinguish between the type of the face t_0.

- *Case 1: t_0 is a 0-triangle.* Observe that $\overline{e}_1$ and $\overline{e}_4$ already have three crossings and $\overline{e}_0$ two crossings. Therefore, $\overline{e}_0$ ends at f_0 or f_4, say w.l.o.g. f_0, so f_0 is a 2-quadrilateral (see Fig. 11a). Then f_0 has an excess of 0.2 after Step 5, as it only contributes in Step 1 charge. Note that the vertex-neighbors of f_0 are f and f_4. Since f_4 is not a 0-face, f_0 contributes its excess of 0.2 charge in Step 6 only to f, and therefore $\mathrm{ch}_6(f) \geq 0$.

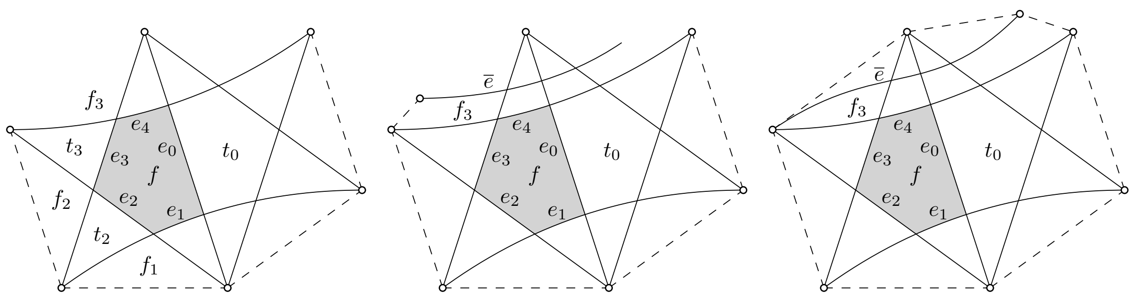
- *Case 2: t_0 is a 0-pentagon.* Again, $\overline{e}_1$ and $\overline{e}_4$ already have three crossings and $\overline{e}_0$ two crossings. Therefore, f_2 is a 2-triangle and also one of f_1 and f_3, say w.l.o.g. f_1 (see Fig. 11b). So we have $\mathrm{ch}_3(t_2) = -0.2$, and therefore $\mathrm{ch}_4(f) \geq 1 - 3 \cdot 0.3 - 0.2 \geq -0.1$.

  Note that the only face besides f that may receive charge from t_0 in Step 5 is f_4. Therefore, we distinguish two cases:



(a) (b) (c)

Figure 11: Illustrations for the proof of Proposition 12 Cases 1 and 2.1.



(a) (b) (c)

Figure 12: Illustrations for the proof of Proposition 12 Case 2.2.

– *Case 2.1: f_4 is a 0-pentagon.* If less than three wedge-neighbors of t_0 are 1-triangles, then $\mathrm{ch}_4(t_0) \geq 0.4$ and f receives enough charge in Step 5. If three wedge-neighbors of t_0 are 1-triangles, then the 1-triangle at the vertex at which $\overline{e}_1$ ends has -0.2 charge after Step 3, as it lies between two 2-triangles (see Fig. 11c). Therefore, we have $\mathrm{ch}_4(t_0) = 1 - 2 \cdot 0.3 - 0.2 = 0.2$ and f can receive half of it in Step 5, which is enough.

– *Case 2.2: f_4 is not a 0-pentagon.* If $\mathrm{ch}_4(t_0) \geq 0.1$, then f receives its missing charge already in Step 5. So assume the opposite, which implies that four wedge-neighbors of t_0 are 1-triangles (see Fig. 12a).

  If now f_3 is a 2-triangle, then $\mathrm{ch}_3(t_3) = -0.2$ and we have $\mathrm{ch}(f) = 1-2\cdot 0.3-2\cdot 0.2 = 0$ and f never has a demand. Otherwise, there is an edge $\overline{e}$ that crosses $\overline{e}_3$ at f_3 and has already three crossings. Thus, f_3 is either a 2-quadrilateral or a 1-triangle. In the first case, f_3 has an excess of at least 0.2 and only contributes it to f in Step 6 (see Fig. 12b). In the second case, we have a planar cycle of length seven, in which all faces except f are satisfied after Step 6 (see Fig. 12c). Here, $\mathrm{ch}_5(t_0) = 0$ and t_0 receives 0.9 charge in Step 6 from its vertex-neighbor that is a 2-quadrilateral. Thus, t_0 has an excess of 0.9, which is contributed to f in Step 7. In all cases f is satisfied after Step 7.

□

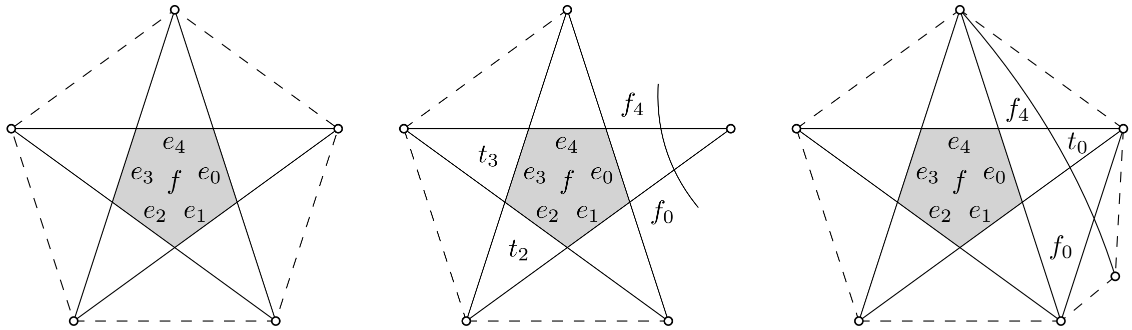


Figure 13: Illustrations for the proof of Proposition 13 Cases 1 and 2.

Proposition 13. *After Step 7, all 0-pentagons that are the wedge-neighbor of five 1-triangles are satisfied.*

Proof: We continue to use the notation introduced in the proof of Proposition 12. Let f be a 0-pentagon with five wedge-neighbors that are 1-triangles. We distinguish the number of 0-neighbors of f that are 0-quadrilaterals. Note that at most two such faces can exist next to a 0-pentagon.

- *Case 1: No 0-neighbor of f is a 0-quadrilateral.* Then all five vertex-neighbors are 2-triangles (see Fig. 13a) and we have $\mathrm{ch}'(f) = \mathrm{ch}_4(f) = 1 - 5 \cdot 0.2 = 0$.

- *Case 2: Exactly one 0-neighbor of f is a 0-quadrilateral.* Assume w.l.o.g. that this 0-quadrilateral lies at e_0. We consider the wedge-neighbors t_2 and t_3 of f (see Fig. 13b). Observe that $\mathrm{ch}_3(t_2) = \mathrm{ch}_3(t_3) = -0.2$, and therefore $\mathrm{ch}_4(f) \geq 1 - 3 \cdot 0.3 - 2 \cdot 0.2 = -0.3$. Note further that f_0 and f_4 cannot be 0-faces, and therefore, by Proposition 11, f receives the missing charge in Step 6 if $|f_0|$ or $|f_4|$ is at least five. The same holds if one of f_0 and f_4 is a 2-quadrilateral, because in this case there is an excess of more than 0.3 charge after Step 5, which is only contributed to f (their other vertex-neighbor is t_0, which does not receive charge in Step 5, see Fig. 13b).

  Further, f_0 and f_4 cannot be 2-triangles or 3-triangles. If both are 1-triangles, then there would be homotopic multi-edges, which is not allowed. If both are 1-quadrilaterals, the existence of an edge with four crossings would be implied. So the last case to consider is when one of them – w.l.o.g. f_4 – is a 1-triangle and the other – therefore f_0 – is a 1-quadrilateral. If f_4 is the only wedge-neighbor, to which f_0 contributes in Steps 1–5, then it contributes its excess of 0.3 units of charge to f in Step 6 and f is satisfied (the other vertex-neighbors of f_0 are not 0-faces and thus receive no charge in Step 6). Otherwise, the second wedge-neighbor of f_0 is also a 1-triangle and we have a planar cycle of length six (see Fig. 13c). Here, f_0 contributes 0.1 units of charge to f in Step 6 and the 1-neighbor of f_0 that is a 2-triangle can contribute its excess of 0.2 to f in Step 7. Therefore, we have $\mathrm{ch}'(f) \geq 0$.

- *Case 3: Exactly two 0-neighbors of f are 0-quadrilaterals.* W.l.o.g., one 0-quadrilateral is at e_0. If the other 0-quadrilateral would be at e_2 (resp. e_3), then $\overline{e}_1$ (resp. $\overline{e}_4$) would have four crossings. Therefore, we can assume w.l.o.g. that the second 0-quadrilateral is at e_4. Here, we have $\mathrm{ch}_3(t_2) = -0.2$ as f_1 and f_2 are 2-triangles, thus $\mathrm{ch}_4(f) \geq 1 - 4 \cdot 0.3 - 0.2 = -0.4$ (see Fig. 14a).

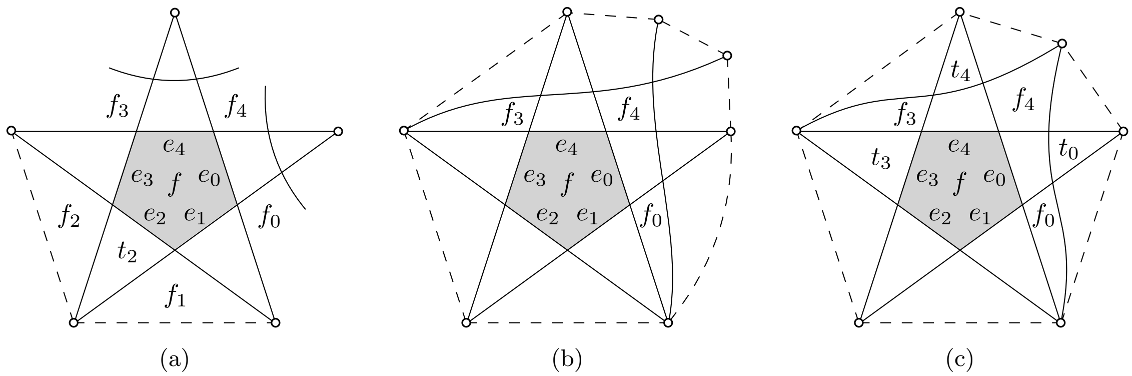
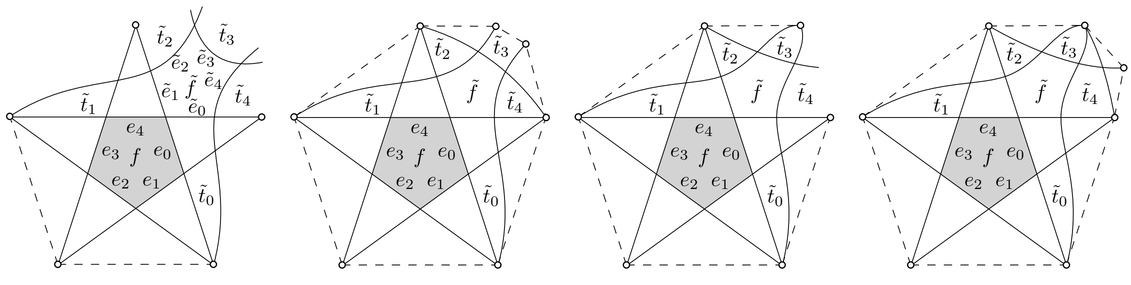


Figure 14: Illustrations for the proof of Proposition 13 Cases 3, 3.1 and 3.2.

We consider the type of the vertex-neighbor f_4. Note that $|f_4| \geq 4$ and f_4 cannot be a 2-quadrilateral. If f_4 is not a 0-quadrilateral, 1-quadrilateral, 0-pentagon or 0-hexagon, then, by Proposition 11, f_4 contributes 0.4 charge to f in Step 6, and therefore f is satisfied. The other cases are more complex, but they all have in common that if one of f_0 and f_3 is a 2-quadrilateral, then it has an excess of at least $2 - 2 \cdot 0.4 - 0.3 = 0.9$ charge after Step 5 and this is enough to ensure $\mathrm{ch}_6(f) \geq 0$.

- *Case 3.1: f_4 is a 0-quadrilateral.* The only case to consider is that f_0 and f_3 are 1-triangles. This directly implies a planar cycle of length seven (see Fig. 14b). We make use of the second part of Step 6 and have two 2-quadrilaterals contributing 0.9 charge each to the 0-neighbors of f at e_0 and e_4, which then is moved to f. Therefore, f is satisfied after Step 6.
- *Case 3.2: f_4 is a 1-quadrilateral.* Then t_0 and t_4 each receive 0.2 charge in Step 3 from a 2-triangle that is a 1-neighbor of f_4. Therefore we have $\mathrm{ch}_4(f) \geq 1-2\cdot 0.3-3\cdot 0.2 = -0.2$. If now f_4 contributes to less than two 1-triangles in Step 4, f receives from f_4 enough charge in Step 6. Otherwise, f_0 and f_3 are 1-triangles, implying a planar cycle of length six (Fig. 14c). Here, $\mathrm{ch}_3(t_0) = \mathrm{ch}_3(t_4) = -0.1$ holds because each triangle additionally receives 0.1 charge from its other 1-neighbor in Step 3. Therefore, f contributes only $2 \cdot 0.1 + 0.2 + 2 \cdot 0.3 = 1$ charge and f never has a demand.
- *Case 3.3: f_4 is a 0-pentagon.* We introduce some new notation for f_4 and its wedge-neighbors, likewise for the 0-pentagon f itself: Let $\tilde{f} := f_4$, $\tilde{e}_0$ the edge-segment of $\overline{e}_4$ at $\tilde{f}$, $\tilde{e}_1$ the edge-segment of $\overline{e}_0$ at $\tilde{f}$ and so on (see Fig. 15a). Analogously, we denote by $\tilde{t}_i$ the wedge-neighbor of $\tilde{f}$ at $\tilde{e}_i$ and by $\tilde{f}_i$ the vertex-neighbor at the crossing of $\overline{\tilde{e}}_i$ and $\overline{\tilde{e}}_{(i+1 \mod 5)}$. Note that $\tilde{t}_0 = f_0$ and $\tilde{t}_1 = f_3$ are 1-triangles or 2-quadrilaterals and, as pointed out above, we only have to consider the case that both are 1-triangles.

  Observe that f is the only vertex-neighbor of f_4 that may receive charge from f_4 in Step 6, as all its other vertex-neighbors cannot be 0-faces. Distinguish the number of 1-triangles that are wedge-neighbors of f_4. The wedge-neighbors of f_4 can never be 0-triangles or 0-pentagons, so, by Proposition 10, they contribute 0.3 charge to f_4 if they are not 1-triangles. If three or less wedge-neighbors of f_4 are 1-triangles, $\mathrm{ch}_5(f_4) \geq 1-3\cdot 0.3+2\cdot 0.3 \geq 0.7$, which then is contributed to f in Step 6 implying $\mathrm{ch}_6(f) \geq 0$. If all five wedge-neighbors of f_4 are 1-triangles, we have the forbidden F_6^3 configuration. In the



(a) (b) (c) (d)

Figure 15: Illustrations for Case 3.3 in the proof of Proposition 13.

last case, four wedge-neighbors of f_4 are 1-triangles. Here, $\mathrm{ch}_5(f_4) \geq 1-4\cdot 0.3+0.3 = 0.1$ holds and this charge is contributed to f in Step 6, so only 0.3 charge is missing for f. By symmetry, $\tilde{t}_2$ is w.l.o.g. a 1-triangle. If $\tilde{t}_3$ is the wedge-neighbor of $\tilde{f}$ that is not a 1-triangle, then it must be 2-quadrilateral and this implies a planar cycle of length seven, in which f is the only face with a demand after Step 6 (see Fig. 15b). The 2-quadrilateral $\tilde{t}_3$ has an excess of 0.9 charge after Step 6 and contributes it in Step 7 to f. Therefore, f is satisfied.

So assume now that $\tilde{t}_3$ is a 1-triangle and $\tilde{t}_4$ is the wedge-neighbor that is not a 1-triangle (see Fig. 15c). Note that $\tilde{t}_4$ is not a 0-face. So for all cases, except that $\tilde{t}_4$ is a 1-quadrilateral or 2-quadrilateral, Proposition 11 guarantees that $\tilde{t}_4$ contributes in Step 6 0.4 charge to all its vertex-neighbors. In particular, the 0-neighbor of f at e_0 receives 0.4 charge and gives it completely to f. Thus, in this case, f is satisfied.

If $\tilde{t}_4$ is a 2-quadrilateral, then we have $\mathrm{ch}_5(t_4) = 0.9$ and it contributes in the same way enough charge to f via the 0-neighbor of f at e_0. This works also if $\tilde{t}_4$ is a 1-quadrilateral contributing to only one wedge-neighbor (namely $\tilde{f}$) in Steps 1–5.

In the last case, $\tilde{t}_4$ is a 1-quadrilateral and it contributes in Steps 1–5 to $\tilde{f}$ and another wedge-neighbor $\tilde{f}_3$ which must be a 1-triangle. This implies a planar cycle of length seven (see Fig. 15d). Here, $\tilde{t}_4$ and its 1-neighbor that is a 2-triangle have an excess of 0.1 resp. 0.2 charge after Step 5 and contribute it to f later on. Thus, f is satisfied.

– *Case 3.4: f_4 is a 0-hexagon.* Note that no wedge-neighbor of f_4 can be a 0-face, so f_4 contributes no charge in Steps 1–3 and 5 (see Fig. 16a). If at most four wedge-neighbors of f_4 are 1-triangles, then $\mathrm{ch}_4(f_4) \geq 2 - 4 \cdot 0.3 = 0.8$ holds by Proposition 10 and $\mathrm{ch}_5(f_4) = \mathrm{ch}_4(f_4)$. In this case, there is at most one other vertex-neighbor of f_4 besides f that can be a 0-quadrilateral or 0-pentagon and f_4 can contribute to both 0.4 charge in Step 6. That is enough to satisfy f.

  If five wedge-neighbors of f_4 are 1-triangles, then no vertex-neighbor of f_4 except f is a 0-face. Therefore, f receives the excess of f_4 in Step 6, which is at least $2-5\cdot 0.3 = 0.5$. So again f is satisfied.

  Assume now that all six wedge-neighbors of f_4 are 1-triangles (see Fig. 16b). Then two of them have a demand of only 0.2 after Step 3 as they have two 1-neighbors that are 2-triangles. Therefore, $\mathrm{ch}_5(f_4) \geq 2 - 4 \cdot 0.3 - 2 \cdot 0.2 = 0.4$. Here, f is the only face to which f_4 contributes in Step 6 and we have $\mathrm{ch}_6(f) \geq 0$.

□

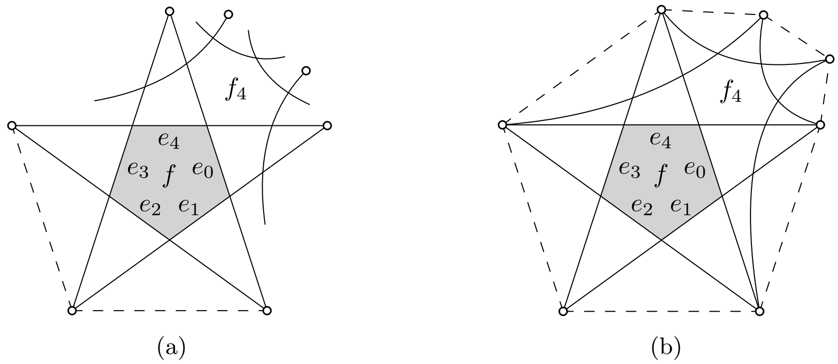


Figure 16: Illustrations for Case 3.4 in the proof of Proposition 13.

B Details for Section 5

We recall the notation. In the first part of the proof of Theorem 4, we give an algorithm where we denote intermediate drawings D_i for different values of i. Let c_5^2 be the number of F_5^2 configurations and c_6^2 the number of F_6^2 configurations in D_2. Let further E_0 be the set of the (planar) boundary edges of the F_6^3 configurations in D_3 resp., of the F_5^2 and F_6^2 configurations in D_2 that do not exist in D_2 and therefore may be added. We denote $|E_0| = m_0$.

Proposition 15. *Let $\mathcal{H}'$ be the set of triangles induced by the triangulation $\mathcal{H}$ so that the edges of the triangles are edge-disjoint from any crossings-dense configuration. Let $c_\triangle = |\mathcal{H}'|$. Then $m_4 \leq m_3 + c_6^2 + 4m_0 + 2c_\triangle$.*

Proof: To prove the result, we use the discharging technique. We assign in total $m_3+c_6^2+4m_0+2c_\Delta$ units of charge to the network. In particular, we assign 1 unit of charge to each F_6^3 and F_6^2 configuration, 4 units to each edge in E_0 and 2 units to every triangle in $\mathcal{H}'$. Here, the number of F_6^3 configurations is equal to m_3. The proposition will follow from a redistribution of the charge in a way that each edge in E_4 has 1 unit of charge and no edges, triangles or configurations have negative charge. We discuss the redistribution and the final charges by case analysis. For that, we use the fact that edges of E_4 do not cross each other and D_4 is 4-planar.

- *Case 1: $e \in E_4$ lies completely in one of the crossings-dense configurations.* This can only be the case in an F_6^2 or F_6^3 configuration as all five edges of an F_5^2 configuration still exist in D_2. In each F_6^2 or F_6^3 configuration all 2-hops exist in D_3. Therefore, e is a 3-hop and crosses the other 3-hops inside the hexagon, which therefore cannot be in E_4. So e is the only edge in E_4 inside the crossings-dense configuration and can receive 1 charge from it.

- *Case 2: $e \in E_4$ starts in a crossings-dense configuration C and ends in another one, say C'.* Let uu' be the edge on the boundary of C that e crosses. We will move 1 charge from uu' to e and argue that $uu' \in E_0$. Let e_1, e_2 and e_1', e_2' resp. be the 2-hop edges of C and C' that enclose the edge uu' (see Fig. 17a). Each of these four edges is crossed at least twice by edges belonging to the same crossings-dense configurations C or C'. Edge e crosses at least two of those four edges. And since those edges must not be crossed more than four times, there are at most four edges of E_4 that receive charge from the same boundary edge uu'. We can also guarantee $uu' \in E_0$, as otherwise e has at least five crossings (two each in the crossings-dense configurations and one with uu').

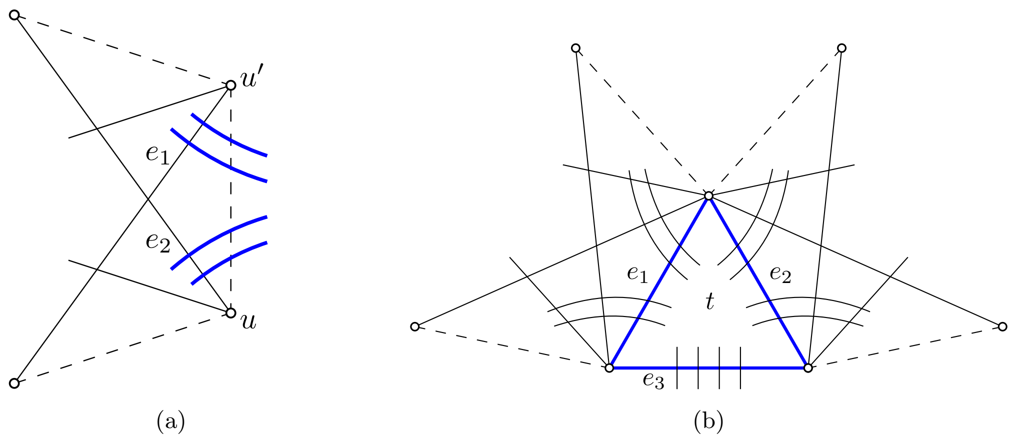


Figure 17: (a) At most four edges of E_4 (blue) can leave a crossings-dense configuration C through the same edge of its boundary, as otherwise one of the 2-hops e_1, e_2 of C has more than four crossings. (b) A triangle of $\mathcal{H}$ with all three edges in E_4 and two edges as boundary edges of crossings-dense configurations leads to a contradiction.

- *Case 3: $e \in E_4$ is completely outside of any crossings-dense configuration.* By the third property of the triangulation $\mathcal{H}$, e is an edge of two neighboring triangles $t, t' \in \mathcal{H}$. If both $t, t' \notin \mathcal{H}'$, then e would be a boundary edge of two crossings-dense configurations and any edge crossing e would have more than four crossings. So assume w.l.o.g. $t \in \mathcal{H}'$. If $t' \notin \mathcal{H}'$, then we move 1 unit of charge from t to e. In the case $t' \in \mathcal{H}'$, t and t' both contribute 0.5 units of charge to e so that e receives 1 unit of charge in total. By this, the only critical case to consider where a triangle of $\mathcal{H}'$ may get negative charge is if all three edges e_1, e_2, e_3 of t are in E_4 and two or three neighbors of t are not in $\mathcal{H}'$.
  Assume w.l.o.g. that the neighboring triangles of t at e_1 and e_2 are not in $\mathcal{H}'$. Observe that in this case the eight edges crossing e_1, e_2 in D_4 must end at t or leave t through e_3 as otherwise they have more than four crossings (see Fig. 17b). Thus, at least four of these edges end at t implying more than four crossings for the other edges. Therefore this case can not occur.

- *Case 4: $e \in E_4$ lies partially in the faces of $\mathcal{H}'$ and a crossings-dense configuration.* This is the remaining case. Let uu' be a boundary edge of the crossings-dense configuration C that is crossed by edge $e \in E_4$. If uu' does not exist in D_2, then e (and at most three other such edges) receive 1 unit of charge each from uu' (as in Case 2).
  Assume now, that uu' exists in D_2. Recall that therefore there exists the charging triangle t of uu' with vertices u, u', a. Note that, by the choice of the triangulation $\mathcal{H}$, Case 4 can only occur on one of the edges of the triangle t, here the edge uu' (an edge in E_4 that enters t through another edge and ends at u or u' would cross the pillar of t, contradiction). We distinguish three cases:

  - *Case 4.1: uu' is crossed by exactly one edge of E_4*. If at most one edge of t is in E_4, edge e (and the other edge) can receive 1 unit of charge from t. Assume now that the two edges $ua, u'a$ of t are in E_4 (see Fig. 18a). As e has at least two crossings at the crossings-dense configuration and one crossing with uu', it has only one possible

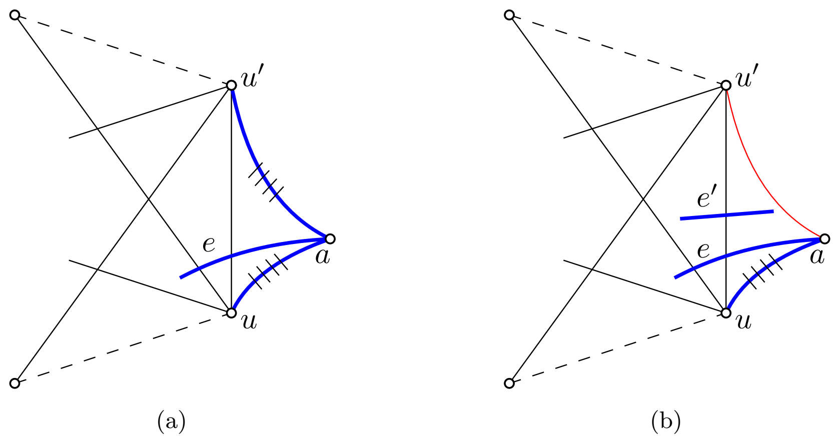


Figure 18: Illustrations for (a) Case 4.1 and (b) Case 4.2. Note that in (b) we have the same contradiction to 4-planarity of e or uu', if $u'a \in E_4$ instead of $ua \in E_4$.

crossing inside t left. Thus three edges crossing ua and three edges crossing $u'a$ must leave triangle t through uu', a contradiction to the fact that D_4 is 4-planar.

– *Case 4.2: uu' is crossed by exactly two edges of E_4.* Let $e, e' \in E_4$ be the edges crossing uu'. If no edge of t is in E_4, then t has enough charge to contribute 1 unit of charge to e, e' each. Assume now the opposite, i.e., there is an edge $e_1 \in E_4$ of t (see Fig. 18b). As in the case above, e, e' can have only one crossing inside t in D_4 and therefore three of the four edges crossing e_1 must leave t through uu'. But this is a contradiction to the fact that D_4 is 4-planar.

– *Case 4.3: uu' is crossed more often.* Using the previous notation, let $\tilde{e}_1$, $\tilde{e}_2$ be the edge-segments enclosing the 2-triangle next to uu' in the crossings-dense configuration C and let $\tilde{e}_1$ be incident to u and $\tilde{e}_2$ incident to u'. Further, let a, a' be the endpoints of e, e' that lie outside of the crossings-dense configuration such that the segments between their endpoints a and a', resp., and their crossing points with uu' are completely outside of C.

  As before, we observe that there are at most four edges crossing uu'. In particular, $\tilde{e}_1$ is crossed by exactly two edges $e, e' \in E_4$ (while $\tilde{e}_2$ is crossed by one or two edges in E_4).

  * *Case 4.3.1: $a = a'$.* Assume first that e also leaves the crossings-dense configuration through another boundary edge vv'. We have $u = v$, as otherwise e is crossed more than four times. Every edge of E_4 crossing uu' or vv' must cross one of the edges $u'v'$ or $\tilde{e}_1$. By 4-planarity, both these edges can have two crossings each additional to the crossings with other edges of the crossings-dense configuration. Thus, there are at most four edges in E_4 that cross uu' or vv' (see Fig. 19a). The charging triangle t can contribute 2 units of charge to these edges, so 2 units more are required. We find that at vv', if it does not exist in D_2, or otherwise at the charging triangle of vv'.

    Assume now that e does not cross another boundary edge. Then it is impossible that e and e' both have exactly four crossings in D_4 while the drawing is still 4-planar (see Fig. 19b).

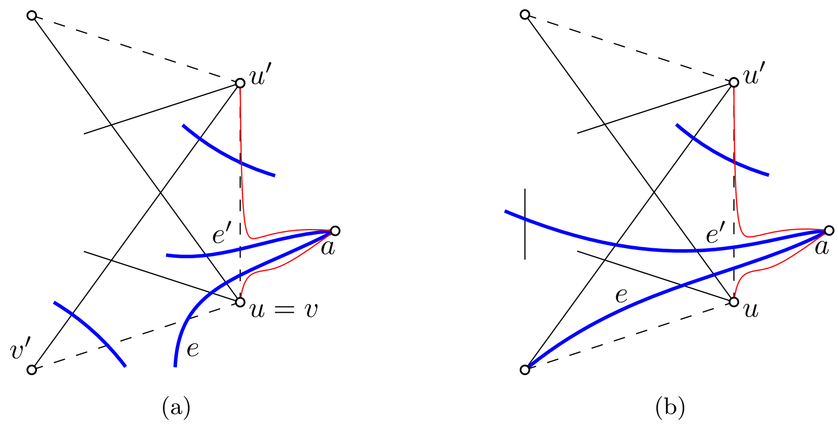


Figure 19: Illustrations for the subcase that uu' is crossed three or four times by edges of E_4. (a) If e crosses uu' and vv' then these edges are crossed by at most four distinct edges of E_4 in total. (b) e' already has four crossings while e has only three. Every additional edge crossing e implies more crossings for e'.

* *Case 4.3.2:* $a \neq a'$. Recall that in this case there exists also the charging by-triangle t' of uu' (see Fig. 9b). For the following reason, no edge of t, t' is in E_4: The edges au' and $a'u'$ cross edges of E_4 and edges of E_4 do not cross each other. The edge au has at most two crossings, because every edge that crosses au must cross e, which has only one crossing left, or uu', which has also at most one crossing left. Finally, for the same reason, aa' has at most two crossings, as all crossing edges must cross one of e, e'. Since t, t' uniquely belong to uu', they can contribute 1 unit of charge each to e and to the at most three other edges in E_4 crossing uu'.

$\square$