Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 28, no. 1, pp. 301–312 (2024)
DOI: 10.7155/jgaa.v28i1.2951

Nash Equilibria in Reverse Temporal Voronoi Games

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Submitted: February 2024	Accepted: June 2024	Published: July 2024
Article type: Regular paper	Communicated by: R. Uehara	

Abstract. We study Voronoi games on temporal graphs as introduced by Boehmer et al. (IJCAI '21) where two players each select a vertex in a temporal graph with the goal of reaching the other vertices earlier than the other player. In this work, we consider the *reverse* temporal Voronoi game, that is, a player wants to maximize the number of vertices reaching her earlier than the other player. Since temporal distances in temporal graphs are not symmetric in general, this yields a different game. We investigate the difference between the two games with respect to the existence of Nash equilibria in various temporal graph classes including temporal trees, cycles, grids, cliques and split graphs. Our extensive results show that the two games indeed behave quite differently depending on the considered temporal graph class.

1 Introduction

The Voronoi game on graphs is an influence maximization game where two or more competitive players try to influence as many vertices as possible by choosing one initial vertex (or more) which then propagates the information to other vertices. The Voronoi game on graphs was introduced by Dürr and Thang [6] and is motivated by modeling the spread of information (e.g. viral marketing) or diseases within social networks. Here, each player chooses an initial vertex and wins all vertices with a shorter distance to her chosen vertex than to any other player. A central game-theoretic question is the existence of a Nash equilibrium, that is, a stable strategy profile where no player has an incentive to deviate, for a Voronoi game on a given graph. This question has been studied for Voronoi games on various classes of graphs such as trees, cycles and grids [9, 7, 12].

Recently, Boehmer et al. [4] introduced the *temporal* Voronoi game which is played on temporal graphs, that is, graphs where the edge set changes over discrete time steps [10]. Due to their dynamic nature, temporal graphs are a more realistic model for social networks and are thus a natural extension for the Voronoi game. Boehmer et al. [4] defined the payoff of a player to be the number of vertices which she reaches earlier (that is, the temporal distance is smaller) than

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any other player and studied Nash equilibria on various forms of temporal paths, trees and cycles. They posed the question how the temporal Voronoi game behaves for other temporal distance notions. Note that the temporal distance between vertices in a temporal graph is not symmetric. Hence, we study the reverse definition where a player gains the vertices that reach her before any other player.

Related Work. The Voronoi game has originally been introduced as a competitive facility location problem in continuous spaces by Ahn et al. [1]. For the discrete Voronoi game on graphs, also complexity-theoretic questions about existence of Nash equilibria have been studied [6, 13, 3]. Another similar game on graphs is the *competitive diffusion game* which was introduced by Alon et al. [2] and for which Nash equilibria have also been studied on various graph classes [11, 5, 8]. Boehmer et al. [4] also studied diffusion games on temporal graphs.

Our Contributions. We study the existence of Nash equilibria in *reverse* temporal Voronoi games with two players on (among others) temporal trees, cycles, cliques, grids, and split graphs (see Table 1 for an overview). Our results answer the question of guaranteed existence of Nash equilibria for a wide range of temporal graph classes. For the sake of completeness, we also obtain results for the "classic" temporal Voronoi game on the corresponding temporal graph classes. It turns out that the two games indeed behave differently on temporal trees, cycles and split graphs. For example, on temporally connected trees there is always a Nash equilibrium in the reverse game but not in the classic game, while on monotonically growing cycles the opposite is true. One of the key differences between the two games is that in the classic game the two players can "catch up" each other while this effect does not exist for the reverse game. This seemingly renders the reverse temporal Voronoi game easier to analyze than the temporal Voronoi game where the catch-up dynamics can cause more complicated situations. An interesting side-observation is that on temporally connected trees the reverse temporal Voronoi game behaves similar to the static Voronoi game on static trees while on monotonically shrinking split graphs the classic temporal Voronoi game behaves analogous to the static case.

Organization of the Paper. Our paper is organized as follows: Section 2 introduces basic definitions of temporal graphs and the (reverse) temporal Voronoi game. The results for the reverse temporal Voronoi game are then presented in Section 3 followed by the results for the temporal Voronoi game in Section 4. We conclude with some open questions in Section 5.

2 Preliminaries

For $a \le b \in \mathbb{N}$, let $[a,b] := \{a, a+1, \ldots, b\}$ and let $[a] := [1,a]$.

Temporal Graphs. A *temporal graph* $\mathcal{G} = (V, (E_t)_{t=1}^{\infty})$ consists of a finite set V of vertices and an infinite sequence $(E_t)_{t=1}^{\infty}$ of edge sets $E_t \subseteq \binom{V}{2}$. If there is an integer i such that $E_t = E_i$ for all $t \ge i$, then we define the *lifetime* $\tau(\mathcal{G})$ of $\mathcal{G}$ to be the minimum such integer. For our considered game, we can assume that all temporal graphs have finite lifetime τ. Hence, we do not specify E_i for $i > \tau$. The (static) graph $G_t := (V, E_t)$ is called the *t-th layer* of $\mathcal{G}$ and $\mathcal{G}_\downarrow := (V, E_\downarrow)$ with $E_\downarrow := \bigcup_{t=1}^{\infty} E_t$ is the *underlying (static) graph* of $\mathcal{G}$.

A *temporal path* (resp. *tree*, *cycle* etc.) is a temporal graph whose underlying graph is a path (resp. tree, cycle etc.). A (static) *$(n \times m)$-grid graph* is a graph which is isomorphic to $([n] \times [m], \{\{(i,j),(i',j')\} \mid |i-i'| + |j-j'| = 1\})$. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. A *threshold graph* is a split graph that can

Table 1: Overview of results. The results in the top rows (marked with *) are by Boehmer et al. [4]. A "✓" indicates guaranteed existence of a Nash equilibrium while an "✗" means that a Nash equilibrium is not guaranteed to exist. Entries in parentheses are implied by other table cells.

	Temporally connected	**Monotonically growing**	**Monotonically shrinking**
Temporal Voronoi			
Temporal Paths*	✗	(✓)	✗
Temporal Trees*	(✗)	✓	(✗)
Temporal Cycles*	✗	✓	✗
Temporal Grids	(✗)	✗ (Theorem 8)	(✗)
Temporal Cliques	(✗)	✗ (Corollary 3)	(✓)
Temp. Complete k-partite ($k \geq 2$)	(✗)	✗ (Corollary 3)	✓ (Theorem 9)
Temporal Threshold	(✗)	(✗)	(✓)
Temporal Split	(✗)	(✗)	✓ (Theorem 10)
Reverse Temporal Voronoi			
Temporal Paths	(✓)	(✓)	✗ (Theorem 4)
Temporal Trees	✓ (Theorem 1)	(✓)	(✗)
Temporal Cycles	(✗)	✗ (Theorem 2)	✗ (Theorem 5)
Temporal Grids	(✗)	✗ (Theorem 3)	(✗)
Temporal Cliques	(✗)	✗ (Corollary 1)	(✓)
Temp. Complete k-partite ($k \geq 2$)	(✗)	✗ (Corollary 2)	✓ (Theorem 7)
Temporal Threshold	(✗)	(✗)	✓ (Theorem 7)
Temporal Split	(✗)	(✗)	✗ (Theorem 6)

be constructed by iteratively adding isolated vertices or dominating vertices (hence, there always exists a vertex which dominates all non-isolated vertices).

In a temporal graph $\mathcal{G} = (V, (E_t)_{t=1}^{\infty})$, a *temporal walk* from a vertex u to a vertex v is a sequence $(\{v_0 \coloneqq u, v_1\}, t_1), (\{v_1, v_2\}, t_2), \ldots, (\{v_{d-1}, v_d \coloneqq v\}, t_d)$ such that $t_i < t_{i+1}$ for all $i \in [d-1]$ and $\{v_{i-1}, v_i\} \in E_{t_i}$ for all $i \in [d]$. We call t_d the *arrival time* of the temporal walk. A temporal walk from u to v is called *foremost* if there is no temporal walk from u to v with earlier arrival time. The *temporal distance* $\operatorname{td}(u, v)$ from u to v is the arrival time of a foremost walk from u to v (we set $\operatorname{td}(u, v) \coloneqq 0$ if $u = v$). If there is no such walk, then $\operatorname{td}(u, v) \coloneqq \infty$. Note that temporal distances are not symmetric, that is, $\operatorname{td}(u, v) \neq \operatorname{td}(v, u)$ is possible. We say that a vertex u *reaches* a vertex v *until* (*at*) step t if $\operatorname{td}(u, v) \leq t$ $(= t)$. A temporal graph $\mathcal{G}$ is *temporally connected* if $\operatorname{td}(u, v) < \infty$ for all vertex pairs u, v. Further, $\mathcal{G}$ is *monotonically growing (shrinking)* if edges do not disappear (appear) over time, that is, $E_t \subseteq E_{t+1}$ $(E_{t+1} \subseteq E_t)$ for all t. Note that, if $\mathcal{G}_\downarrow$ is connected, then monotonic growth of $\mathcal{G}$ implies temporal connectedness.

(Reverse) Temporal Voronoi Games. For a temporal graph $\mathcal{G} = (V, (E_t)_{t=1}^{\infty})$ and a number $k \in \mathbb{N}$ of players, Boehmer et al. [4] introduced the k-player *temporal Voronoi game* $\operatorname{Vor}(\mathcal{G}, k)$ on $\mathcal{G}$. The *strategy space* of each player $i \in [k]$ is the vertex set V, that is, each player i selects a single vertex $p_i \in V$ (also called *position*). A *strategy profile* is a tuple $(p_1, \ldots, p_k) \in V^k$ containing the chosen vertices of all players. In $\mathrm{Vor}(\mathcal{G}, k)$, the strategy profile $(p_1, \dots, p_k)$ determines a vertex subset $U_i(p_1, \dots, p_k) \coloneqq \{v \in V \mid \forall j \neq i : \mathrm{td}(p_i, v) < \mathrm{td}(p_j, v)\}$ for each player i. The *payoff* of player i is then $u_i(p_1, \dots, p_k) \coloneqq |U_i(p_1, \dots, p_k)|$. That is, each player "wins" those vertices which she reaches earlier than all other players. In the *reverse temporal Voronoi game* $\mathrm{rVor}(\mathcal{G}, k)$, we define the set $U_i(p_1, \dots, p_k) \coloneqq \{v \in V \mid \forall j \neq i : \mathrm{td}(v, p_i) < \mathrm{td}(v, p_j)\}$, that is, each player "wins" those vertices which reach her earlier than any other player.

In both games, the players aim to maximize their payoffs. Hence, player i plays a *best response* to the other players in $(p_1, \dots, p_k)$ if for all vertices $p' \in V$ it holds that

$$u_i(p_1, \dots, p_{i-1}, p', p_{i+1}, \dots, p_k) \leq u_i(p_1, \dots, p_k).$$

A strategy profile $(p_1, \dots, p_k)$ is a *Nash equilibrium* if every player plays a best response to the other players. In this paper, we only consider $k = 2$ players.

3 Reverse Temporal Voronoi Games (rVor)

In this section we prove the results for the reverse temporal Voronoi game shown in Table 1.

3.1 Temporally Connected Graphs

For temporally connected graphs, a Nash equilibrium is only guaranteed if the underlying graph is a tree. In fact the Nash equilibrium is analogous to the Voronoi game on static graphs. Note that for all other temporal graph classes considered in this paper, there are examples without a Nash equilibrium already for monotonically growing graphs (as shown in Section 3.2).

Theorem 1 *On every temporally connected tree $\mathcal{T}$, there exists a Nash equilibrium in* $\mathrm{rVor}(\mathcal{T}, 2)$.

Proof: Let $T \coloneqq \mathcal{T}_\downarrow = (V, E)$ with $|V| \geq 2$ (the case $|V| = 1$ is trivial). Let p_1 be a centroid of T (that is, a vertex v that minimizes the maximum size of any connected component in $T - v$) and let p_2 be a neighbor of p_1 in a maximum-size component $C = (V', E')$ of $T - p_1$. Then (p_1, p_2) is a Nash equilibrium. Note that $U_1(p_1, p_2) = V \setminus V'$ since $\mathcal{T}$ is temporally connected and all vertices in $V \setminus V'$ reach p_1 before p_2. Analogously, it holds $U_2(p_1, p_2) = V'$. Since p_1 is a centroid, we have $u_1(p_1, p_2) \geq |V|/2$ and $u_2(p_1, p_2) \leq |V|/2$. Clearly, player 2 cannot improve since she could only win vertices within a component of $T - p_1$ and C is already maximal. Also player 1 cannot improve since she could only win a subset of vertices of $V \setminus V'$ or V'. □

3.2 Monotonically Growing Graphs

In the following, we show that disallowing edges to disappear does not guarantee a Nash equilibrium (except for trees). The following theorem is in contrast to the classic temporal Voronoi game where a Nash equilibrium always exists for monotonically growing cycles as shown by Boehmer et al. [4].

Theorem 2 *There is a monotonically growing cycle $\mathcal{C}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{C}, 2)$.

Proof: Consider the temporal cycle $\mathcal{C} \coloneqq ([7], E_1, E_2)$ where $E_1 \coloneqq \{\{i, i+1\} \mid i \in [6]\} \setminus \{\{2, 3\}\}$ and $E_2 \coloneqq E_1 \cup \{\{2, 3\}, \{7, 1\}\}$ (see Figure 1). To show that there is no Nash equilibrium, we show that both players can always win at least 4 vertices regardless of the choice of the other player. Since there are only 7 vertices in total, there cannot be a Nash equilibrium. To show the above claim, we assume that $p_1 \in [2, 5]$ (by symmetry of $\mathcal{C}$). The following cases are easily verified:

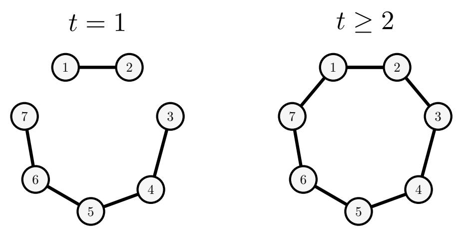

Figure 1: A monotonically growing temporal cycle without a Nash equilibrium.

- If $p_1 = 2$, then, for $p_2 = 5$, it holds $[4, 7] \subseteq U_2(p_1, p_2)$.
- If $p_1 = 3$, then, for $p_2 = 4$, it holds $[4, 7] \subseteq U_2(p_1, p_2)$.
- If $p_1 = 4$, then, for $p_2 = 7$, it holds $\{1, 2, 6, 7\} \subseteq U_2(p_1, p_2)$.
- If $p_1 = 5$, then, for $p_2 = 4$, it holds $[1, 4] \subseteq U_2(p_1, p_2)$.

□

Recall that a Nash equilibrium is guaranteed for the temporal Voronoi game on monotonically growing cycles. An example for a Nash equilibrium in the temporal Voronoi game $\mathrm{Vor}(\mathcal{C}, 2)$ on the cycle from the proof above, is $(5, 4)$. Here, both players win three vertices.

From Theorem 2, we easily obtain an analogous result for monotonically growing cliques (and with it also for split and threshold graphs.).

Corollary 1 *There is a monotonically growing clique $\mathcal{Q}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{Q}, 2)$.

Proof: Consider the monotonically growing cycle $\mathcal{C}$ from Theorem 2 where no Nash equilibrium exists. Since $\mathcal{C}$ is temporally connected, all pairwise temporal distances are finite, that is, $\mathrm{td}(u, v) \leq d$ for all $u, v \in [7]$ and some $d \in \mathbb{N}$. Hence, any further edge appearing after time d is not changing the temporal distance of any vertex pair. Therefore, $\mathcal{Q} := ([7], E'_1, \ldots, E'_{d+1})$ with $E'_t := E_t$ for all $t \leq d$ and $E'_{d+1} := \binom{[7]}{2}$ is a monotonically growing clique without a Nash equilibrium. □

Next, we show that also on monotonically growing grids there is no guarantee for a Nash equilibrium.

Theorem 3 *There is a monotonically growing grid $\mathcal{G}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{G}, 2)$.

Proof: Consider the graph $\mathcal{G} := ([6], E_1, E_2)$ (depicted in Figure 2) with

$$E_1 := \{\{1, 2\}, \{1, 4\}, \{3, 6\}, \{5, 6\}\} \text{ and}$$
$$E_2 := E_1 \cup \{\{2, 3\}, \{2, 5\}, \{4, 5\}\}.$$

By symmetry, assume that $p_1 \in \{1, 2, 4\}$.

- If $p_1 = 1$, then the best response by player 2 is $p_2 = 2$, where $U_1(1, 2) = \{1, 4\}$ and $U_2(1, 2) = \{2, 3, 5, 6\}$. But $p_1 = 6$ yields $U_1(6, 2) = \{3, 5, 6\}$.
- If $p_1 = 2$, then $p_2 = 6$ is the best response with $U_1(2, 6) = \{1, 2, 4\}$ and $U_2(2, 6) = \{3, 5, 6\}$. But then $p_1 = 5$ is the best response with $U_1(5, 6) = \{1, 2, 4, 5\}$.

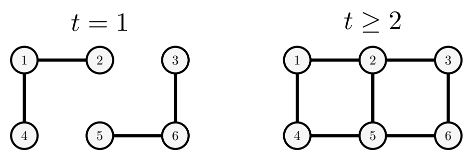

Figure 2: A monotonically growing temporal grid without a Nash equilibrium.

- If $p_1 = 4$, then a best response is $p_2 = 6$ with $U_1(4,6) = \{1,2,4\}$ and $U_2(4,6) = \{3,5,6\}$. But then the best response is $p_1 = 5$ again. Also $p_2 = 5$ is a best response, where $U_1(4,5) = \{1,4\}$ and $U_2(4,5) = \{3,5,6\}$. Then, p_1 can improve with $U_1(1,5) = \{1,2,4\}$.

□

Interestingly, note that (1,6) is a Nash equilibrium for the classic temporal Voronoi game on the above grid of Theorem 3. (However, this is not the case for all monotonically growing grids as shown in Theorem 8.)

Again, from Theorem 3, we easily obtain the following corollary.

Corollary 2 *For every $k \geq 2$, there is a monotonically growing complete k-partite graph $\mathcal{K}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{K}, 2)$.

Proof: Consider the monotonically growing grid $\mathcal{G}$ from Theorem 3 (which is bipartite) where no Nash equilibrium exists. Since all pairwise temporal vertex distances are finite (at most some $d \in \mathbb{N}$), we can modify $\mathcal{G}$ as follows without introducing a Nash equilibrium: $\mathcal{K} := ([4+k], E'_1, \ldots, E'_{d+1})$, where $E'_t := E_t$ for all $t \leq d$ and

$$E'_{d+1} := E'_d \cup \{\{1,6\},\{3,4\}\} \cup \bigcup_{j=7}^{4+k} \{\{j,i\} \mid i < j\}.$$

Note that $\mathcal{K}$ is a monotonically growing complete k-partite graph where all temporal distances between vertices in $[6]$ are the same as in $\mathcal{G}$. It remains to check that there is no Nash equilibrium. If both players pick vertices in $[6]$, then the outcome is exactly the same as in $\mathcal{G}$ (all newly introduced vertices have equal temporal distance to both players and hence are not won by any player). Hence, this is not a Nash equilibrium. If a player picks one of the new vertices, then this is never optimal, since she only wins this single vertex, whereas she could win at least two vertices by choosing some vertex in $[6]$. □

3.3 Monotonically Shrinking Graphs

We now consider temporal graphs where no edges are allowed to appear over time. It turns out that among the graph classes we considered, a Nash equilibrium is only guaranteed if the game is essentially "decided" in the first layer, that is, on temporal complete k-partite and threshold graphs. We start with excluding Nash equilibria from all other considered temporal graph classes.

Theorem 4 *There is a monotonically shrinking path $\mathcal{P}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{P}, 2)$.

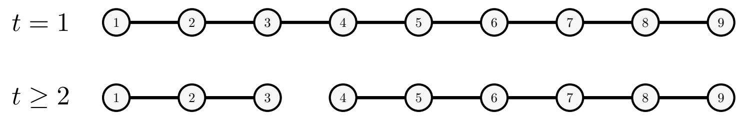

Figure 3: A monotonically shrinking temporal path without a Nash equilibrium.

Proof: Let $\mathcal{P} := ([9], E_1, E_2)$ with $E_1 := \{\{i, i+1\} \mid i \in [8]\}$ and $E_2 := E_1 \setminus \{\{3, 4\}\}$ (Figure 3). Clearly, if $\{p_1, p_2\} \subseteq [3]$, then this is not optimal, since each player can win at most three vertices, whereas choosing vertex 4 yields at least six vertices. If $\{p_1, p_2\} \subseteq [4, 9]$, then we can assume without loss of generality that $4 \leq p_1 < p_2 = p_1 + 1$.

- If $p_1 = 4$, then $u_1(p_1, p_2) = 2$ and $u_1(6, p_2) = 4$.
- If $p_1 = 5$, then $u_1(p_1, p_2) = 3$ and $u_1(3, p_2) = 4$.
- If $p_1 \geq 6$, then $u_2(p_1, p_2) \leq 3$ and $u_2(p_1, 3) \geq 4$.

Finally, let $p_1 \leq 3 < p_2$ (wlog).

- If $p_2 \leq 5$, then $u_1(p_1, p_2) \leq 3$ and $u_1(6, p_2) = 4$.
- If $p_2 > 5$ and $4 \in U_1(p_1, p_2)$, then $u_2(p_1, p_2) \leq 5$ and $u_2(p_1, 4) = 6$.
- If $p_2 > 5$ and $4 \notin U_1(p_1, p_2)$, then $u_1(p_1, p_2) \leq 3$ and $u_1(3, p_2) = 4$.

□

Theorem 5 *There is a monotonically shrinking cycle $\mathcal{C}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{C}, 2)$.

Proof: Let $\mathcal{C} := ([10], E_1, E_2)$ with $E_1 := \{\{i, i+1\} \mid i \in [9]\} \cup \{\{10, 1\}\}$ and $E_2 := E_1 \setminus \{\{3, 4\}, \{10, 1\}\}$ (see Figure 4). Clearly, if $\{p_1, p_2\} \subseteq [3]$, then this is not a Nash equilibrium since each player wins at most four vertices and choosing vertex 4 would yield at least six vertices. Now consider the case if $\{p_1, p_2\} \subseteq [4, 10]$. Note that vertex 2 does not reach any of the players. Hence, the remaining graph behaves like a path. Therefore, we can assume that $p_2 = p_1 + 1$.

- If $p_1 \geq 8$, then $u_2(p_1, p_2) \leq 3$ and $u_2(p_1, 7) \geq 5$.
- If $p_1 \leq 5$, then $u_1(p_1, p_2) \leq 3$ and $u_1(7, p_2) \geq 5$.
- If $p_1 = 6$, then $u_1(6, 7) = 4$ and $u_1(2, 7) = 5$.
- If $p_1 = 7$, then $u_2(7, 8) = 4$ and $u_2(7, 2) = 5$.

Finally, assume $p_1 \leq 3 < p_2 \leq 7$ (by symmetry of the cycle).

- If $p_2 \leq 5$, then $u_1(p_1, p_2) \leq 4$ and $u_1(6, p_2) = 6$.
- If $p_2 = 6$, then $U_2(p_1, 6) \subseteq [4, 9]$. If $4 \notin U_2(p_1, 6)$, then $p_1 \geq 2$ and thus player 2 can improve with $p_2 = 4$ giving $U_2(p_1, 4) = [4, 9]$. If $4 \in U_2(p_1, 6)$, then $p_1 = 1$ and player 2 can improve with $p_2 = 4$ to $U_2(1, 4) = [3, 9]$.
- If $p_2 = 7$, then $u_2(p_1, 7) \leq 5$ and $u_2(p_1, 4) \geq 6$.

□

Notably, for temporal paths already one disappearing edge is enough to exclude a Nash equilibrium while the counterexample for cycles has two disappearing edges. In fact, one can show that for cycles a Nash equilibrium always exists if at most one edge disappears.

It remains to exclude Nash equilibria for monotonically shrinking split graphs.

Theorem 6 *There exists a monotonically shrinking split graph $\mathcal{S}$ such that there is no Nash equilibrium in* $\mathrm{rVor}(\mathcal{S}, 2)$.

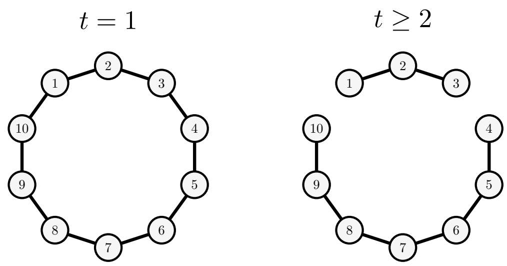

Figure 4: A monotonically shrinking temporal cycle without a Nash equilibrium.

Proof: Let $\mathcal{S} \coloneqq (V = C \cup I, E_1, E_2)$ with $C \coloneqq [4,7]$, $I \coloneqq \{1,2,3,8\}$, and

$$E_1 \coloneqq \binom{C}{2} \cup \{\{1,4\},\{2,4\},\{2,5\},\{3,5\},\{6,8\},\{7,8\}\},$$
$$E_2 \coloneqq \{\{2,4\},\{2,5\},\{4,6\},\{5,7\}\}.$$

Figure 5 shows $\mathcal{S}$. By symmetry of $\mathcal{S}$, let $p_1 \in \{1,2,4,6,8\}$.

- If $p_1 = 1$, then clearly $p_2 = 4$ is the best response. But player 1's best response is then $p_1 = 7$ which yields $U_1(7,4) = \{3,7,8\}$.
- If $p_1 = 2$, then we can assume $p_2 \in \{1,4,6,8\}$ by symmetry. The best response is $p_2 = 4$, where $U_1(2,4) = \{2,3\}$. Again, player 1 can improve with $p_1 = 7$.
- If $p_1 = 4$, then $p_2 = 7$ is the best response with $U_1(4,7) = \{1,2,4\}$. But then $p_1 = 5$ yields $U_1(5,7) = \{1,2,3,5\}$.
- If $p_1 = 6$, then $p_2 = 4$ is the best response with $U_1(6,4) = \{6,8\}$. Again, player 1 improves with $p_1 = 7$.
- If $p_1 = 8$, then $p_2 = 6$ is the best response (up to symmetry). Player 1 can improve with $p_1 = 5$ which yields $U_1(5,6) = \{2,3,5\}$.

□

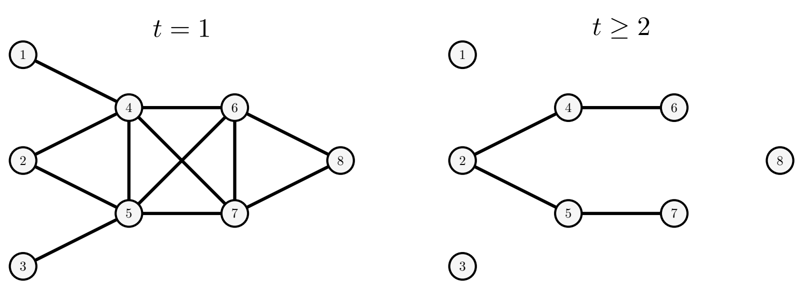

Figure 5: A monotonically shrinking temporal split graph without a Nash equilibrium.

Note that the classic temporal Voronoi game always has a Nash equilibrium on monotonically shrinking split graphs (as we show in Theorem 10). For example, (4,5) is a Nash equilibrium for the temporal split graph in the proof of Theorem 6.

Contrasting Theorem 6, we finish this section with a positive result for threshold graphs (and thus also cliques) and complete k-partite graphs.

Theorem 7 *There exists a Nash equilibrium in* rVor$(\mathcal{G}, 2)$ *if* $\mathcal{G}$ *is a monotonically shrinking*
(1) complete k*-partite graph with* $k \geq 1$ *or*
(2) threshold graph.

Proof: (1) Let $\mathcal{G} = (V_1 \cup \cdots \cup V_k, (E_t)_{t=1}^{\infty})$ be a monotonically shrinking temporal complete k-partite graph with $k \geq 2$ (the case $k = 1$ is trivial). Let $p_1 \in V_1$ and $p_2 \in V_2$. Then, (p_1, p_2) is a Nash equilibrium. Note that $u_1(p, p_2) = |V_2|$ for all $p \in V \setminus V_2$ and $u_1(p, p_2) \leq 1$ if $p \in V_2$ (and symmetrically for $u_2(p_1, p) = |V_1|$ if $p \in V \setminus V_1$ and $u_2(p_1, p) \leq 1$ if $p \in V_1$) since G_1 is complete k-partite. Hence, no player can improve.

(2) Let $\mathcal{G} = (V, (E_t)_{t=1}^{\infty})$ be a monotonically shrinking temporal threshold graph with $|V| \geq 2$ (the case $|V| = 1$ is trivial). If all vertices are isolated in G_1, then a Nash equilibrium trivially exists. Otherwise, there exists a vertex v which dominates all non-isolated vertices in G_1. Then, (v, w) with $w \neq v$ is a Nash equilibrium. Clearly, player 1 cannot improve since all non-isolated vertices already reach v no later than time step 1. Hence, also player 2 cannot improve. □

4 Temporal Voronoi Games (Vor)

We complement the results for the reverse temporal Voronoi game from Section 3 with the missing results for the remaining graph classes for the classic temporal Voronoi game.

4.1 Monotonically Growing Graphs

We first show that for grids a Nash equilibrium is also not guaranteed.

Theorem 8 *There exists a monotonically growing grid* $\mathcal{G}$ *such that there is no Nash equilibrium in* Vor$(\mathcal{G}, 2)$.

Proof: Consider the (3×4)-grid $\mathcal{G}$ with vertex set [12] given in Figure 6 (left). To see that there is no Nash equilibrium, we consider the best responses (Figure 6 (right)) which are straightforward to verify.

- For $p_1 = 1$, the best response is $p_2 = 6$ with $u_2(1, 6) = 10$.
- For $p_1 = 2$, the best response is $p_2 = 6$ with $u_2(2, 6) = 5$.
- For $p_1 = 3$, a best response is $p_2 \in \{6, 7\}$ with $u_2(3, p_2) = 8$.
- For $p_1 = 4$, the best response is $p_2 = 3$ with $u_2(4, 3) = 9$.
- For $p_1 = 5$, a best response is $p_2 \in \{2, 6, 10\}$ with $u_2(5, p_2) = 9$.
- For $p_1 = 6$, the best response is $p_2 = 8$ with $u_2(6, 8) = 3$.
- For $p_1 = 7$, a best response is $p_2 \in \{2, 6, 10\}$ with $u_2(7, p_2) = 6$.
- For $p_1 = 8$, the best response is $p_2 = 7$ with $u_2(8, 7) = 9$.
- The case $p_1 \in \{9, 10, 11, 12\}$ is symmetric to $p_1 \in \{1, 2, 3, 4\}$.

Note that the above best responses always run into a cycle $6 \to 8 \to 7 \to 6$ or $6 \to 8 \to 7 \to 2(10) \to 6$. Hence, there exists no Nash equilibrium. □

For monotonically growing cliques (and also threshold and split graphs) and complete k-partite graphs, the same arguments as in Corollaries 1 and 2 for the reverse Voronoi game apply. Hence, from Theorem 8, we also obtain the following.

Corollary 3 *There exists a monotonically growing clique* $\mathcal{C}$ *and a monotonically growing complete* k*-partite graph* $\mathcal{K}$ *for each* $k \geq 2$ *such that* Vor$(\mathcal{C}, 2)$ *and* Vor$(\mathcal{K}, 2)$ *have no Nash equilibrium.*

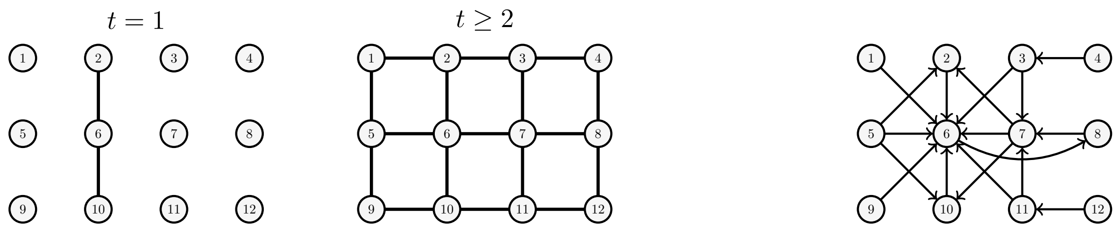

Figure 6: (Left) A monotonically growing temporal grid $\mathcal{G}$ without a Nash equilibrium in $\mathrm{Vor}(\mathcal{G}, 2)$. (Right) The best response graph of $\mathrm{Vor}(\mathcal{G}, 2)$.

4.2 Monotonically Shrinking Graphs

For monotonically shrinking graphs, the following theorem is easily obtained from analogous arguments as for Theorem 7. Hence, we omit a formal proof.

Theorem 9 *For every monotonically shrinking complete k-partite graph $\mathcal{K}$ with $k \geq 1$, there exists a Nash equilibrium in* $\mathrm{Vor}(\mathcal{K}, 2)$.

Finally, for the temporal Voronoi game, a Nash equilibrium is guaranteed even for monotonically shrinking split graphs (as opposed to the reverse Voronoi game). Notably, this case is analogous to the Diffusion game on static split graphs [8].

Theorem 10 *For every monotonically shrinking split graph $\mathcal{S}$, there exists a Nash equilibrium in* $\mathrm{Vor}(\mathcal{S}, 2)$.

Proof: Let $\mathcal{S} = (V, (E_t)_{t=1}^{\infty})$ with $V = C \cup I$, where C forms a clique in $S_1 = \mathcal{S}_{\downarrow}$ and I an independent set. We assume for each vertex $v \in I$ that it is not adjacent to all vertices in C, since otherwise we could remove v from I and add it to C. Hence, we can also assume $|C| \geq 2$, since otherwise a Nash equilibrium trivially exists. We now show that there exists a Nash equilibrium (p_1, p_2) with $\{p_1, p_2\} \subseteq C$ and $p_1 \neq p_2$. To this end, observe that in this case we have $U_1(p_1, p_2) = \{p_1\} \cup N_I(p_1) \setminus N_I(p_2)$ and $U_2(p_1, p_2) = \{p_2\} \cup N_I(p_2) \setminus N_I(p_1)$, where $N_I(v)$ denotes the set of neighbors of v in I. Clearly, no player can improve by choosing a vertex in I since the payoff then is 1. Next, we show that the players cannot improve arbitrarily often with vertices in C. Assume towards a contradiction that there exists an infinite sequence $(v_1, w_1), (v_2, w_1), (v_2, w_2), \ldots$ of profiles with $u_1(v_{i+1}, w_i) > u_1(v_i, w_i)$ and $u_2(v_{i+1}, w_{i+1}) > u_2(v_{i+1}, w_i)$ for all $i \geq 1$. Note that this is equivalent to

$$|N_I(v_i)| - |N_I(v_i) \cap N_I(w_i)| < |N_I(v_{i+1})| - |N_I(v_{i+1}) \cap N_I(w_i)| \text{ and}$$
$$|N_I(w_i)| - |N_I(v_{i+1}) \cap N_I(w_i)| < |N_I(w_{i+1})| - |N_I(v_{i+1}) \cap N_I(w_{i+1})|.$$

Since the number of different profiles is finite, there exists a subsequence $(v_1, w_1), \ldots, (v_i, w_j)$ with $(v_i, w_j) = (v_1, w_1)$ (wlog). But this yields the contradiction

$$\begin{aligned}
|N_I(v_1)| + |N_I(w_1)| - |N_I(v_1) \cap N_I(w_1)| &< |N_I(v_2)| + |N_I(w_1)| - |N_I(v_2) \cap N_I(w_1)| \\
&< |N_I(v_2)| + |N_I(w_2)| - |N_I(v_2) \cap N_I(w_2)| \\
&< \cdots \\
&< |N_I(v_i)| + |N_I(w_j)| - |N_I(v_i) \cap N_I(w_j)| \\
&= |N_I(v_1)| + |N_I(w_1)| - |N_I(v_1) \cap N_I(w_1)|.
\end{aligned}$$

Hence, there exists a profile where both players cannot improve, that is, a Nash equilibrium. $\square$

5 Conclusion

We analyzed Nash equilibria for the classic and the reverse temporal Voronoi game and highlighted some major differences depending on the considered temporal graph.

As regards open questions, note that the classes of temporal graphs we considered already settle the question of guaranteed existence of a Nash equilibrium for most graph classes commonly considered in the literature. A possible direction for future work would be to further restrict the temporal behavior of the temporal graph to grow or shrink in a more specific way. For example, it can be shown that on temporal cycles where at most one edge changes a Nash equilibrium always exists. Another direction is to study other variants of the temporal Voronoi game. Here, a natural question is whether Nash equilibria exist for more than two players. It is also interesting to study the game when the players are allowed to choose more than one vertex initially or if the temporal distance is defined differently (e.g. with faster arrival instead of earlier). Finally, it might also be fruitful to investigate the existence of other forms of equilibria, e.g. when introducing a certain cost for changing.

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