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Holes in Convex and Simple Drawings

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Abstract. Gons and holes in point sets have been extensively studied in the literature. For simple drawings of the complete graph a generalization of the Erdős–Szekeres theorem is known and empty triangles have been investigated. We introduce a notion of k-holes for simple drawings and survey generalizations thereof, like empty k-cycles. We present a family of simple drawings without 4-holes and prove a generalization of Gerken's empty hexagon theorem for convex drawings. A crucial intermediate step is the structural investigation of pseudolinear subdrawings in convex drawings. With respect to empty k-cycles, we show the existence of empty 4-cycles in every simple drawing of K_n and give a construction that admits only $\Theta(n^2)$ of them.

1 Introduction

A classic theorem from combinatorial geometry is the Erdős–Szekeres theorem [19]. It states that for every $k \in \mathbb{N}$ every sufficiently large point set in general position (that is, no three points on a line) contains a subset of k points that are the vertices of a convex polygon, a so called *k-gon*. In this article we focus on a prominent variant of the Erdős–Szekeres theorem suggested by Erdős himself [18], which asks for the existence of *empty* k-gons, also known as k-holes. A *k-hole* H in a point set P is a k-gon with the property that there are no points of P in the interior of the convex

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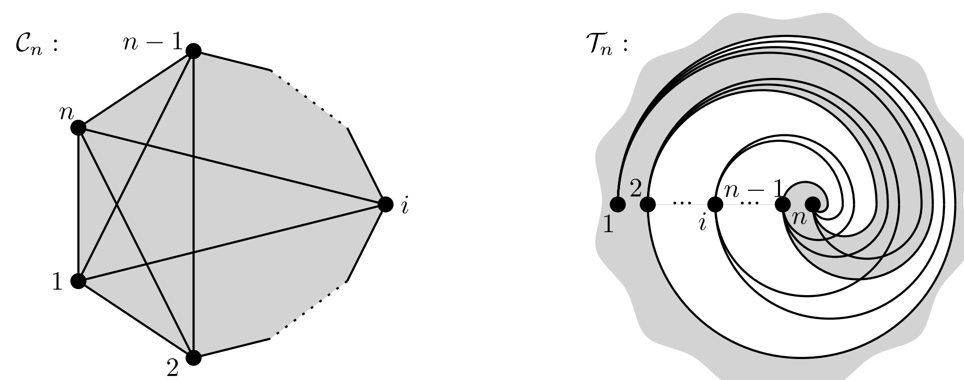
hull of H. It is known that every sufficiently large point set contains a 6-hole [23, 30] and that there are arbitrarily large point sets without 7-holes [28].

Point sets in general position are in correspondence with *geometric drawings* of the complete graph where vertices are mapped to points and edges are drawn as straight-line segments between the vertices. In this article we generalize the notion of holes to simple drawings of the complete graph K_n. In a *simple drawing*, vertices are mapped to distinct points in the plane (or on the sphere) and edges are mapped to simple curves connecting the two corresponding vertices such that two edges have at most one point in common, which is either a common vertex or a proper crossing. In the course of this article, we will see that many important properties do not depend on the full drawing but only on the underlying combinatorics, more specifically, on the isomorphism class of a drawing. We call two simple drawings of the same graph *isomorphic*[1] if there is a bijection between their vertex sets such that the corresponding pairs of edges cross. Note that this isomorphism is independent of the choice of the outer cell and thus only encodes the simple drawing on the sphere.

To study k-holes, we first extend the notion of k-gons to simple drawings of K_n. A *k-gon*[2] $\mathcal{C}_k$ is a subdrawing isomorphic to the geometric drawing on k points in convex position; see Figure 1(a) for a depiction of an n-gon. In terms of crossings, a k-gon $\mathcal{C}_k$ is a (sub)drawing with vertices $v_1, \ldots, v_k$ such that $\{v_i, v_\ell\}$ crosses $\{v_j, v_m\}$ if and only if $i < j < \ell < m$. In contrast to the geometric setting where every sufficiently large geometric drawing contains a k-gon, simple drawings of complete graphs do not necessarily contain k-gons [25]. For example, the twisted drawing $\mathcal{T}_n$ depicted in Figure 1(b) does not contain any 5-gon. In terms of crossings, $\mathcal{T}_n$ can be characterized as a drawing of K_n with vertices $v_1, \ldots, v_n$ such that $\{v_i, v_m\}$ crosses $\{v_j, v_\ell\}$ exactly if $i < j < \ell < m$. A theorem by Pach, Solymosi and Tóth [33] states that, for every k, every sufficiently large simple drawing of K_n contains $\mathcal{C}_k$ or $\mathcal{T}_k$. The currently best known estimate is due to Suk and Zeng [39] who showed that every simple drawing of K_n with $n > 2^{9 \cdot \log_2(a) \log_2(b) a^2 b^2}$ contains $\mathcal{C}_a$ or $\mathcal{T}_b$. Convex drawings, which we define in the next paragraph, are a class of drawings nested between geometric drawings and simple drawings. In particular, convex drawings do not contain $\mathcal{T}_5$ as a subdrawing. Hence every convex drawing of K_n contains a k-gon $\mathcal{C}_k$ for some $k = (\log n)^{1/2 - o(1)}$.



(a) (b)

Figure 1: A drawing of (a) an n-gon $\mathcal{C}_n$ and (b) a twisted $\mathcal{T}_n$ for $n \geq 4$. The largest hole in these drawings, an n-hole and a 4-hole respectively, is shaded in gray.

[1] This isomorphism is often referred to as "weak isomorphism" since there also exist stronger notions.
[2] We keep the terminology from the geometric setting and trust that this does not lead to any confusion.

In the last decades, holes were intensively studied for the setting of point sets. Our focus lies on determining the existence of holes in convex drawings, the most general class of the convexity hierarchy introduced by Arroyo, McQuillan, Richter, and Salazar [6], which gives a more fine-grained layering between geometric drawings and simple drawings. The basis to define convexity are *triangles*, which are subdrawings induced by three vertices. Since in a simple drawing incident edges do not cross, a triangle separates the plane (respectively the sphere) into two connected components. The closure of each of the components is called a *side* of the triangle. A side S is *convex* if, for every pair of vertices in S, the connecting edge is fully contained in S. A simple drawing $\mathcal{D}$ of K_n is

- *convex* if every triangle in $\mathcal{D}$ has a convex side;
- *h-convex* (hereditarily convex) if there is a choice of a convex side S_T for every triangle T such that, for every triangle T' contained in S_T, it holds that $S_{T'} \subseteq S_T$;
- *f-convex* (face convex) if there is a marking face F in the plane such that for all triangles the side not containing F is convex.

The class of f-convex drawings is related to pseudolinear drawings. A *pseudolinear drawing* is a simple drawing in the plane such that the edges can be extended to an arrangement of pseudolines. A *pseudoline* is a simple curve partitioning the plane into two unbounded components and in an *arrangement* each pair of pseudolines has exactly one point in common, which is a proper crossing. As shown by Arroyo, McQuillan, Richter, and Salazar [5], a simple drawing of K_n is pseudolinear if and only if it is f-convex and the marking face F is the unbounded face. For more details on the convexity hierarchy and the classes it contains, we refer the reader to [5, 6, 7, 13].

Before we define k-holes, consider the case of 3-holes, also known as empty triangles. A triangle is *empty* if one of its two sides does not contain any vertex in its interior. Harborth [25] proved that every simple drawing of K_n contains at least two empty triangles and conjectured that the minimum among all simple drawings of K_n is $2n - 4$. While $2n - 4$ is obtained by $\mathcal{T}_n$ and all generalized twisted drawings [21], the best known lower bound is n [4].

In the geometric setting, the number of empty triangles behaves differently: every point set has $\Omega(n^2)$ empty triangles, and this bound is asymptotically optimal [9]. Note that the notion of empty triangles in point sets slightly differs from the one in simple drawings, where the complement of the convex hull of a point set can be an empty triangle as well. The class of convex drawings behaves similarly to the geometric setting: the minimum number of empty triangles is asymptotically quadratic [5, Theorem 5].

Holes in Simple Drawings. In the drawing $\mathcal{C}_k$ with $k \geq 4$, every triangle has exactly one empty side, which is also its unique convex side. The *convex side* of $\mathcal{C}_k$ is the union of convex sides of its triangles; see the gray shaded regions in Figure 1. Given a k-gon $\mathcal{C}_k$ in a simple drawing of K_n, we call vertices in the interior of the convex side of $\mathcal{C}_k$ *interior vertices*. A *k-hole* in a simple drawing of K_n is a k-gon that has no interior vertices. For example, the vertices 1, 2, $n-1$, and n in $\mathcal{T}_n$ form a 4-hole; marked gray in Figure 1(b). In convex drawings, as in the geometric setting, edges from an interior vertex to a vertex of $\mathcal{C}_k$ and edges between two interior vertices are contained in the convex side of $\mathcal{C}_k$ [6, Lemma 3.5]; see also Section 2.

In this paper, using the notion of k-holes in simple drawings defined above, we resolve the questions of existence of 4-, 5- and 6-holes in simple and convex drawings of K_n. In particular, we show the existence of 6-holes in sufficiently large convex drawings (Theorem 1), generalizing Gerken's empty hexagon theorem [23]. The key ingredient of the proof is that in a convex drawing every subdrawing induced by a minimal k-gon together with its interior vertices is f-convex (Lemma 5). This allows to transfer various existential results from the geometric, pseudolinear, and f-convex settings to convex drawings. Besides the existence of 6-holes, we also show the existence of monochromatic generalized 4-holes in two-colored convex drawings (Corollary 7), generalizing a result on bichromatic point sets [3]. For this we discuss two variants of generalized holes (Section 3) in the setting of simple drawings of K_n and show the existence of empty 4-cycles, that is, plane cycles of length 4 such that one side does not contain any interior vertices (Theorem 10). Furthermore, we construct a simple drawing of K_n that does not contain any two interior-disjoint empty triangles sharing an edge (Proposition 8) and another one containing only $\Theta(n^2)$ empty 4-cycles (Proposition 12).

2 Holes in Convex Drawings

In this section, we show that convex drawings behave similarly to geometric point sets when it comes to the existence of holes. We show that every sufficiently large convex drawing contains a 6-hole and hence a 5-hole and a 4-hole. This is tight, as the construction by Horton [28] gives arbitrarily large point sets, that is, geometric drawings without 7-holes.

Theorem 1 (Empty hexagon theorem for convex drawings)**.** *For every sufficiently large n, every convex drawing of K_n contains a 6-hole.*

For the proof we use the existence of k-gons in sufficiently large convex drawings [33, 39]. Our key lemma is that the subdrawing induced by a minimal k-gon together with its interior vertices is f-convex, a fact that had been known only for h-convex drawings [6, Lemma 4.7]. For k fixed, a k-gon is *minimal* if its convex side does not contain the convex side of another k-gon.

Arroyo, McQuillan, Richter and Salazar [6, Section 3] started the investigations of interior vertices of k-gons. An important part is their Lemma 3.5, which we use in the following.

Lemma 2 (cf. [6, Lemma 3.5])**.** *Let $\mathcal{C}_k$ be a k-gon in a convex drawing of K_n with vertices $v_1, \ldots, v_k$ and $k \geq 4$. Then for every two vertices u, v contained in the convex side of $\mathcal{C}_k$ the edge $\{u, v\}$ is contained in the convex side of $\mathcal{C}_k$.*

Note that in a k-gon $\mathcal{C}_k$ the edges on its convex hull form a *plane k-cycle*, that is, a cycle of length k that does not cross itself. This plane cycle divides the plane into two connected components whose closures we call *sides*. Furthermore, all *chords* of that cycle, that is, edges between non-adjacent vertices of the cycle lie on the same side of the cycle. On the other hand, if all chords of a plane k-cycle lie on the same side of it, then they cross each other in the exact same pattern as in a k-gon $\mathcal{C}_k$.

Observation 3. *A k-gon $\mathcal{C}_k$ is equivalent to a plane k-cycle that has all chords on the same side, which is the convex side of $\mathcal{C}_k$.*

For the sake of readability, we refer to the vertices $v_1, \ldots, v_k$ of a k-gon with indices modulo k.

Lemma 4. *Let $\mathcal{C}_k$ be a minimal k-gon in a convex drawing $\mathcal{D}$ of K_n with vertices $v_1, \ldots, v_k$ and $k \geq 3$. Then for all i there are no interior vertices in the convex side of the triangle $\{v_i, v_{i+1}, v_{i+2}\}$. In particular, every minimal 4-gon is a 4-hole and every minimal 3-gon is an empty triangle.*

Proof: Assume there is an interior vertex v in the convex side of the triangle determined by $\{v_i, v_{i+1}, v_{i+2}\}$. If $k = 3$ then, by minimality of $\mathcal{C}_k = \{v_1, v_2, v_3\}$, the side S_N of the triangle $\{v_1, v_2, v\}$ contained in the convex side of $\mathcal{C}_k$ cannot be convex. Hence, there exists a vertex z in the interior of S_N such that the subdrawing induced by $\{v_1, v_2, v, z\}$ has a crossing [6, Corollary 2.5]. The edge $\{z, v\}$ cannot cross $\{v_1, v_2\}$ since that would contradict the convex side of $\mathcal{C}_k$. Hence, without loss of generality, let the edge $\{z, v_1\}$ cross $\{v, v_2\}$; Figure 2(a) gives an illustration. Then, however, the edge $\{z, v_1\}$ shows that the triangles $\{v_2, v_3, v\}$ and $\{v_1, v, v_3\}$ both have a unique convex side, which is the side contained in the convex side of $\mathcal{C}_k$. This is a contradiction to the minimality of $\mathcal{C}_k$. Thus, a minimal 3-gon is an empty triangle.

For $k \geq 4$, clearly the vertices $v_1, \ldots, v_i, v_{i+2}, \ldots, v_k$ span a $(k-1)$-gon and the triangle v_i, v, v_{i+2} is not contained in the convex side of that $(k-1)$-gon. Moreover all chords of $\mathcal{C}_k$ not involving v_{i+1} lie in the convex side of that $(k-1)$-gon. It remains to consider edges incident to v. Let $j \in [k] \setminus \{i, i+1, i+2\}$ be arbitrary but fixed. By Lemma 2, the edge $\{v, v_j\}$ does not leave the convex side of $\mathcal{C}_k$ and, since $\mathcal{D}$ is a simple drawing, $\{v, v_j\}$ crosses $\{v_i, v_{i+2}\}$ and therefore lies in the convex side of the 4-gon v, v_i, v_j, v_{i+2}. Figure 2(b) gives an illustration. This shows that $v_1, \ldots, v_i$, $v, v_{i+2}, \ldots, v_k$ span a plane k-cycle with all chords on the same side and hence, by Observation 3, they span a k-gon $\mathcal{C}'_k$. Furthermore, the convex side of $\mathcal{C}'_k$ is contained in the convex side of $\mathcal{C}_k$, implying that $\mathcal{C}_k$ was not minimal; a contradiction. □

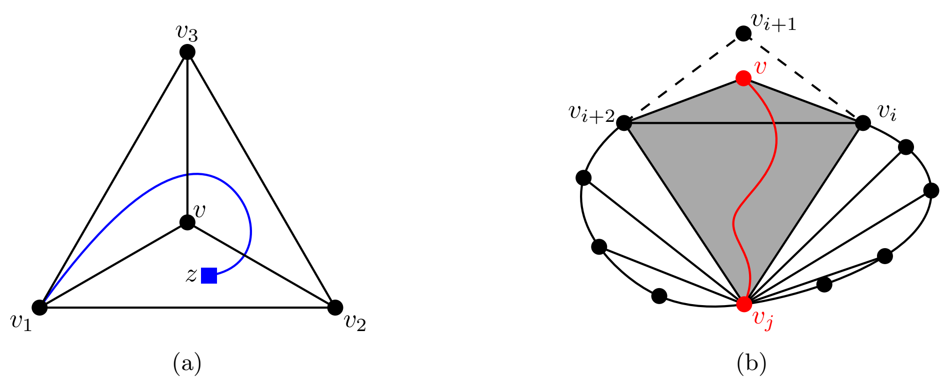


Figure 2: (a) If the convex side of a triangle is not empty, it contains the convex side of another triangle. (b) A k-gon with an interior vertex v in the convex side of the triangle v_i, v_{i+1}, v_{i+2}.

Lemma 5 (Key lemma)**.** *Let $\mathcal{C}_k$ be a minimal k-gon in a convex drawing $\mathcal{D}$ of K_n with $n \geq k \geq 3$. Then the subdrawing $\mathcal{D}'$ induced by the vertices in the convex side of $\mathcal{C}_k$ is f-convex.*

Proof: For $k \leq 4$, by Lemma 4, a minimal k-gon is empty and thus $\mathcal{D}'$ is clearly f-convex.

So let $k \geq 5$, let $v_1, \ldots, v_k$ be the vertices of the minimal k-gon $\mathcal{C}_k$ in $\mathcal{D}$, and let F be a face contained in the non-convex side of $\mathcal{C}_k$. We show that for every triangle spanned by three vertices of the convex side of $\mathcal{C}_k$, the side not containing F is convex and hence $\mathcal{D}'$ is f-convex. Suppose towards a contradiction that there exists a triangle T spanned by vertices t_1, t_2, t_3 from the convex side of $\mathcal{C}_k$, such that the side not containing F is not convex. Then this non-convex side S_N of T is the side contained in the convex side of $\mathcal{C}_k$. Since $\mathcal{D}$ is convex, the other side of T, containing F and all vertices $v_1, \ldots, v_k$, is convex and is denoted by S_C. If we additionally assume that S_N is not contained in (the closure of) a single cell of the subdrawing induced by $\mathcal{C}_k$, then some edge

$\{v_i, v_j\}$ has a crossing with one of the edges $\{t_\ell, t_m\}$ of T. This shows that S_C is not convex; a contradiction. Hence, S_N lies in (the closure of) a cell of $\mathcal{C}_k$.

Since $\mathcal{C}_k$ is minimal, by Lemma 4, there are no interior vertices in the convex side of a triangle $\{v_i, v_{i+1}, v_{i+2}\}$.

Since all cells in the convex side of $\mathcal{C}_k$ incident to the vertex v_{i+1} are inside this triangle, the vertex v_{i+1} is not part of the triangle T spanned by t_1, t_2, t_3. This holds for every $i = 1, \ldots, k$ and hence the vertices t_1, t_2, t_3 are interior vertices of $\mathcal{C}_k$ and S_N lies in a cell of the convex side of $\mathcal{C}_k$ that is not covered by the convex side of any triangle $\{v_i, v_{i+1}, v_{i+2}\}$. Since S_N is not convex, there exists a vertex z in the interior of S_N such that the subdrawing induced by $\{t_1, t_2, t_3, z\}$ has a crossing [6, Corollary 2.5]. We assume without loss of generality that the edge $\{t_1, z\}$ crosses $\{t_2, t_3\}$. Moreover, exactly one of the following two conditions holds: Either the triangle $\{t_1, t_3, z\}$ separates t_2 and F or the triangle $\{t_1, t_2, z\}$ separates t_3 and F. We assume that the former holds as otherwise we exchange the roles of t_2 and t_3. Figure 3 gives an illustration.

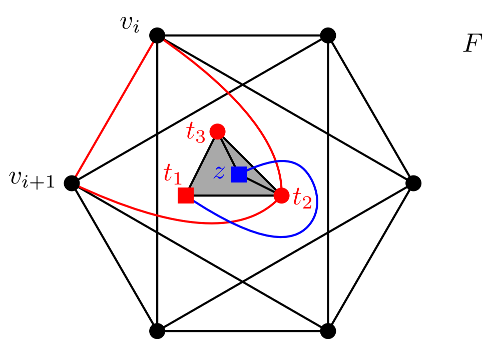


Figure 3: The non-convex side (shaded gray) of the triangle $\{t_1, t_2, t_3\}$ (red vertices) is witnessed by the edge $\{t_1, z\}$ (blue), and the triangle $\{t_1, t_3, z\}$ separates t_2 and F. Then the triangle $\{t_2, v_i, v_{i+1}\}$ (red edges) has no convex side.

Now we consider all edges from t_2 to the vertices $v_1, \ldots, v_k$ of $\mathcal{C}_k$. Since S_C is convex and contains $v_1, \ldots, v_k$, the edges $\{t_2, v_i\}$ are contained in S_C. This shows that none of the edges $\{t_2, v_i\}$ crosses any of the triangle edges and, in particular, they do not cross $\{t_1, t_3\}$.

The edges $\{t_2, v_1\}, \ldots, \{t_2, v_k\}$ partition the convex side of $\mathcal{C}_k$ into triangles $\{t_2, v_i, v_{i+1}\}$. Hence there is an index i such that the three vertices t_1, t_3, z lie in the side of the triangle $\{t_2, v_i, v_{i+1}\}$, which is contained in the convex side of $\mathcal{C}_k$. However, the edge $\{t_1, z\}$ is not fully contained in this side; a contradiction to its convexity. Moreover, the other side of that triangle is not convex either: Since t_2 is not inside the triangle $\{v_i, v_{i+1}, v_{i+2}\}$ the edge $\{v_i, v_{i+2}\}$ crosses $\{t_2, v_{i+1}\}$, so it is not fully contained in this side. This is a contradiction to the convexity of the drawing and thus completes the proof. □

Recently, Heule and Scheucher [27] used SAT to show that every set of 30 points has a 6-hole. Since their result is about the more general case of pseudoconfigurations of points, it holds for pseudolinear drawings. To prove Theorem 1, we combine this fact with Lemma 5.

Proof: [Theorem 1] Let $\mathcal{D}$ be a convex drawing of K_n with $n > 2^{9 \cdot 5^2 \log_2(5) \cdot 30^2 \log_2(30)}$. Since convex drawings do not contain the twisted drawing $\mathcal{T}_5$, it follows from [39] that $\mathcal{D}$ contains a 30-gon. To find a 6-hole in $\mathcal{D}$, we choose a minimal 30-gon $\mathcal{C}_{30}$. By Lemma 5, the subdrawing $\mathcal{D}'$ induced by $\mathcal{C}_{30}$ and its interior vertices is f-convex. Since the existence of holes is invariant under the choice of the outer cell, we can assume without loss of generality that $\mathcal{D}'$ is pseudolinear as we may otherwise choose the face F as the unbounded face. According to [8], $\mathcal{D}'$ corresponds to a pseudoconfiguration of points, and hence there exists a 6-hole $\mathcal{C}_6$ in $\mathcal{D}'$ [27]. Hence the convex side of $\mathcal{C}_6$ does not contain any vertex of $\mathcal{D}'$. Moreover, every vertex of $\mathcal{D}$ in the convex side of $\mathcal{C}_6$ would be an interior vertex of $\mathcal{C}_{30}$ and therefore belong to $\mathcal{D}'$. This shows that $\mathcal{C}_6$ is also a 6-hole in $\mathcal{D}$. □

The existence of 6-holes further implies the existence of 4- and 5-holes. However, it remains a challenging task to determine the smallest integer $n(k)$ such that every convex drawing of K_n with $n \geq n(k)$ contains a k-hole for $k \in \{5, 6\}$. The case $k = 4$ we resolve below.

For 6-holes, one can slightly improve the estimate from Theorem 1 by utilizing the fact that every 9-gon in a point set yields a 6-hole [23]. As shown in [37], this result transfers to pseudolinear drawings. It then follows from [39] and Lemma 5 that every convex drawing of K_n with $n > 5^{18\,225 \cdot \log_2(9)}$ contains a 6-hole.

A similar improvement is possible for 5-holes: as the textbook proof for the existence of 5-holes in every 6-gon of a point set (see for example Section 3.2 in [29]) applies to pseudolinear drawings, every convex drawing with more than $5^{8100 \cdot \log_2(6)}$ vertices contains a 5-hole.

For 4-holes, we can combine the proof of Bárány and Füredi [9, Theorem 3.3] for the quadratic number of 4-holes in point sets and the proof of Arroyo, McQuillan, Richter, and Salazar [5, Theorem 5] for the quadratic number of empty triangles in convex drawings to obtain:

Lemma 6. *Every crossed edge in a convex drawing of K_n is a chord of a 4-hole, that is, it is one of the crossing edges of a 4-hole.*

Proof: Let $\mathcal{D}$ be a convex drawing of K_n. Let e be an edge that is crossed by another edge f. The subdrawing induced by the four end-vertices of e and f is a 4-gon, and we denote it by $\mathcal{C}_4$. We assume that the vertices are labeled with v_1, v_2, v_3, v_4 such that $e = \{v_1, v_3\}$ and $f = \{v_2, v_4\}$. If $\mathcal{C}_4$ is minimal, it is a 4-hole by Lemma 4.

Hence, we assume that there is an interior vertex x of $\mathcal{C}_4$ as illustrated in Figure 4. By the properties of a 4-gon, x lies in the convex side of exactly two of its triangles. Without loss of generality, we assume that x is in the convex side of the two triangles $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$. By Lemma 2, the edges $\{x, v_i\}$ are fully contained in the convex side of $\mathcal{C}_4$. Since the edge $\{x, v_4\}$ is fully contained in the convex side of $\{v_2, v_3, v_4\}$, but has to leave the triangle $\{v_1, v_2, v_3\}$ to get to v_4, it crosses the edge $e = \{v_1, v_3\}$. Hence v_1, x, v_3, v_4 span another 4-gon in which $\{v_1, v_3\}$ is one of the crossing edges. Furthermore, since the edges $\{x, v_1\}$, $\{x, v_2\}$, and $\{x, v_3\}$ are fully contained in the convex side of $\mathcal{C}_4$, the convex side of the 4-gon $\{v_1, x, v_3, v_4\}$ is fully contained in the convex side of $\mathcal{C}_4$.

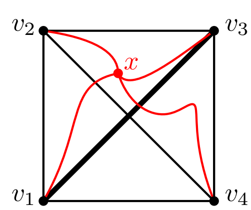


Figure 4: If a 4-gon $\{v_1, v_2, v_3, v_4\}$ with chord $e = \{v_1, v_3\}$ is not empty, then it contains a smaller 4-gon $\{v_1, x, v_3, v_4\}$ still with chord e.

This shows that every 4-gon that is minimal subject to the restriction that e is one of its chords is actually a minimal 4-gon without restriction. Consequently, by Lemma 4, every crossed edge e gives a 4-hole whose diagonal is e, which completes the proof. □

Note that there are $\binom{n}{2}$ edges in a drawing of the complete graph, at most $2n-2$ of which are uncrossed [36]. Since every 4-hole is counted at most twice, the total number of 4-holes in a convex drawing of K_n is at least $\frac{1}{2}\left(\binom{n}{2}-2n+2\right)=\frac{1}{4}n^2-\frac{5}{4}n+1$.

Since every drawing of K_5 contains a crossing, Lemma 6 also implies that every convex drawing of K_n with $n \geq 5$ contains a 4-hole. In contrast to the convex setting, 4-holes can be avoided in simple drawings as we show in the next section.

3 Generalized Holes

Devillers, Hurtado, Károlyi, and Seara [17] showed that sufficiently large two-colored point sets in general position contain a monochromatic 3-hole and constructed arbitrarily large two-colored sets without monochromatic 5-holes. The existence of monochromatic 4-holes, however, remains a longstanding open problem [16, Problem 8.2.7]. A weaker version was shown by Aichholzer, Hackl, Huemer, Hurtado, and Vogtenhuber [3]. They proved that every two-colored point set $P = A \dot\cup B$ contains a monochromatic generalized 4-hole. A *generalized k-hole* is a simple polygon (not necessarily convex) which is spanned by k points of P and does not contain any point of P in its interior.

To define generalized k-holes in simple drawings we consider plane cycles. Recall that a plane cycle divides the plane into two connected components whose closures we call sides. An *empty k-cycle* in a simple drawing is a plane cycle of length k such that one of its sides contains no vertices in its interior. For $k = 3$ this definition coincides with empty triangles. Since polygons in point sets can be triangulated, we say that an empty k-cycle is an *empty k-triangulation* if its empty side is the disjoint union of empty triangles.

Since the proof in [3] only relies on triangle orientations and not on the exact geometry of the point set, their result transfers to the pseudolinear setting. This allows us to generalize it to convex drawings in the same way as the Empty Hexagon Theorem (Theorem 1) using Lemma 5.

Corollary 7. *Every sufficiently large convex drawing on vertices $V = A \dot\cup B$ has an empty 4-triangulation induced only by vertices from A or only by vertices from B.*

As the following construction (Figure 5) shows, there exist simple drawings of K_n without any empty 4-triangulation. For the construction, we start with the twisted drawing $\mathcal{T}_n$ and reroute some edges such that the drawing is still crossing maximal, that is, every 4-tuple contains a crossing. The resulting drawing $\mathcal{T}'_n$ then does not contain any empty 4-triangulation and thus no 4-hole.

Proposition 8. *For $n \geq 6$ the simple drawing $\mathcal{T}'_n$ contains no empty 4-triangulation.*

Proof: We start by giving the exact crossing edge pairs in $\mathcal{T}'_n$ and thus describing the drawing up to isomorphism. The vertices $1, 3, 4, \ldots, n$ form a twisted drawing $\mathcal{T}_{n-1}$ and hence every 4-tuple from $[n]\backslash\{2\}$ contains a crossing, giving $\binom{n-1}{4}$ crossings. More specifically, the edges $\{i, \ell\}$ and $\{j, k\}$ cross for $i, j, k, \ell \in [n] \setminus \{2\}$ with $i < j < k < \ell$.

It remains to describe the crossings in 4-tuples which do contain vertex 2. The edge $\{2, 1\}$ crosses the edges $\{3, n\}$, $\{4, i\}$ for $i = 5, \ldots n$, and $\{3, 4\}$, which are $n-2$ crossings. The edge $\{2, 3\}$ has no crossings and the edge $\{2, 4\}$ crosses only the edge $\{3, n\}$. For $j = 5, \ldots, n-1$ the edge $\{2, j\}$ crosses the two edges $\{3, n\}$ and $\{1, 3\}$, the $n-j$ edges $\{1, j+1\}, \ldots, \{1, n\}$, and the edges $\{i, k\}$ for $2 < i < k < j$, of which there are $\binom{j-3}{2}$. Finally, the edge $\{2, n\}$ crosses the $\binom{n-4}{2}$ edges $\{i, j\}$ for $3 < i < j < n$.

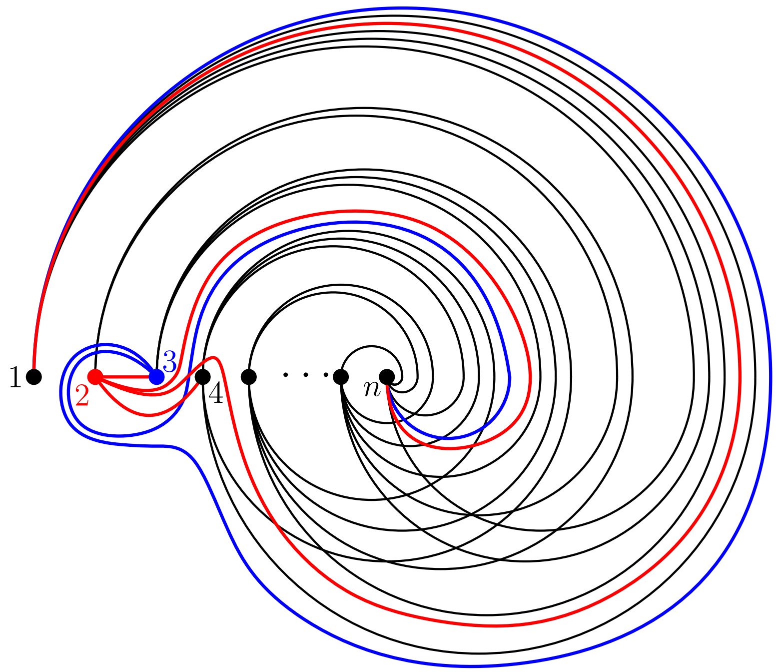


Figure 5: The drawing $\mathcal{T}'_n$ without empty 4-triangulations for $n \geq 6$.

In total there are

$$\begin{aligned}
&\binom{n-1}{4} + (n-2) + 1 + \sum_{j=5}^{n-1}\left(2 + (n-j) + \binom{j-3}{2}\right) + \binom{n-4}{2} \\
=&\binom{n-1}{4} + 3n - 11 + \binom{n-4}{2} + \sum_{j=2}^{n-4}\binom{j}{2} + \binom{n-4}{2} \\
=&\binom{n-1}{4} + 2n - 7 + \binom{n-3}{2} + \binom{n-3}{3} + \binom{n-4}{2} \\
=&\binom{n-1}{4} + 1 + (n-4) + \binom{n-3}{2} + \binom{n-2}{3} \\
=&\binom{n-1}{4} + \binom{n-1}{3} = \binom{n}{4}
\end{aligned}$$

crossings because of the well-known identities $\sum_{j=r}^{n}\binom{j}{r} = \binom{n+1}{r+1}$ and $\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$. Hence $\mathcal{T}'_n$ is crossing maximal.

Because of this crossing maximality every empty 4-triangulation is a 4-hole since the drawing of the four induced vertices has a crossing. In the twisted subdrawing $\mathcal{T}_{n-1}$ induced by $1, 3, \ldots, n$ the empty triangles are $\{1, 3, i\}$ for $i = 4, \ldots, n$ and $\{i, n-1, n\}$ for $i = 1, 3 \ldots n-2$ and the only 4-hole is $\{1, 3, n-1, n\}$, which is not a 4-hole in $\mathcal{T}'_n$ because we placed the vertex 2 into the triangle $\{3, n-1, n\}$. Hence if there is a 4-hole, it consists of the vertex 2 and the three vertices of an empty triangle of the induced subdrawing $\mathcal{T}_{n-1}$. However, since all empty triangles from the induced subdrawing $\mathcal{T}_{n-1}$ have either both 1 and 3 or both $n-1$ and n as vertices, at least one of the two triangles $\{1,2,3\}$ or $\{2,n-1,n\}$ must be empty. This is not the case in the constructed drawing; the triangle $\{1,2,3\}$ has 4 on one side and all other vertices on the other side and the triangle $\{2,n-1,n\}$ has 3 on one side and all other vertices on the other side. This completes the proof. □

In contrast to this construction, if instead of empty 4-triangulations we only ask for empty 4-cycles, then we can actually guarantee their existence in all simple drawings of K_n. This resolves one case of a recent conjecture by Bergold, Felsner, M. Reddy, Orthaber, and Scheucher [11]. They showed that every convex drawing contains an empty k-cycle for all $3 \le k \le n$ and conjectured that this also holds for simple drawings.

Conjecture 9 ([11]). *Every simple drawing of K_n contains an empty k-cycle for each $3 \le k \le n$.*

While the case $k=3$ follows by Harborth's result [25], the case $k=n$ coincides with Rafla's conjecture concerning the existence of plane Hamiltonian cycles in all simple drawings of K_n [35]. For the proof of the case $k=4$ we use results on plane subdrawings by García, Pilz, and Tejel [20].

Theorem 10. *Let $\mathcal{D}$ be a simple drawing of K_n with $n \ge 4$ and let v be a vertex of $\mathcal{D}$. Then $\mathcal{D}$ contains an empty 4-cycle passing through v.*

Proof: For a fixed vertex v, we consider the spanning star S_v centered at v. By [20, Corollary 3.4], there is a plane subdrawing $\mathcal{D}'$ of $\mathcal{D}$ that consists of the star S_v and some spanning tree T on the other $n-1$ vertices. Note that $\mathcal{D}'$ has exactly $2n-3$ edges and $n-1$ faces. Every face F of $\mathcal{D}'$ contains v on its boundary because the tree T is cycle-free and since $\mathcal{D}'$ is 2-connected [20, Theorem 3.1], F is bounded by exactly two edges of S_v.

If there is a face of $\mathcal{D}'$ with exactly 4 boundary edges or if there are two adjacent triangular faces, we obtain an empty 4-cycle passing through v and the statement follows. Otherwise we count the number of edges $|E|$ in $\mathcal{D}'$: At most half of the $n-1$ faces are triangles so that none of them are adjacent. All other faces have at least 5 boundary edges. Since every edge is incident to exactly two faces, we have $|E| \ge \frac{1}{2}\left(5(n-1) - 2\left\lfloor\frac{n-1}{2}\right\rfloor\right) \ge 2n-2$. This is a contradiction to the fact that $\mathcal{D}'$ contains exactly $2n-3$ edges. □

The above theorem implies a linear lower bound on the number of empty 4-cycles. This is similar to the minimum number of empty triangles which is asymptotically linear as well [4].

Corollary 11. *Every simple drawing of K_n with $n \ge 4$ contains at least $\lceil \frac{n}{4} \rceil$ empty 4-cycles.*

While the twisted drawing $\mathcal{T}_n$ is conjectured to minimize the number of empty triangles, it contains $\Theta(n^3)$ empty 4-cycles, since the cycles (i,j,l,k) for $i<j<k<l$ separate the elements between j and k from the rest, whereas all other 4-cycles are crossing themselves. This is certainly not minimal as, in the following, we construct drawings with $\Theta(n^2)$ empty 4-cycles; see Figure 6.

Proposition 12. *There is a simple drawing of K_n that admits $\frac{1}{8}n^2 + O(n)$ empty 4-cycles.*

Note that this is strictly less than the lower bound of $\frac{5}{2}n^2 - \Theta(n)$ for the number of empty 4-cycles in geometric drawings shown in [2]. Moreover, in the geometric setting, the number of empty k-cycles with $k \ge 4$ is actually conjectured to be super-quadratic [1].

Proof: We start with the drawing $\mathcal{D}_5 = \mathcal{C}_5$ with vertices $1,\dots,5$ labeled counter-clockwise. We then recursively construct the drawing $\mathcal{D}_{n+1}$ from $\mathcal{D}_n$ as follows: We add a new vertex $n+1 \ge 6$ close to the vertex n in a chosen cell c_n next to the edge $e_n := \{n, i_n\}$ for some choice of i_n. We connect $n+1$ and n with an uncrossed edge. Then we add the edges $\{n+1, j\}$ for $j < n$ by making them cross all edges between e_n and $\{n, j\}$ incident to n close to n and then follow the edge $\{n, j\}$ from n to j. In particular, the edge $\{n+1, i_n\}$ crosses all edges incident to n except e_n and $\{n+1, n\}$ before following e_n to i_n. As shown by Harborth and Mengersen [26] the resulting drawing $\mathcal{D}_{n+1}$ is crossing maximal for all choices of i_n and cells c_n next to it. Also note that by construction $n+1$ and n have the same rotation (ignoring the edge between them).

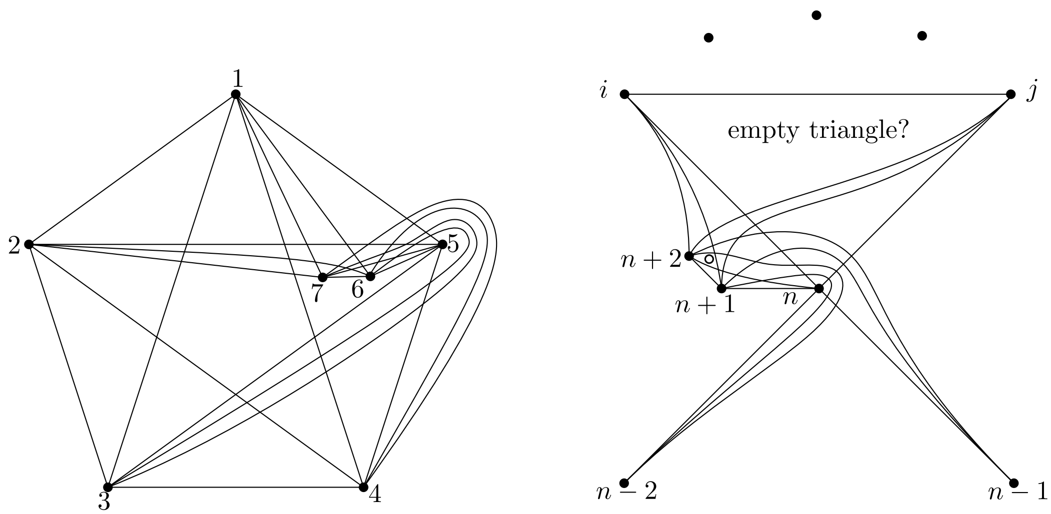


Figure 6: Constructing the drawing $\mathcal{D}_{n+2}$ of K_{n+2} with few empty 4-cycles for n odd. The small circle indicates the cell where the additional vertices $n+3$ and $n+4$ will be put in the next step.

For our construction, we assume that n is odd and we perform two steps at once, hence producing drawings $\mathcal{D}_{n+1}$ and $\mathcal{D}_{n+2}$. In the first step from n to $n+1$, we choose $i_n = n-2$ and the cell c_n not to be incident to $\{n, n-1\}$. This is well-defined for the base case $n = 5$ and also for all larger odd n, as we make sure in the following that n and $n+1$ are consecutive in the rotation of $n+2$. In the second step from $n+1$ to $n+2$, we choose $i_{n+1} = n$ and c_{n+1} to be the cell not incident to $\{n+1, n-2\}$. This cell is well-defined as we added the previous vertex $n+1$ in the cell incident to $e_n = \{n, n-2\}$. We start with some general observations:

- Since the drawings are crossing maximal, every 4-tuple of vertices can produce at most one empty 4-cycle.
- The vertices n, $n+1$, and $n+2$ have the exact same rotation (ignoring the edges between them), that is, removing a non-trivial subset of them from $\mathcal{D}_{n+2}$ results in a drawing isomorphic to $\mathcal{D}_n$ or $\mathcal{D}_{n+1}$.
- Every empty 4-cycle in $\mathcal{D}_n$ involving vertex n that is still empty in $\mathcal{D}_{n+2}$ produces two other empty 4-cycles in $\mathcal{D}_{n+2}$ involving vertex $n+1$ and $n+2$ respectively. These are the only 4-cycles involving exactly one of n, $n+1$, and $n+2$.
- An empty 4-cycle in $\mathcal{D}_n$ still exists in $\mathcal{D}_{n+2}$ if and only if it does not contain the cell c_{n+1}. In particular, only empty 4-cycles of $\mathcal{D}_n$ incident to n are destroyed by $n+1$ and $n+2$ as c_{n+1} is incident to n and any other 4-cycle containing them would contain n as well.

We are left to characterize the empty 4-cycles involving at least 2 of n, $n+1$, and $n+2$: The cycle $(i, n, n+1, n+2)$ is empty for all $i \leq n-1$; it is actually a 4-hole. In particular, the empty side contains $\{n, n+2\}$ completely, so this empty 4-cycle will be destroyed in the next step when we introduce vertices just next to that edge. See Figure 7 for an illustration.

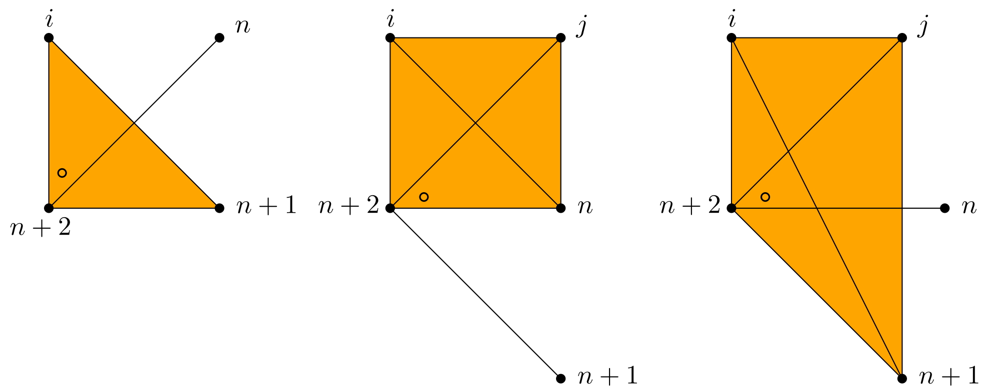


Figure 7: The empty triangles and 4-cycles incident to $n+2$ that will not be empty anymore, once $n+3$ is added to the drawing. The empty side of these cycles is orange, while $n+3$ will be put at the location of the small circle.

Finally the 4-cycles $(i, j, x, y), i < j < n \leq x < y$ are empty if and only if $\{i, j, n\}$ is an empty triangle in $\mathcal{D}_{n+2}$. See Figure 6 for an illustration. Notably, empty triangles $\{i, j, n\}$ from $\mathcal{D}_n$ that contain $n+1$ or $n+2$ also do not leave empty triangles nor empty 4-cycles with $n+1$ and/or $n+2$ since those contain n.

It is therefore important to also consider which empty triangles are incident to $n+2$ after one step. The ones of the form $\{i, j, n+2\}$, $i < j < n$ exist if and only if triangle $\{i, j, n\}$ is still empty in $\mathcal{D}_{n+2}$ because of the second observation and the argumentation in the last paragraph. Since $\{i, n, n+1, n+2\}$ is a 4-hole for all $i < n$, all other triangles are empty as well.

However, the empty 4-cycles of the form $(i, j, x, n+2)$ as well as triangles of the form $\{i, x, n+2\}$ are again destroyed in the next step. See Figure 7 for an illustration. If $x = n$, this is true because for $n+1$ not to be in the empty side, since $\{n+1, n+2\}$ is uncrossed, the empty side has to be on the other side of $\{n, n+2\} = e_{n+2}$. If $x = n+1$, we know $\{n, n+2\} = e_{n+2}$ is crossed only by edges incident to $n+1$ such as $\{i, n+1\}$ and $\{j, n+1\}$ so the empty side has to contain the first segment of it completely.

Thus the only empty 4-cycles introduced in this step, which will stay empty through the next step are of the form $(i, j, n, n+1)$ for each empty triangle $\{i, j, n\}$, whereas the empty triangles $\{i, j, n\}$ that are still empty after that step give rise to empty triangles $\{i, j, n+2\}$ and there is a single additional empty triangle $\{n, n+1, n+2\}$. From Figure 6 it is easy to convince yourself, that after the first step, the only empty triangles that vertex 5 is still incident to are $\{1, 2, 5\}$ and $\{3, 4, 5\}$. Thus the empty triangles incident to the last vertex n are going to be all triangles of the kind $\{2k-1, 2k, n\}$ for some $k < \frac{n}{2}$ and the empty 4-cycles that will stay are of the form $(2i-1, 2i, 2j-1, 2j)$ for $i < j \leq \frac{n}{2}$. These are only $\binom{\lfloor \frac{n}{2} \rfloor}{2} = \frac{1}{8}n^2 + O(n)$ empty 4-cycles and the linear number of additional empty 4-cycles incident to n of the forms $(i, n-2, n-1, n), i < n-2$ and $(2k-1, 2k, x, n), k < \frac{n-2}{2}, x \in \{n-2, n-1\}$ do not change these asymptotics. $\square$

4 Conclusion

We have shown that every convex drawing of K_n with $n \geq 5$ contains a quadratic number of 4-holes and that sufficiently large convex drawings contain 5- and 6-holes, while 7-holes do not exist in general. For $k \in \{5, 6\}$ given, it remains an interesting open question to determine the smallest integer $n(k)$ such that every convex drawing of K_n with $n \geq n(k)$ contains a k-hole.

In the geometric setting, Harborth [24] showed that 10 points are sufficient to contain a 5-hole and since recently it is known that 30 points in general position always contain a 6-hole [27, 32]. Note that for larger point sets we can find a 6-hole in the 30 leftmost points, which is still a 6-hole in the whole point set. By this argument, containing a k-hole is a monotone property for point sets. In contrast to that, we used the SAT framework from [15] to find convex drawings for $n \leq 10$ and $n = 12$ without 5-holes, while proving that every convex drawing for $n = 11$ and $13 \leq n \leq 16$ contains a 5-hole. This shows that containing a k-hole is in general not a monotone property for convex drawings. Based on the computational data, however, we conjecture that every convex drawing on at least 13 vertices contains a 5-hole.

It would further be interesting to obtain better bounds on the size of a largest k-gon and on the size of a largest f-convex subdrawing in a convex drawing of K_n. The currently best estimate for a k-gon is by Suk and Zeng [39], which yields $\Omega((\log n)^{1/2-o(1)})$, and combining this with Lemma 5 yields an f-convex drawing of the same size.

Furthermore, it would be intriguing to define some kind of twisted hole of size k based on the twisted drawing $\mathcal{T}_k$. Then, in the flavor of the result that every simple drawing of K_n contains either a k-gon or twisted subdrawing of a certain size [33, 39], one could try to show the existence of a 6-hole or twisted hole of size 6 in every simple drawing of K_n. Clearly a twisted hole of size k would have to be defined via a plane k-cycle. However, while in a k-gon the plane k-cycle is unique, the twisted drawing $\mathcal{T}_k$ contains exponentially many plane k-cycles [31] and it is unclear which of them would be a somehow canonical choice.

Finally, while our construction for few empty 4-cycles shows a clear difference to the geometric setting, it remains an open question whether a sub-quadratic or even just a linear number of empty 4-cycles is possible in a simple drawing of K_n.

References

[1] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. *Computational Geometry*, 48(7):528–537, 2015. doi:10.1016/j.comgeo.2014.12.007.

[2] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *Computational Geometry*, 47(6):644–650, 2014. doi:10.1016/j.comgeo.2013.12.004.

[3] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. *SIAM Journal on Discrete Mathematics*, 23(4):2147–2155, 2010. doi:10.1137/090767947.

[4] O. Aichholzer, T. Hackl, A. Pilz, P. Ramos, V. Sacristán, and B. Vogtenhuber. Empty triangles in good drawings of the complete graph. *Graphs and Combinatorics*, 31(2):335–345, 2015. doi:10.1007/s00373-015-1550-5.

[5] A. Arroyo, D. McQuillan, R. B. Richter, and G. Salazar. Levi's lemma, pseudolinear drawings of K_n, and empty triangles. *Journal of Graph Theory*, 87(4):443–459, 2018. doi:10.1002/jgt.22167.

[6] A. Arroyo, D. McQuillan, R. B. Richter, and G. Salazar. Convex drawings of the complete graph: Topology meets geometry. *Ars Mathematica Contemporanea*, 22(3):4:1–4:27, 2022. doi:10.26493/1855-3974.2134.ac9.

[7] A. Arroyo, R. B. Richter, and M. Sunohara. Extending drawings of complete graphs into arrangements of pseudocircles. *SIAM Journal on Discrete Mathematics*, 35(2):1050–1076, 2021. doi:10.1137/20M1313234.

[8] M. Balko, R. Fulek, and J. Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete & Computational Geometry*, 53(1):107–143, 2015. doi:10.1007/s00454-014-9644-z.

[9] I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canadian Mathematical Bulletin*, 30(4):436–445, 1987. doi:10.4153/cmb-1987-064-1.

[10] H. Bergold, S. Felsner, M. M. Reddy, J. Orthaber, and M. Scheucher. Plane Hamiltonian cycles in convex drawings. In *Proceedings of the 40th International Symposium on Computational Geometry (SoCG 2024)*, pages 18:1–18:16, 2024. doi:10.4230/LIPIcs.SoCG.2024.18.

[11] H. Bergold, S. Felsner, M. M. Reddy, J. Orthaber, and M. Scheucher. Plane Hamiltonian cycles in convex drawings. *Discrete & Computational Geometry*, pages 1–28, 2025. A conference version of this paper appeared at SoCG 2024 [10]. doi:10.1007/s00454-025-00752-3.

[12] H. Bergold, S. Felsner, M. Scheucher, F. Schröder, and R. Steiner. Topological drawings meet classical theorems from convex geometry. In *Proceedings of the 28th International Symposium on Graph Drawing and Network Visualization (GD 2020)*, pages 281–294, 2020. doi:10.1007/978-3-030-68766-3_22.

[13] H. Bergold, S. Felsner, M. Scheucher, F. Schröder, and R. Steiner. Topological drawings meet classical theorems from convex geometry. *Discrete & Computational Geometry*, 70(3):1121–1143, 2023. A conference version of this paper appeared at GD 2020 [12]. doi:10.1007/s00454-022-00408-6.

[14] H. Bergold, J. Orthaber, M. Scheucher, and F. Schröder. Holes in convex and simple drawings. In *Proceedings of the 32nd International Symposium on Graph Drawing and Network Visualization (GD 2024)*, pages 5:1–5:9, 2024. doi:10.4230/LIPIcs.GD.2024.5.

[15] H. Bergold and M. Scheucher. Investigating simple drawings of K_n using SAT, 2025. arXiv:2504.02650.

[16] P. Brass, W. O. J. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, 2005. doi:10.1007/0-387-29929-7.

[17] O. Devillers, F. Hurtado, G. Károlyi, and C. Seara. Chromatic variants of the Erdős–Szekeres theorem on points in convex position. *Computational Geometry*, 26(3):193–208, 2003. doi:10/bb9h93.

[18] P. Erdős. Some more problems on elementary geometry. *Australian Mathematical Society Gazette*, 5(2):52–54, 1978. URL: https://users.renyi.hu/~p_erdos/1978-44.pdf.

[19] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935. URL: http://www.renyi.hu/~p_erdos/1935-01.pdf.

[20] A. García, A. Pilz, and J. Tejel. On plane subgraphs of complete topological drawings. *Ars Mathematica Contemporanea*, 20(1):69–87, 2021. doi:10.26493/1855-3974.2226.e93.

[21] A. García, J. Tejel, B. Vogtenhuber, and A. Weinberger. Empty triangles in generalized twisted drawings of K_n. *Journal of Graph Algorithms and Applications*, 27(8):721–735, 2023. Special Issue of few Selected Papers from GD 2022 [22]. doi:10.7155/jgaa.00637.

[22] A. García, J. Tejel, B. Vogtenhuber, and A. Weinberger. Empty triangles in generalized twisted drawings of K_n. In *Proceedings of the 30th International Symposium on Graph Drawing and Network Visualization (GD 2022)*, pages 40–48, 2023. doi:10.1007/978-3-031-22203-0_4.

[23] T. Gerken. Empty convex hexagons in planar point sets. *Discrete & Computational Geometry*, 39(1):239–272, 2008. doi:10.1007/s00454-007-9018-x.

[24] H. Harborth. Konvexe fünfecke in ebenen punktmengen. *Elemente der Mathematik*, 33(5):116–118, 1978. doi:10.5169/seals-32945.

[25] H. Harborth. Empty triangles in drawings of the complete graph. *Discrete Mathematics*, 191(1–3):109–111, 1998. doi:10.1016/S0012-365X(98)00098-3.

[26] H. Harborth and I. Mengersen. Drawings of the complete graph with maximum number of crossings. *Congressus Numerantium*, 88:225–228, 1992.

[27] M. J. H. Heule and M. Scheucher. Happy ending: An empty hexagon in every set of 30 points. In *Proceedings of the 30th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2024)*, pages 61–80, 2024. doi:10.1007/978-3-031-57246-3_5.

[28] J. D. Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26(4):482–484, 1983. doi:10.4153/CMB-1983-077-8.

[29] J. Matoušek. *Lectures on Discrete Geometry*. Springer, 2002. doi:10.1007/978-1-4613-0039-7.

[30] C. M. Nicolás. The empty hexagon theorem. *Discrete & Computational Geometry*, 38(2):389–397, 2007. doi:10.1007/s00454-007-1343-6.

[31] J. Orthaber. Crossing-free Hamiltonian cycles in simple drawings of the complete graph (and what we found along the way). Master's thesis, Graz University of Technology, Austria, 2022. doi:10.3217/012rz-4m722.

[32] M. H. Overmars. Finding sets of points without empty convex 6-gons. *Discrete & Computational Geometry*, 29(1):153–158, 2002. doi:10.1007/s00454-002-2829-x.

[33] J. Pach, J. Solymosi, and G. Tóth. Unavoidable configurations in complete topological graphs. *Discrete & Computational Geometry*, 30(2):311–320, 2003. A conference version of this paper appeared at GD 2000 [34]. doi:10.1007/s00454-003-0012-9.

[34] J. Pach and G. Tóth. Unavoidable configurations in complete topological graphs. In *Proceedings of the 8th International Symposium on Graph Drawing (GD 2000)*, pages 328–337, 2001. doi:10.1007/3-540-44541-2_31.

[35] N. H. Rafla. *The Good Drawings D_n of the Complete Graph K_n*. PhD thesis, McGill University, Montreal, 1988. URL: https://escholarship.mcgill.ca/concern/theses/x346d4920.

[36] G. Ringel. Extremal problems in the theory of graphs. In *Theory of Graphs and Its Applications: Proceedings of the Symposium Held in Smolenice in June 1963*, pages 85–90, 1964.

[37] M. Scheucher. A SAT attack on Erdős–Szekeres numbers in $\mathbb{R}^d$ and the empty hexagon theorem. *Computing in Geometry and Topology*, 2(1):2:1–2:13, 2023. doi:10.57717/cgt.v2i1.12.

[38] A. Suk and J. Zeng. Unavoidable patterns in complete simple topological graphs. In *Proceedings of the 30th International Symposium on Graph Drawing and Network Visualization (GD 2022)*, pages 3–15, 2023. doi:10.1007/978-3-031-22203-0_1.

[39] A. Suk and J. Zeng. Unavoidable patterns in complete simple topological graphs. *Discrete & Computational Geometry*, pages 1–13, 2024. A conference version of this paper appeared at GD 2022 [38]. doi:10.1007/s00454-024-00658-6.