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The Complexity of the Fixed Clique Property

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Abstract.

We prove that the following decision problems are co-NP-complete: Determine whether a finite reflexive graph has the fixed clique property. Determine whether a finite simplicial complex has the fixed simplex property. Determine whether a finite truncated lattice has the fixed point property.

1 Introduction

An **ordered set** is a set equipped with a reflexive, antisymmetric and transitive relation $\leq$, the order relation. A function f from one ordered set to another is **order-preserving** iff $x \leq y$ implies $f(x) \leq f(y)$. The task to classify the finite ordered sets with the **fixed point property** (every order-preserving self-map has a fixed point) was one of the original problems on the journal ORDER's list of leading open problems, until it was shown in [5] that the decision problem whether a given finite ordered set has the fixed point property is co-NP-complete.

A **reflexive graph** is a pair (V, E) of a set V of **vertices** and a set E of subsets of V, called **edges**, such that all singleton subsets of V are in E and all edges are singleton or doubleton subsets of V. The vertices in an edge are called **adjacent**. For vertices v, w in a reflexive graph, we write $v \simeq w$ iff $\{v, w\} \in E$ and we write $v \sim w$ iff $v \simeq w$ and $v \neq w$. For two reflexive graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a function $f : V_G \to V_H$ is called a **homomorphism** iff, for all vertices x, y with $x \simeq_G y$, we have $f(x) \simeq_H f(y)$. A homomorphism for which domain and codomain are equal is called an **endomorphism**. Most homomorphisms in this paper will be endomorphisms. All graphs in this paper will be reflexive, and consequently all endomorphisms will be allowed to map an edge between two distinct vertices to a single vertex.[1]

A **clique** is a set of vertices such that any two distinct vertices are adjacent, and a graph has the **fixed clique property** (see [3], p.10; for earlier results, see [4], Section 1, [10, 12, 13]; for more recent contributions, see [8, 11]) iff every endomorphism maps a clique to itself.

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[1] This property is crucial. Readers who prefer graphs without loops need to replace the endomorphisms here with simplicial endomorphisms, see Definition 6.1 in [15].

The fixed clique property is a natural graph-theoretical analogue of the fixed point property. On one hand, as first observed on page 346 in [10], for any reflexive graph with an edge $\{v, w\}$ between distinct vertices $v \neq w$, mapping $V \setminus \{v\}$ to w and w to v produces a fixed-point-free endomorphism, which means that a fixed vertex property for endomorphisms of reflexive graphs is futile. Note that, as shown in [14], the fixed vertex property for endomorphisms of *irreflexive graphs* is quite rich, however.

On the other hand, and much more importantly, the fixed clique property serves as a waystation in the connection between the order-theoretical fixed point property and the topological fixed point property, which was first investigated in [2]. Said connection (also see Figure 9.2 in [15]) passes from the fixed point property for the ordered set through the fixed clique property for the **(reflexive) comparability graph** (two points x and y are adjacent iff $x \leq y$ or $x \geq y$, which produces a reflexive graph), or, equivalently through the fixed simplex property of the clique complex of the comparability graph (every clique is considered to be a simplex), to the topological fixed point property of the topological realization of the clique complex of the comparability graph. By Theorem 2.3 in [2], the topological fixed point property of the topological realization of the clique complex implies the fixed simplex property for the clique complex: Although the theorem concludes the fixed point property for the ordered set, the proof shows there is a fixed simplex for every endomorphism of the simplicial complex. An explicit proof is indicated on page 240 in [15]. Example 2.4 in [2] shows that the converse does not hold, as the clique complex given there has the fixed simplex property; a more explicit example is in Exercise 9-27 in [15]. Because we work with the same functions in each case, the fixed clique property for a comparability graph is equivalent to the fixed simplex property for the comparability graph's clique complex. It is easy to see that the fixed clique property for a comparability graph implies the fixed point property for an ordered set, see [3], remark before Corollaire 3.3, or Proposition 6.7 in [15]. However, the converse does not hold, see Example 6.8 in [15].

The connection between order and topology continues to attract attention, see [1], and recently, see Corollary 4.5 in [7], it led to an advance on the long-standing evasiveness conjecture.

Given the interest in these properties, it is natural to investigate the complexity of the corresponding decision problems. Theorem 3 and Corollaries 12 and 13, though possibly unsurprising, settle the complexity status of the following decision problems, thereby closing obvious gaps in the theory for three interesting properties.

- Determine whether a finite reflexive graph has the fixed clique property.
- Determine whether a finite simplicial complex has the fixed simplex property.
- Determine whether a finite truncated lattice has the fixed point property.

Because of the above mentioned ability to translate problems into new settings, our primary focus will be on the fixed clique property for reflexive graphs, with the remaining results being simple corollaries. Our approach is a technical, though nontrivial, modification of Duffus and Goddard's construction from [5].

2 Background

This section presents fundamental definitions as well as results that will be needed in the main proof, which is presented in Section 3. We start with retractions, which are a standard tool in fixed point theory and which also play an important role in the investigation of the fixed clique property.

Definition 1 *Let $G = (V, E)$ be a reflexive graph. An endomorphism $r : V \to V$ such that $r^2 = r$ is called a* **retraction**. *The induced graph $G[r[V]]$ is called a* **retract** *of G.*

Lemma 2 *Let $G = (V, E)$ be a reflexive graph with the fixed clique property and let $r : V \to V$ be a retraction. Then $G[r[V]]$ has the fixed clique property.*

Proof. For any endomorphism $f : r[V] \to r[V]$ of $G[r[V]]$, we have that $f \circ r$ is an endomorphism of G. Hence $f \circ r$ fixes a clique $C \subseteq r[V]$, which means that $C = f[r[C]] = f[C]$. $\square$

Lemma 3 *Let $G = (V, E)$ be a finite reflexive graph and let $f : V \to V$ be an endomorphism. Then there are a retract R of G and an automorphism Φ of R such that Φ fixes a clique of R iff f fixes a clique of G.*

Proof. It is easy to check, by considering individual vertices, (see, for example, Proposition 4.4 in [15]) that the function $r := f^{|V|!}$ is a retraction of G. Let $R := G\left[f^{|V|!}[V]\right]$ and let $\Phi := f|_{f^{|V|!}[V]}$. It is easy to check (similar to part 2 of Theorem 4.8 in [15]) that Φ is an automorphism that fixes a clique of R iff f fixes a clique of G. $\square$

The **complement** of a reflexive graph (V, E) is the graph (V, E^c), where $E^c = \Big\{\{v, w\} : v, w \in V, v \neq w, \{v, w\} \notin E\Big\} \cup \Big\{\{v\} : v \in V\Big\}$. A graph is called **codisconnected** iff its complement is disconnected. The components of the complement will be called the graph's **cocomponents**. Proposition 4 below characterizes the fixed clique property for codisconnected graphs.

Proposition 4 *Let $G = (V, E)$ be a finite codisconnected reflexive graph. Then G has the fixed clique property iff G has a cocomponent K such that $G[K]$ has the fixed clique property.*

Proof. For the implication "$\Rightarrow$," simply note that if, for every cocomponent K, there were an endomorphism $f_K : K \to K$ that does not fix any clique in $G[K]$, then any union of such maps would be an endomorphism that does not fix any clique in G.

For the implication "$\Leftarrow$," let $f : V \to V$ be an endomorphism. By Lemma 3 and because a retract of a codisconnected graph is codisconnected or a singleton, we can assume that f is an automorphism.

Let $m \in \mathbb{N}$ be the smallest positive integer such that $f^m[K] = K$. Then f^m fixes a clique $C \subseteq K$. Now $\bigcup_{j=1}^{m} f^j[C]$ is a clique fixed by f. $\square$

For two disjoint nonempty sets of vertices A and B, we will write $A \sim B$ iff, for all $a \in A$ and $b \in B$, we have $a \sim b$. If either set is a singleton, $A = \{a\}$ or $B = \{b\}$, respectively, we will omit the set braces and write $a \sim B$ or $A \sim b$, respectively. A **universal vertex** is a vertex $u \in V$ such that $u \sim V \setminus \{u\}$.

Lemma 5 *Let $G = (V, E)$ be a reflexive graph, and let $r : V \to V_R$ be a retraction to the retract $R = G[V_R]$. Let $C \subseteq V$ and $t \in V_R$ such that $r[C] = \{t\}$ and such that, for every $v \in V$, there is a vertex $w \in C \cup \{t\}$ such that $v \simeq w$. Then t is a universal vertex for R and R has the fixed clique property.*

Proof. Let $v \in V_R$ be a vertex in R. By hypothesis, we have $t \simeq v$ or there is a vertex $w \in C$ with $v \simeq w$. In the latter case, we have $v = r(v) \simeq r(w) = t$. Thus t is adjacent to all vertices in V_R, which means it is a universal vertex for R. Because R has a universal vertex t, by Proposition 4, R has the fixed clique property. □

Theorem 1 below shows that the fixed clique property is invariant under the addition or removal of dominated vertices.

Definition 6 *Let $G = (V, E)$ be a graph and let $a, b \in V$ be two distinct vertices. Then a is* **dominated** *by b iff $a \sim b$ and, for all $v \in V \setminus \{a, b\}$, we have that $a \sim v$ implies $b \sim v$.*

Theorem 1 *(See [3], Théorème 3.1.) Let $G = (V, E)$ be a finite reflexive graph and let $a \in V$ be a dominated vertex. Then G has the fixed clique property iff the induced graph $G[V \setminus \{a\}]$ has the fixed clique property.* □

Retractable vertices are a generalization of dominated vertices. Their addition or removal affects the fixed clique property as stated in Theorem 2 below.

Definition 7 *Let $G = (V, E)$ be a graph and let $a, b \in V$ be two distinct vertices. Then a is* **retractable** *to b iff, for all $v \in V \setminus \{a, b\}$, we have that $v \sim a$ implies $v \sim b$.*

Definition 8 *Let $G = (V, E)$ be a finite graph and let $a \in V$. We define the* **neighborhood** *of a to be $N(a) := \{v \in V : a \sim v\} \setminus \{a\}$.*

Theorem 2 *(See [16], Theorem 8.9.2) Let $G = (V, E)$ be a finite reflexive graph and let $a \in V$ be retractable to $b \in V$. Then G has the fixed clique property iff the induced subgraphs $G[V \setminus \{a\}]$ and $G[N(a)]$ both have the fixed clique property.* □

3 A Modification of Duffus and Goddard's Construction

Perhaps unsurprisingly, it is a close analysis and subsequent modification of the argument in [5] that provides the key to the results in this paper. Although the author did indeed look for a more direct argument to settle the complexity status of deciding the fixed clique property for reflexive graphs, it was to no avail. Duffus and Goddard's approach apparently provides the most feasible, and so far the only, path. For the following, recall that the **height** of an ordered set is one less than the size of the largest clique in the comparability graph and that a $2n$**-crown** is an ordered set with elements $c_1 < c_2 > c_3 < \cdots > c_{2n-1} < c_{2n} > c_1$ and no further comparabilities beyond what is indicated and reflexivity. Recall that a **maximal** element of an ordered set is an element that is not strictly below any other element of the ordered set and that order relations are inherited by subsets.

Although Definition 9 is primarily in terms of ordered sets, this is purely for the convenience of being able to use transitivity as we establish the comparabilities in the comparability graph. The argument itself is purely graph theoretical.

Definition 9 (Compare with [5], Section 3.) *Throughout this paper, we let $(X, \leq_X)$, $(Y, \leq_Y)$, and $(Z, \leq_Z)$ be finite ordered sets of height 1, we let $(D_6, \leq_{D_6})$ be a 6-crown and we let $(C_{2n}, \leq_{C_{2n}})$ be a $2n$-crown. We let $H = (V, \leq)$ be an ordered set with the following properties (also see Figure 1), and we let $G = (V, E)$ be the reflexive comparability graph of H.*

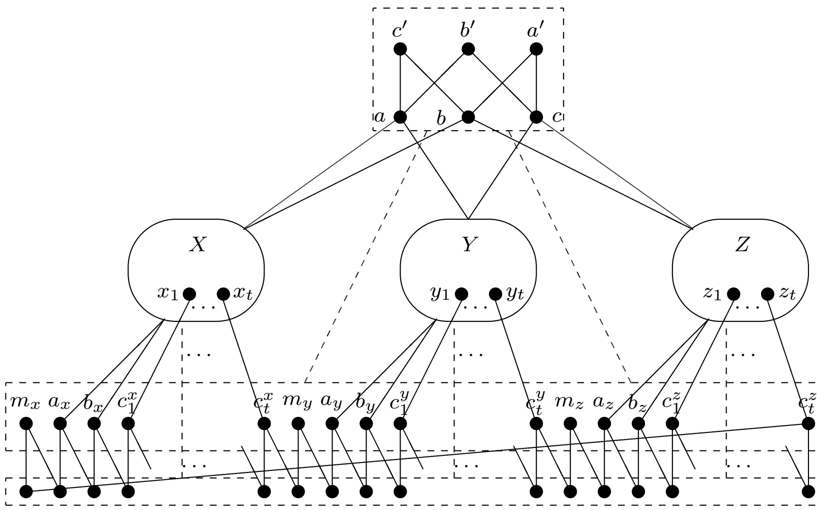

Figure 1: Modification of the construction in [5].

1. *$V = C_{2n} \cup X \cup Y \cup Z \cup D_6$.*

2. *The set D_6 is $\{a, b, c, a', b', c'\}$, and it is ordered by $a < b', c'$; $b < a', c'$; and $c < a', b'$.*

3. *X, Y, Z, and C_{2n} carry their original orders.*

4. *$X < \{a, b\}$, $Y < \{a, c\}$, $Z < \{b, c\}$, and $C_{2n} < \{a, b, c\}$.*

5. *Between the elements of X, Y, Z and the elements of C_{2n}, we have the following comparabilities.*

   (a) *The maximal elements of C_{2n} are m_x, a_x, b_x, $c_1^x, \ldots, c_t^x$, m_y, a_y, b_y, $c_1^y, \ldots, c_t^y$, m_z, a_z, b_z, $c_1^z, \ldots, c_t^z$, where $t > 1$, and any two consecutively listed elements (as well as c_t^z and m_x) have a common lower bound.*

   (b) *For every $S \in \{X, Y, Z\}$, the maximal elements a_s and b_s of C_{2n} are lower bounds of S.*

   (c) *For every $S \in \{X, Y, Z\}$ and every $j \in \{1, \ldots, t\}$, each of the maximal elements c_j^s of C_{2n} is below exactly one (necessarily maximal) element of S, which will be denoted s_j.*

   (d) *The 3 maximal elements m_x, m_y, m_z of C_{2n} are not comparable to any element of X, Y, or Z.*

   (e) *Let M be the set of minimal elements of C_{2n}. Then $M < X \cup Y \cup Z$.*

6. *The only further comparabilities are those added to the above by transitivity.*

Moreover, we make the following additional assumptions on X, Y and Z.

A. *None of X, Y, Z contains a 6-crown.*

B. *For every $S \in \{X, Y, Z\}$, and any two distinct elements s_i and s_j (see Part 5c above), the distance in the induced subgraph $G[S]$ is at least 4.*

Remark 10 The construction in Definition 9 is a modification of the construction from Section 3 in [5], using the structure from Proposition 7.26 in [15]. Because the explicitly stated additional conditions A and B in Proposition 11 are easily checked to be satisfied by the sets $X := P$, $Y := P$ and $Z := Q$, with P and Q as constructed in Section 4 in [5], the only difference between Proposition 11 and Duffus and Goddard's construction is that we demand in condition 5e that $M < X \cup Y \cup Z$. □

Proposition 11 (Compare with [5], Section 3.) *If G has an endomorphism that does not fix any clique in G, then G has an endomorphism $f : V \to V$ that does not fix any clique in G and for which the following hold.*

I. The restrictions $f|_{D_6}$ and $f|_{C_{2n}}$ are automorphisms.

II. We either have $f[X] \subseteq Y$, $f[Y] \subseteq Z$, $f[Z] \subseteq X$, or $f[X] \subseteq Z$, $f[Z] \subseteq Y$, $f[Y] \subseteq X$.

III. For every $S \in \{X, Y, Z\}$, with $U \in \{X, Y, Z\}$ such that $f[S] \subseteq U$, we have $f(s_1) = u_1, \ldots, f(s_t) = u_t$.

Proof. Let $f : V \to V$ be an endomorphism of G that does not fix any clique in G. By Lemma 3, we can assume without loss of generality, that $f = \Phi \circ r$, where $r : V \to V$ is a retraction onto the induced subgraph $R := G[r[V]] = (V_R, E_R)$, and $\Phi : V_R \to V_R$ is an automorphism that does not fix any cliques in R.

Because, by Theorem 1, the presence or absence of dominated vertices does not affect whether a finite reflexive graph has the fixed clique property, we can assume that no vertex in R is dominated by another vertex in R. In particular, this means that R does not have a universal vertex.

Claim 1. $r[C_{2n}] \cap \{a', b', c'\} = \emptyset$.

Suppose, for a contradiction, that there is an $x \in C_{2n}$ that is retracted by r to a'. Because $a' \sim C_{2n} \cup X \cup Y \cup Z$ and $x \sim D_6$, every vertex of R is adjacent to one of x or a'. Application of Lemma 5 with $C = \{x\}$ and $t = a'$ leads to existence of a universal vertex in R, a contradiction. Similarly, we exclude elements of C_{2n} being retracted to b' or c'.

Claim 2. $r[D_6] \cap M = \emptyset$.

Suppose, for a contradiction, that there is an $x \in D_6$ that is retracted by r to a vertex $m \in M$. Because $m \sim X \cup Y \cup Z \cup D_6$ and $x \sim C_{2n}$, every vertex of R is adjacent to one of x or m. Application of Lemma 5 with $C = \{x\}$ and $t = m$ leads to existence of a universal vertex in R, a contradiction.

Claim 3. $r[M] \cap D_6 = \emptyset$.

Suppose, for a contradiction, that $r[M] \cap D_6 \neq \emptyset$. By *Claim 1*, we have $r[M] \cap D_6 \subseteq \{a, b, c\}$. Because $M \sim \{a, b, c\}$ and r is a retraction, we obtain that $r[M] \cap D_6 \subseteq \{a, b, c\}$ is a singleton $\{t\}$. Now note that $t \sim C_{2n}$ and that $r^{-1}(t)$ contains an element of M, which is adjacent to $X \cup Y \cup Z \cup D_6$. By Lemma 5 with $C = r^{-1}(t)$, R has a universal vertex, a contradiction.

Claim 4. V_R intersects at least one of X, Y, Z.

Suppose, for a contradiction, that $V_R \subseteq C_{2n} \cup D_6$. Note that this means that $r[C_{2n}]$ and $r[D_6]$ are unions of cocomponents of R. Because R does not have the fixed clique property, by Proposition 4, no cocomponent of $R[r[C_{2n}]]$ and $R[r[D_6]]$ has the fixed clique property. Because every component of a proper subgraph of a cycle is a path, this means that both $R[r[C_{2n}]]$ and $R[r[D_6]]$ must be cycles, and hence $r[C_{2n}] = C_{2n}$ and $r[D_6] = D_6$. Now, however, because every vertex in $X \cup Y \cup Z$ is adjacent to 5 vertices in D_6 and to more than 5 vertices in C_{2n}, and because vertices in a cycle only have two neighbors in the cycle, no vertex in $X \cup Y \cup Z$ can be retracted to any vertex in $C_{2n} \cup D_6$, a contradiction.

Claim 5. If V_R intersects exactly one of X, Y, Z, then $r[D_6] \subseteq D_6$.

Without loss of generality, assume that V_R only intersects X, that is, $V_R \subseteq C_{2n} \cup X \cup D_6$.

Suppose, for a contradiction, that $c \notin V_R$. The path $D_6 \setminus \{c\}$ is a cocomponent of the induced subgraph $I := G[C_{2n} \cup X \cup (D_6 \setminus \{c\})]$. Because paths have the fixed clique property, by Proposition 4, I has the fixed clique property. Now, via the restriction $r|_{C_{2n} \cup X \cup (D_6 \setminus \{c\})}$, the graph R is a retract of I and hence it has the fixed clique property, a contradiction. Thus $c \in V_R$.

Suppose, for a contradiction, that $r(a) \notin D_6$. Because $a \sim X \cup C_{2n}$, we have $r(a) \simeq r[X \cup C_{2n}]$. Because every vertex in $X \cup C_{2n}$ is adjacent to the vertices a, b, a', b', c', by Lemma 5 applied to $R - c$ with $C = \{a\}$ and $t = r(a)$, we obtain that $r(a)$ is a universal vertex for $R - c$. In particular, this means that $R - c$ has the fixed clique property and that c is, in R, retractable to $r(a)$.

If $r(a) \sim c$, then $r(a)$ would be a universal vertex for R, which cannot be. Thus $r(a) \not\sim c$. Consequently $r(a) \notin C_{2n}$, and hence $r(a) \in X$.

Now, because every vertex of C_{2n} is adjacent to a and to c, we obtain $r[C_{2n}] \subseteq (N(r(a)) \cup \{r(a)\}) \cap (N(c) \cup \{c\}) \subseteq C_{2n} \cup \{a', b'\}$. By Claim 1, $r[C_{2n}]$ does not intersect $\{a', b', c'\}$ and hence $r[C_{2n}] \subseteq C_{2n}$. Because $r(a)$ is not adjacent to all vertices in C_{2n}, we have $r[C_{2n}] \neq C_{2n}$ and hence $r[C_{2n}]$ is a path.

Because $r[C_{2n}]$ is a cocomponent of the neighborhood of c in R, by Proposition 4, the graph induced by R on the neighborhood of c in R has the fixed clique property. By applying Theorem 2 to the graph R and the retractable vertex c, we conclude that R has the fixed clique property, a contradiction. Thus $r(a) \in D_6$.

The exact same argument can be used to prove $r(b), r(c') \in D_6$. The argument can also be used to prove $r(a'), r(b') \in D_6$, and it is indeed simpler for these vertices: Because $a' \sim c$ ($b' \sim c$) leads to $r(a') \sim c$ ($r(b') \sim c$), we immediately obtain a universal vertex and thus the desired contradiction. Consequently $r[D_6] \subseteq D_6$, which completes the proof of *Claim 5*.

Claim 6. $r[D_6] \subseteq D_6$.

We first show $r(a) \in D_6$. Suppose, for a contradiction, that $r(a) \notin D_6$. By *Claim 2*, $r(a) \notin M$. Therefore, there is an $S \in \{X, Y, Z\}$ such that $r(a) \in S \cup \{m_s, a_s, b_s, c_1^s, \ldots, c_t^s\}$. In case $r(a) = m_s$, we would have that $r(a) = m_s$ is not adjacent to any element of $X \cup Y$, which, because $a \sim X \cup Y$, means $V_R \cap (X \cup Y) = r[X \cup Y] \cap (X \cup Y) = \emptyset$. By *Claim 5*, we would have $r[D_6] \subseteq D_6$, which cannot be. Thus $r(a) \in S \cup \{a_s, b_s, c_1^s, \ldots, c_t^s\}$.

Now $r[M] \simeq r(a)$ and (*Claim 3*) $r[M] \cap D_6 = \emptyset$ means that no element of $r[M]$ is adjacent to any element in $(X \cup Y \cup Z) \setminus S$. Because every element of $X \cup Y \cup Z$ is adjacent to an element of M, we obtain $V_R \cap (X \cup Y \cup Z) \setminus S = \emptyset$. Now, by *Claim 5*, $r[D_6] \subseteq D_6$, a contradiction.

The same argument shows $r(a') \in D_6$ and by symmetry we conclude $r[D_6] \subseteq D_6$.

Claim 7. $r[D_6] = D_6$.

Suppose, for a contradiction, that $r[D_6] \neq D_6$. Recall that, by *Claim 6*, $r[D_6] \subseteq D_6$. If $r[D_6]$ was a singleton $\{t\}$, then, via Lemma 5 with $C = D_6$, R would have a universal vertex, which cannot be. Hence $r[D_6]$ has at least 2 vertices. Because D_6 is a cycle, $r[D_6] \subsetneq D_6$, must be a path in D_6 with at most 4 vertices.

Thus at least one of a, b, c is not in $r[D_6]$. Without loss of generality, assume that $a \notin r[D_6]$. Now, $b', c' \in r[D_6]$ would imply $r(a) \sim b', c'$ and hence $r(a) = a \in D_6$, which cannot be. Without loss of generality, we can assume that $c' \notin r[D_6]$. If $b \in r[D_6]$, then, because $r[D_6]$ is a path in D_6 with at least 2 and at most 4 vertices, $a' \in r[D_6]$, and, in R, the vertex b would be dominated by a', which cannot be. Thus $b \notin r[D_6]$, and $r[D_6] \subseteq \{a', b', c\}$. Because R cannot have a universal vertex, we must have $a', b' \in r[D_6]$ and hence $r[D_6] = \{a', b', c\}$.

Now r must retract a vertex $v \in \{a, b, c'\}$ to c. Because $X \sim v$, we obtain $r[X] \simeq r(v) = c$ and c is a universal vertex for R, a contradiction. This completes the proof of *Claim 7*.

Claim 8. $r[C_{2n}] \subseteq C_{2n}$*, and, in case* $r[C_{2n}] \neq C_{2n}$*, we have that* $r[C_{2n}]$ *is a path.*

From $r[D_6] = D_6$, we immediately infer $r[C_{2n}] \subseteq C_{2n}$. In particular, in case $r[C_{2n}] \neq C_{2n}$, we have that $r[C_{2n}]$, being a retract of a cycle, is a path.

Claim 9. No induced 6*-cycle in* G *intersects* M*.*

Suppose, for a contradiction, that B is an induced 6-cycle that intersects M. Then, because every vertex in a cycle has degree 2, $B \cap (X \cup Y \cup Z \cup D_6)$ has at most two vertices, and, because B cannot be contained in C_{2n}, $B \cap (X \cup Y \cup Z \cup D_6)$ has at least one vertex. Therefore, because every vertex in a cycle has degree 2 and $|B \cap (X \cup Y \cup Z \cup D_6)| \geq 1$, we have that $B \cap M$ has at most two vertices, and, because B cannot contain a 4-cycle, $|B \cap (M \cup X \cup Y \cup Z \cup D_6)| \leq 3$. Consequently, B contains at least 3 vertices from $C_{2n} \setminus M$. Because no two vertices from $C_{2n} \setminus M$ are adjacent, B contains exactly 3 vertices from $C_{2n} \setminus M$. Consequently B contains exactly 3 vertices from $M \cup X \cup Y \cup Z \cup D_6$ and no two of these vertices are adjacent. By assumption 5e, this contradicts the fact that B contains at least one vertex each from M and from $X \cup Y \cup Z \cup D_6$.

Claim 10. Every induced 6*-cycle in* G *intersects* D_6*.*

Suppose, for a contradiction that there is an induced 6-cycle B in G that does not intersect D_6.

Clearly, B is not contained in C_{2n}, and, by assumption A, the set B is not contained in any of X, Y, Z. Moreover, by *Claim 9*, we have $B \cap M = \emptyset$. Because, in $X \cup Y \cup Z \cup C_{2n} \setminus M$, no two distinct sets X, Y, or Z have a path from a vertex in one set to a vertex in the other set, we can assume, without loss of generality, that $B \subseteq X \cup \{a_x, b_x, c_1^x, \ldots, c_t^x\}$. Because each c_j^x has degree 1 in the induced graph $G[X \cup \{a_x, b_x, c_1^x, \ldots, c_t^x\}]$, we obtain $B \subseteq X \cup \{a_x, b_x\}$. Because $\{a_x, b_x\}$ is a cocomponent of the induced graph $G[X \cup \{a_x, b_x\}]$, and because 6-cycles are coconnected, we conclude that $B \subseteq X$, a contradiction.

Claim 11. The only induced 6*-cycles in* G *are* D_6 *and* 6*-cycles that contain* a, b, c *and one element each from* X, Y, Z*.*

Let B be an induced 6-cycle in G that contains a'. Because the degree of a' in $G[B]$ is 2, B contains exactly two vertices from $C_{2n} \cup X \cup Y \cup Z \cup \{b, c\}$. Therefore, the remaining 3 vertices of B must be a, b' and c'. Now, because $\{a', b', c'\} \sim C_{2n} \cup X \cup Y \cup Z$ and again because the $G[B]$-degree of every vertex in B must be 2, we conclude that $B \cap (C_{2n} \cup X \cup Y \cup Z) = \emptyset$ and hence B is contained in, and hence equal to, D_6. Similarly, any induced 6-cycle that contains b' or c' must be equal to D_6.

Now let B be an induced 6-cycle in G that does not intersect $\{a', b', c'\}$. By *Claim 10*, without loss of generality, we can assume that $a \in B$. By Claim 9, the two neighbors of a in B are in $X \cup Y \cup C_{2n} \setminus M$ and B must contain exactly two vertices from this set. Thus, the remaining 3 vertices of B are in $Z \cup \{b, c\}$.

Suppose, for a contradiction, that $B \cap \{b, c\} = \emptyset$. Then B contains 3 vertices in Z. Consequently, $a_z, b_z \notin B$, and the two neighbors of a in B are vertices c_i^z and c_j^z with $i \neq j$. The second neighbor of c_i^z in B must be z_i and the second neighbor of c_j^z in B must be z_j. However, this means that, via the shortest path from z_i to z_j in B, that the distance from z_i to z_j in Z is 2, contradicting the additional assumption B.

Thus, without loss of generality, $b \in B$. Now B contains at most one vertex in $X \cup C_{2n} \setminus M$ and at least 3 remaining vertices of B are in $Y \cup Z \cup \{c\}$.

Suppose, for a contradiction, that $c \notin B$. Then B contains 2 vertices in Y or in Z, say, $B \cap Y \supseteq \{v_1, v_2\}$. Then v_1 and v_2 are neighbors of a and the second neighbor of v_1 in B would be in $\{a_y, b_y, c_1^y, \ldots, c_t^y\}$, leading to a having degree at least 3 in B, a contradiction. Thus $c \in B$.

We have established that $B \cap \{a', b', c'\} = \emptyset$ implies $\{a, b, c\} \subset B$. Consequently, the 3 remaining vertices of B must be so that one is adjacent to a and b, but not c, which means this vertex is in

X, one is adjacent to a and c, but not b, which means this vertex is in Y, and one is adjacent to b and c, but not a, which means this vertex is in Z. This completes the proof of *Claim 11.*

We now turn to the automorphism Φ on R.

Claim 12. $\Phi[D_6] = D_6$.

Suppose, for a contradiction, that $\Phi[D_6] \neq D_6$. The image of D_6 under Φ must be an induced 6-cycle. By Claim 11, this means that $\Phi[D_6]$ consists of a, b, c and one element each from X, Y, and Z. Moreover, $\Phi[D_6] \sim \Phi[r[C_{2n}]]$. However, the only vertices that are adjacent to a, b, c and one vertex each from X, Y, Z are the vertices of M. We conclude that $\Phi[r[C_{2n}]] \subset M$, which means that there is an $m \in M$ such that $\Phi[r[C_{2n}]] = \{m\}$. However, then, because Φ is an automorphism, $r[C_{2n}]$ has exactly one element, and, because r is a retraction, $r[C_{2n}] = \{m\}$. Now the vertex m is fixed by Φ, a contradiction to Φ not fixing any cliques.

Claim 13. $r[C_{2n}] = C_{2n}$ *and part I of the conclusion holds.*

Because $C_{2n} = \{v \in V : v \sim D_6\}$ and because $\Phi[D_6] = D_6$, we have that $\Phi[r[C_{2n}]] = r[C_{2n}]$. Because $r[C_{2n}]$ is either a path or equal to C_{2n}, and because paths have the fixed clique property, we infer that $r[C_{2n}] = C_{2n}$, and then $\Phi[C_{2n}] = C_{2n}$. Together with Claim 12, this establishes part I of the conclusion.

To prove part II of the conclusion, note that, by part I of the conclusion, the restrictions $f|_{D_6}$ and $f|_{C_{2n}}$ are automorphisms of the respective induced subgraphs. Every vertex $v \in X \cup Y \cup Z$ is adjacent to 5 vertices in D_6 and to more than 5 vertices in C_{2n}. Therefore, for every $v \in X \cup Y \cup Z$, we have that $f(v)$ cannot be in either of D_6 or C_{2n}. Thus $f[X \cup Y \cup Z] \subseteq X \cup Y \cup Z$.

If there was an $x \in X$ such that $f(x) \in X$, then $f[\{a, b, a', b', c'\}] \subseteq \{a, b, a', b', c'\}$ and f would fix a clique. Thus $f[X] \cap X = \emptyset$. Now, if we had $f[X] \cap Y \neq \emptyset$ and $f[X] \cap Z \neq \emptyset$, then $f[\{a, b, a', b', c'\}] \subseteq \{c, a', b', c'\}$, contradicting that f is an automorphism on D_6. Hence, we either have $f[X] \subseteq Y$ or $f[X] \subseteq Z$. Similarly, we either have $f[Y] \subseteq X$ or $f[Y] \subseteq Z$ and we either have $f[Z] \subseteq X$ or $f[Z] \subseteq Y$.

If we had $f[X], f[Y] \subseteq Z$, then we would have $f[D_6] \subseteq \{b, c, a', b', c'\}$, contradicting that f is an automorphism on D_6. Thus X and Y are not both mapped into Z. We conclude similarly that no two distinct sets among X, Y, Z are mapped into the third set among X, Y, Z. Therefore we either have $f[X] \subseteq Y$, $f[Y] \subseteq Z$, $f[Z] \subseteq X$, or $f[X] \subseteq Z$, $f[Z] \subseteq Y$, $f[Y] \subseteq X$, which is part II of the conclusion.

To prove part III of the conclusion, we only consider the case that $f[X] \subseteq Y$, $f[Y] \subseteq Z$, $f[Z] \subseteq X$, as the other case is similar. Because $M \sim X \cup Y \cup Z$ and because no vertex in $C_{2n} \setminus M$ is adjacent to vertices in more than one set among X, Y, Z, we conclude that $f[M] = M$ and $f[C_{2n} \setminus M] = C_{2n} \setminus M$. Because $f[X] \subseteq Y$, and because every element of $\{a_x, b_x, c_1^x, \ldots, c_t^x\}$ is adjacent to an element of X, every element of $f[\{a_x, b_x, c_1^x, \ldots, c_t^x\}]$ must be adjacent to an element of Y. Therefore, $f[\{a_x, b_x, c_1^x, \ldots, c_t^x\}] \subseteq \{a_y, b_y, c_1^y, \ldots, c_t^y\}$, and, because $f|_{C_{2n}}$ is an automorphism, $f[\{a_x, b_x, c_1^x, \ldots, c_t^x\}] = \{a_y, b_y, c_1^y, \ldots, c_t^y\}$.

Let $j \in \{1, \ldots, t\}$. Then there are vertices $p \in \{a_x, b_x, c_1^x, \ldots, c_t^x\}$ and $x_i \in X$ such that $p \sim x_i$ and $f(p) = c_j^y$. Thus $c_j^y = f(p) \sim f(x_i) \in Y$, and, because the only vertex in Y that is adjacent to c_j^y is y_j, we conclude that $y_j = f(x_i) \in f[X]$. We have shown $f[\{x_1, \ldots, x_t\}] \supseteq \{y_1, \ldots, y_t\}$ which implies $f[\{x_1, \ldots, x_t\}] = \{y_1, \ldots, y_t\}$. Because $t > 1$ and a_x, b_x are adjacent to all x_i, we obtain $f[\{a_x, b_x\}] \subseteq \{a_y, b_y\}$ and then $f[\{a_x, b_x\}] = \{a_y, b_y\}$ and $f[\{c_1^x, \ldots, c_t^x\}] = \{c_1^y, \ldots, c_t^y\}$. Finally, because the image of the path in C_{2n} from a_x to c_t^x that goes through c_1^x must be a path in C_{2n}, we obtain that, for all $j \in \{1, \ldots, t\}$, we have $f(c_j^x) = c_j^y$. Now, for all j, we have $x_j \sim c_j^x$ and $Y \ni f(x_j) \sim f(c_j^x) = c_j^y$, which implies $f(x_j) = y_j$, which is part III of the conclusion. □

4 Main Results

We are now ready to prove the main results in this paper. We first note that any proposed solution for a decision problem in this section can be validated or rejected in polynomial time: For simplicity of language, we consider order-preserving functions to be endomorphisms of ordered sets. Let n be the number of vertices of the structure. Because there are $\frac{n}{2}(n+1)$ pairs of distinct vertices, for any self map f, it can be checked in polynomial, indeed quadratic, time whether it is an endomorphism. Moreover, for every vertex x, it takes n steps to compute the set $\{f^k(x) : k = 1, \ldots, n\}$. It then takes fewer than n^2 steps to check whether this set is a clique or a singleton, which means that, independent of maximum clique size, we can check in fewer than n^4 steps if f maps a clique (or a point) to itself. Consequently, it takes polynomial time to check whether a function is an endomorphism that does not fix any cliques or points. Hence all decision problems in this section are NP.

Theorem 3 *The following decision problem is NP-complete.*
Given. *A finite reflexive comparability graph G in which no clique has more than* 6 *vertices.*
Question. *Is there an endomorphism that does not fix any cliques of G?*

Proof. Let G be a reflexive graph as in Definition 9 with the following additional properties.

i. $G[X]$ is isomorphic to $G[Y]$ via an isomorphism $\Psi : X \to Y$ such that, for $j = 1, \ldots, t$, we have $\Psi(x_j) = y_j$.

ii. There is a homomorphism g from $G[Z]$ to $G[X]$ such that, for $j = 1, \ldots, t$, we have $g(z_j) = x_j$.

Note that no clique in G has more than 6 vertices.

Claim. *G has an endomorphism that does not fix any cliques iff the partial map f^p_{YZ} : $\{y_1, \ldots, y_t\} \to Z$ defined by $f^p_{YZ}(y_j) := z_j$ for $j = 1, \ldots, t$ can be extended to a homomorphism from $G[Y]$ to $G[Z]$.*

Let $\Delta : D_6 \to D_6$ be the automorphism that maps $a \mapsto c \mapsto b \mapsto a$ and $a' \mapsto c' \mapsto b' \mapsto a'$, and let $\Gamma : C_{2n} \to C_{2n}$ be the automorphism that maps $m_x \mapsto m_y \mapsto m_z \mapsto m_x$, $a_x \mapsto a_y \mapsto a_z \mapsto a_x$, $b_x \mapsto b_y \mapsto b_z \mapsto b_x$, and, for $j = 1, \ldots, t$, $c^x_j \mapsto c^y_j \mapsto c^z_j \mapsto c^x_j$, with the natural extension to the minimal elements. If there is a homomorphism $f_{YZ} : Y \to Z$ such that, for $j = 1, \ldots, t$, we have $f_{YZ}(y_j) = z_j$, then

$$f(v) := \begin{cases} \Gamma(v); & \text{if } v \in C_{2n}, \\ \Psi(v); & \text{if } v \in X, \\ f_{YZ}(v); & \text{if } v \in Y, \\ g(v); & \text{if } v \in Z, \\ \Delta(v); & \text{if } v \in D_6, \end{cases}$$

is an endomorphism for G that does not fix any cliques.

Conversely, assume that G has an endomorphism f that does not fix any cliques. By Proposition 11, we have that $f|_{D_6}$ and $f|_{C_{2n}}$ are automorphisms, and either $f[X] \subseteq Y$, $f[Y] \subseteq Z$, $f[Z] \subseteq X$, or $f[X] \subseteq Z$, $f[Z] \subseteq Y$, $f[Y] \subseteq X$. Because X and Y are isomorphic via Ψ and because they play symmetric roles in G, we are free to rename X as Y and Y as X, that is, we can assume that $f[X] \subseteq Y$, $f[Y] \subseteq Z$, and $f[Z] \subseteq X$. Now, by part III of Proposition 11, for $j = 1, \ldots, t$, we have that $f(y_j) = f(z_j)$, which means that $f|_Y$ is an extension of the partial map $f^p_{YZ} : \{y_1, \ldots, y_t\} \to Z$ which, for $j = 1, \ldots, t$, satisfies $f^p_{YZ}(y_j) = z_j$. This proves the *Claim.*

The decision problem OPEXT (see Section 4 of [5]) asks whether, for given ordered sets P and Q, an order-preserving function from a subset of P into Q that satisfies certain conditions can be extended to an order-preserving map from all of P into Q. The proof that OPEXT is NP-complete in Section 4 of [5] actually proves that a certain subproblem is NP-complete. This subproblem, which we shall call SOPEXT here, asks whether, for two given connected ordered sets P and Q of height 1, certain functions that map a subset of the maximal elements of P bijectively to maximal elements of Q can be extended to order-preserving functions from P to Q.

The comparability graph of an ordered set of height 1 is bipartite. Conversely, if the vertices of a bipartite graph B are bipartitioned into the discrete classes L and U, then there is a unique ordered set of height 1 whose comparability graph is B and whose set of maximal elements is U. Consequently, there is a unique way for ordered sets of height 1 to be embedded into and extracted from G as $G[Y]$ or $G[Z]$, as long as we demand that the elements $y_1, \ldots, y_t$ and $z_1, \ldots, z_t$ be maximal elements. We shall assume this is the case from now on. Under this assumption, any function from Y to Z that maps $\{y_1, \ldots, y_t\}$ to $\{z_1, \ldots, z_t\}$ is a homomorphism from $G[Y]$ to $G[Z]$ iff it is an order-preserving map between the corresponding ordered sets.

Consider an instance of SOPEXT as in Section 4 of [5] with ordered sets P and Q, and with pairwise distinct maximal elements $p_1, \ldots, p_t \in P$ and $q_1, \ldots, q_t \in Q$. Let $G[Y]$ be the comparability graph of the ordered set P from SOPEXT and let $y_j := p_j$. Let $G[Z]$ be the comparability graph of the ordered set Q from SOPEXT and let $z_j := q_j$. Finally, let $G[X]$ be another copy of the comparability graph of the ordered set P from SOPEXT, assume that corresponding vertices are denoted with primes, and let $x_j := p'_j$. Then the isomorphism $\Psi : X \to Y$ from assumption i here is $\Psi(p) := p'$.

The additional conditions A and B from Proposition 11 and assumption ii here are easily checked by examining the construction in Section 4 of [5]. To not repeat all the details, we defer the exact details to the reader. Briefly speaking, the ordered sets P and Q are constructed as splits of certain graphs (use the vertices as the maximal elements, the doubleton edges as the minimal elements, and order $v > e$ iff $v \in e$) which do not contain 3-cycles (hence no 6-crowns in P or Q, which is condition A), and in which any two of the distinguished vertices $p_i \neq p_j$ ($q_i \neq q_j$, respectiviely) have graph distance at least 2, which means the ordered set distance is at least 4 (which is condition B). Assumption ii here is satisfied, because P and Q are constructed such that there is an order-preserving map that maps Q to P and each q_j to p_j.

By the *Claim*, the thus constructed graph G has an endomorphism that does not fix any cliques iff the given instance of SOPEXT has a solution. Because the instances of SOPEXT in Section 4 of [5] are in bijective correspondence with instances of 3SAT, we have proved NP-completeness. □

Because no clique in the reflexive graph G from Proposition 11 has more than 6 vertices, the number of simplices in the clique complex is bounded by $c \cdot |V|^6$. Hence the following is a straightforward translation of Theorem 3.

Corollary 12 *The following decision problem is NP-complete.*
Given. *A finite simplicial complex Σ that is the clique complex of a finite comparability graph and in which all simplices have at most 6 points.*
Question. *Is there an endomorphism that does not fix any simplex of Σ?* □

The canonical translation between simplicial complexes and truncated lattices yields the following refinement of Duffus and Goddard's result from [5], which settles a natural problem in the fixed point theory of ordered sets.

Corollary 13 *The following decision problem is NP-complete.*
Given. *A finite truncated lattice T of height ≤ 5.*
Question. *Is there an order-preserving self map that does not fix any of the points in T?* $\square$

Remark 14 By directly computing the homology groups, we can determine whether an ordered set/graph/simplicial complex is acyclic. As long as the maximum chain/clique/simplex size is uniformly bounded, this computation can be done in polynomial time. Hence, although an analogue of Theorem 2 exists for acyclicity, see Corollary A.19 in [15], the arguments here cannot be applied to an investigation of the complexity status of acyclicity. At best, large parts of the proof of Proposition 11 could be used to show that certain retracts of the structure must be acyclic. $\square$

4.1 The Adjacent Vertex Property

A reflexive graph $G = (V, E)$ has the **adjacent vertex property** iff, for each endomorphism $f : V \to V$, there is a $v \in V$ such that $f(v) \simeq v$. The spectacular failure of the Abian-Brown Theorem for the fixed clique property in Exercise 6-7 in [15] is indeed an example of a graph that has the adjacent vertex property and not the fixed clique property. Because of the central role of the Abian-Brown Theorem in the fixed point theory for ordered sets, it is natural to investigate the adjacent vertex property, too.

Because existence of an endomorphism such that, for all vertices v of the graph, we have $f(v) \not\simeq v$ implies existence of an endomorphism that does not fix any cliques, Proposition 11 can be applied in this situation. Moreover, for the reflexive graphs G from Proposition 11 for which endomorphisms without fixed cliques exist, these endomorphisms satisfy $f(v) \not\simeq v$ for all vertices. Therefore, the argument for the fixed clique property can be replicated for the adjacent vertex property, and we conclude the following.

Theorem 4 *The following decision problem is NP-complete.*
Given. *A finite reflexive comparability graph G in which no clique has more than 6 vertices.*
Question. *Is there an endomorphism such that, for all $v \in V$, we have $v \not\simeq f(v)$?* $\square$

Note that $D(P) := \max\limits_{f \in \mathrm{End}(P)} \min\limits_{p \in P} \mathrm{dist}(p, f(p))$ is the **distortion** (see [6, 9]) of an ordered set P, where $\mathrm{End}(P)$ is the set of order-preserving self maps and dist is the distance in the comparability graph. Theorem 4 shows (via the usual translation) that the question whether a truncated lattice has a map with distortion ≤ 2 is co-NP-complete.

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