Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 28, no. 1, pp. 95–127 (2024)
DOI: 10.7155/jgaa.v28i1.2930

Thrackles, Superthrackles and the Hanani-Tutte Theorem

Hooman R. Dehkordi *Graham Farr*

Department of Data Science and AI, Faculty of Information Technology,
Monash University, Clayton, Victoria 3800, Australia

Submitted: October 2021 Accepted: January 2024 Published: May 2024

Article type: Regular paper Communicated by: C. Tóth

Abstract. A thrackle is a drawing of a graph in which any two vertex-disjoint edges cross exactly once and incident edges do not cross. A graph that has a thrackle drawing is thracklable. In the past three decades, mathematicians investigated a number of variations of thrackles by relaxing or changing the required number of crossings on edges or restricting the placement of vertices on the surface (e.g., restricting to outerdrawings, in which vertices are restricted to a boundary curve of the surface).

A graph is superthracklable if it can be drawn so that any two edges cross exactly once. We provide a simple proof for the following: a graph can be drawn on the plane so that every two edges cross an odd number of times if and only if it is superthracklable on the plane.

A graph is outersuperthracklable if it has a drawing on a disc in which every vertex is on the boundary of the disc and every pair of edges cross exactly once. We introduce some natural generalisations of outersuperthracklable graphs and we show that these classes of graphs are equivalent.

Lastly, using the Hanani-Tutte characterisation of planar graphs we show that for any surface Σ, there is a relationship between the class of graphs that are not embeddable on Σ and the class of graphs that are not superthracklable with respect to Σ. More specifically, we show how to construct, from any forbidden minor G for embeddability in Σ, two infinite families of graphs that are not superthracklable with respect to Σ. This sheds further light on the relationship between some of Archdeacon and Stor's forbidden configurations for superthrackles and Kuratowski's forbidden minors for planarity.

1 Introduction

When trying to study and understand graphs, it is often useful to construct drawings of them, in which vertices are represented by distinct points and edges by curves that do not intersect except possibly at their endpoints and proper edge crossings.[1]

A *simple drawing* of a graph is a drawing in which:

E-mail addresses: hooman.dehkordi@monash.edu (Hooman R. Dehkordi) graham.farr@monash.edu (Graham Farr)

This work is licensed under the terms of the CC-BY license.

[1] We give a formal definition in §1.1.

- an edge does not cross itself.
- incident edges do not cross.
- two edges do not cross more than once.

Now, a *thrackle* is a drawing of a graph in which any two edges have exactly one point in common [31]. In other words, in a thrackle, any two vertex-disjoint edges cross exactly once and incident edges do not cross (see for example, Figure 1a). Note that by definition a thrackle is a simple drawing.

Having the constraints of simple drawings in mind, a drawing with no crossings (e.g., a planar drawing) is a best drawing that we can hope to construct for a graph and a thrackle is a worst drawing that we may be able to construct for a graph such that it meets all of the above constraints.

Thus, thrackles serve as extremities of the class of simple drawings of a graph. So studying thrackles should contribute to understanding that class.

1.1 Definitions

A *surface* or a *two-dimensional manifold* Σ is a connected compact topological space that is locally homeomorphic to an open disk in the plane and for any two distinct points $x, y \in \Sigma$, there exist open neighbourhoods N_x of x and N_y of y such that $N_x \cap N_y = \emptyset$ [23].

A *drawing* of a graph G in a topological space (usually a surface or a disc) Σ is a mapping η that assigns:

- to each vertex u of G, a distinct point $\eta(u)$ in Σ, and
- to each edge (v_1, v_2) of G, a simple continuous curve $\zeta = \eta((v_1, v_2))$ in Σ connecting $\eta(v_1)$ to $\eta(v_2)$ such that ζ does not pass through the image under η of any vertex.

In this paper, we sometimes refer to a point $\eta(u)$ in a drawing η (of a graph G) that represents a vertex u in G as vertex u of η and similarly, we may refer to a curve ζ in η that represents an edge (u, v) in G as edge (u, v) of η. Moreover, we sometimes use η to denote the set of all points in the drawing η.

In this paper we assume that a drawing η satisfies the following conditions:

- An edge does not contain a vertex other than its endpoints.
- Edges must either properly cross (a transversal intersection) or not cross at all (for example, they must not meet tangentially). More precisely, for a crossing point p on two edges e_1 and e_2, the cyclic order of the edges around p is e_1, e_2, e_1, e_2.
- Any two edges cross a finite number of times and the intersection of the arcs representing them must be a finite set of points.
- No point represents more than one crossing. (It follows that a single edge cannot pass through the same crossing twice.)

We denote the cyclic order around vertex v of the edges incident at a vertex v of a drawing η by $\pi_\eta(v)$ and we put $\Pi_\eta := (\pi_{\eta(v)} : v \in V(G))$. Moreover for any disk d we denote the boundary of d by $\partial(d)$.

We defined thrackles earlier in this paper. Now we define thracklable graphs. Any graph that has a thrackle drawing on a surface Σ is *thracklable* with respect to Σ.

	# crossings when two edges are:	
	vertex-disjoint	incident
thrackle	1	0
superthrackle	1	1
weak thrackle	1	any
generalised thrackle	odd	even
generalised superthrackle	odd	odd
weak generalised thrackle	odd	any

Table 1: Variations of thrackles and the number of crossings on vertex-disjoint and incident edges

A drawing η of a graph G on a surface Σ is a *superthrackle* if any two edges in η cross each other exactly once (see for example, Figure 1b) [3]. Note that in superthrackles any two edges cross once as opposed to thrackles in which only vertex-disjoint edges cross. Any graph that has a superthrackle drawing on a surface Σ is *superthracklable* with respect to Σ.

A drawing η of a graph G on a surface Σ is a *weak thrackle* if any two vertex-disjoint edges in η cross each other exactly once (see for example, Figure 1c). In a weak thrackle, two edges that are incident to one vertex may or may not cross, with no restriction on the number of crossings if they do cross. This compares with a thrackle, in which incident edges do not cross. Any graph that has a weak thrackle drawing on a surface Σ is *weak thracklable* with respect to Σ.

A drawing η of a graph G on a surface Σ is a *generalised thrackle* if any two edges in η have an odd number of points in common (i.e., crossings or shared vertices) (see for example, Figure 1d) [8]. Any graph with a generalised thrackle drawing on a surface Σ is a *generalised thracklable* graph with respect to Σ.

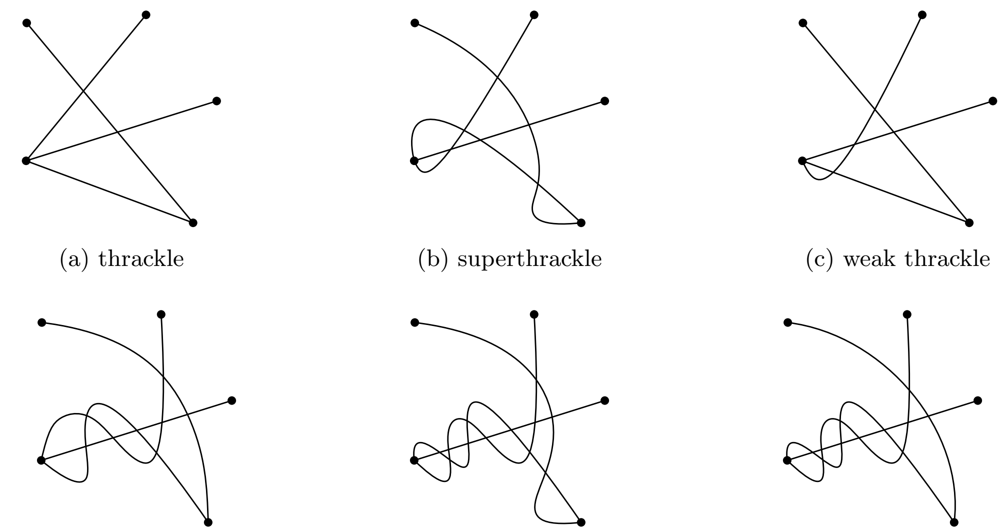
A drawing η of a graph G on a surface Σ is a *generalised superthrackle* if any two edges in η cross each other an odd number of times (see for example, Figure 1e). Any graph with a generalised superthrackle drawing on a surface Σ is a *generalised superthracklable* graph with respect to Σ.

A drawing η of a graph G on a surface Σ is a *weak generalised thrackle* if any two vertex-disjoint edges in η cross each other an odd number of times. (There are no restrictions on the number of crossings of incident edges in weak generalised thrackles. See for example, Figure 1f.) Any graph that can be drawn as a weak generalised thrackle on a surface Σ is a *weak generalised thracklable* graph with respect to Σ.

In summary: the adjective *super* means that the number of times two incident edges cross changes from 0 to 1 or from even to odd, with the result that the number of crossings between two incident edges follows the same rule as the number of crossings between two vertex-disjoint edges; the adjective *generalised* means that the number of times two edges cross is relaxed from zero or one (as the case may be) to even or odd, respectively; and the adjective *weak* means that the number of times two incident edges cross is unrestricted. Table 1 provides a quick reference to the definitions of variations of thrackles.

From the above definitions we can immediately deduce that:

- any thrackle is both a generalised thrackle and a weak thrackle,
- any superthrackle is a generalised superthrackle, and
- any generalised superthrackle is a weak generalised thrackle.

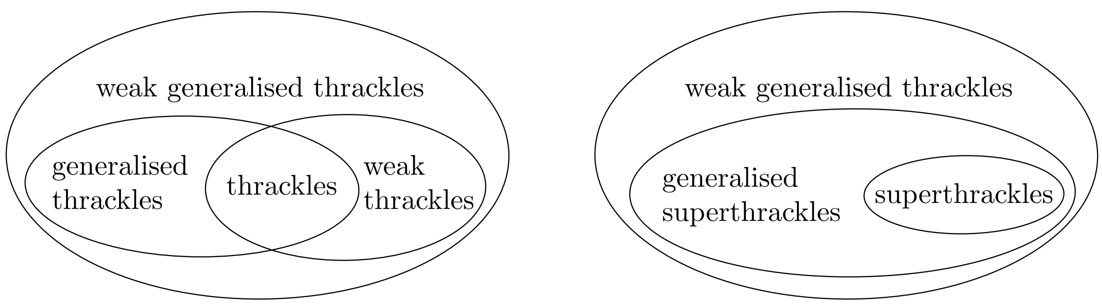


(a) thrackle (b) superthrackle (c) weak thrackle

(d) generalised thrackle (e) generalised superthrackle (f) weak generalised thrackle

Figure 1: Examples of different variations of thrackles

The relationships among these classes are shown by Venn diagrams in Figure 2. All the regions shown in these diagrams are nonempty. We will see in Theorem 3 that every generalised superthracklable graph is superthracklable, but nonetheless there are generalised superthrackles that are not superthrackles.



(a) (b)

Figure 2: Relationship between different variations of thrackles and superthrackles

In this paper, drawings are on the plane unless otherwise stated.

An *outerdrawing* of a graph G is a drawing of G on a disc such that all the edges are drawn on the disk and all the vertices of the drawing are on the boundary of the disc.

A graph that has a thrackle outerdrawing is *outerthracklable*. For each adjective $X \in \{\text{super}, \text{generalised}, \text{weak}\}$, a graph that has an X-thrackle outerdrawing is said to be X-*outerthracklable* or *outer*-X-*thracklable*. An appropriate pair of these adjectives can also be used together (weak generalised; generalised super), with the natural meaning.

An edge $e = (u, v)$ in a graph G is *subdivided* by replacing it with two edges $(u, w), (w, v)$ where

w is not a vertex of G. A graph G or any other graph that can be obtained from G by a sequence of subdivisions is a *subdivision* of G.

Any graph G' that can be obtained from a graph G by a series of edge deletions, vertex deletions and edge contractions is a *minor* of G.

A *caterpillar* is a tree for which removal of all vertices of degree 1 gives a path graph.

An *odd wreath* is a graph containing an odd cycle C such that every edge not in C joins a vertex of C to a vertex of degree 1.

Lastly, we denote the set of all the neighbours of a vertex v by $N(v)$.

1.2 Our Results

In this paper we first characterise weak generalised outerthracklable graphs. More specifically we prove the following theorem:

Theorem 1 *Any graph G is weak generalised outerthracklable if and only if G does not contain any of:*

1. *2-claw graph as a minor (see Figure 3a),*
2. *$K_2 \cup K_3$ as a minor (see Figure 3b),*
3. *any cycle of even length with four or more vertices.*

(a) the 2-claw graph (b) $K_2 \cup K_3$

Figure 3: Two forbidden minors for outersuperthracklable graphs.

We then use Theorem 1 to prove the following.

Corollary 1 *G is weak generalised outerthracklable if and only if it is a subgraph of an odd wreath.*

We also show that all the variants of outerthracklability defined above are equivalent. In other words:

Theorem 2 *For any graph G, the following are equivalent:*

1. *G is outerthracklable,*
2. *G is outersuperthracklable,*
3. *G is weak outerthracklable,*
4. *G is generalised outersuperthracklable,*
5. *G is weak generalised outerthracklable.*

We then provide a simple and direct proof, for the plane, for equivalence of generalised superthracklable graphs and superthracklable graphs. This theorem was originally proved by Archdeacon and Stor [3].

Theorem 3 *Any generalised superthracklable graph is superthracklable.*

A *planar drawing* or a *planar embedding* is a drawing in which no two edges cross. *Planar graphs* are the graphs that have a planar drawing.

The Hanani-Tutte Theorem is a famous result in graph theory which can be thought of as a characterisation of planar graphs.

Theorem 4 (Strong Hanani-Tutte Theorem, [33]) *A graph is planar if it has a drawing η on the plane such that any two vertex-disjoint edges in η cross each other an even number of times.*

The Hanani-Tutte Theorem is usually used in a weaker form.

Theorem 5 (Weak Hanani-Tutte Theorem, [33]) *A graph is planar if it has a drawing η on the plane such that any two edges in η cross an even number of times.*

For any graph $G = (V, E)$ and any subset E' of E, let $\mathcal{G}(G, E')$ be the family of all the graphs that are obtained from G as follows:

- Replace any edge $e = (u, v) \in E'$ with a (u, v)-path P of even length,
- Replace any edge $e = (u, v) \notin E'$ with any (u, v)-path P,

such that the edges and internal vertices of these paths are new, with all paths so introduced being internally disjoint from each other.

We investigate the relationship between the Weak Hanani-Tutte Theorem and superthrackles. We use the weak Hanani-Tutte Theorem to prove Theorem 6.

Theorem 6 *Let Σ be a surface. Let G be a graph such that in any drawing of G on Σ there are two edges that cross each other an odd number of times. Let x be any edge of G and let G' be a graph in $\mathcal{G}(G, E \setminus \{x\})$. Then in any drawing of G' on Σ, there are two edges that cross each other an even number of times.*

This Theorem provides us with a connection between the Hanani-Tutte Theorem and Theorem 3 which we shall discuss in more detail later in this paper.

The rest of this paper is organised as follows. Section 2 provides the reader with the background for this research. Section 3 is dedicated to characterisation of weak generalised outerthracklable graphs. Section 4 investigates the relationship between outerthracklable graphs, outersuperthracklable graphs, weak outerthracklable graphs, generalised thracklable graphs and weak generalised thracklable graphs. Then in Section 5 we prove equivalence of generalised superthracklable graphs and superthracklable graphs on the plane. Section 6 is dedicated to the relationship between the Hanani-Tutte Theorem and thrackles. Lastly, Section 7 summarises our results and points out future directions for research.

2 Related Work

Kuratowski's Theorem is a well-known characterisation of planar graphs in terms of two forbidden subdivisions.

Theorem 7 (Kuratowski's Theorem, [19]) *A graph is planar if and only if it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ as a subgraph.*

Hanani proved the following in 1934.

Theorem 8 (Hanani [10]) [2] *Any drawing of K_5 or $K_{3,3}$ on the plane contains two vertex-disjoint edges that cross each other an odd number of times.*

By Kuratowski's Theorem and Hanani's Theorem, it is straightforward to see that in any drawing of a non-planar graph there are two vertex-disjoint paths that cross an odd number of times and hence there are two vertex-disjoint edges that cross an odd number of times. This is known as the Strong Hanani-Tutte Theorem (which we stated earlier). The Hanani-Tutte Theorem though is usually used in its weaker form (as was stated above as well).

The Hanani-Tutte Theorem in its weak form has been generalised to all 2-manifolds [28]. This is especially interesting as we do not yet have forbidden minor characterisations for the graphs that can be drawn without crossings on surfaces other than the sphere and the projective plane [2, 19, 34].

Theorem 9 ([28]) *A graph G has a drawing on the surface Σ with no crossings if it has a drawing η on Σ such that any two edges cross an even number of times in η.*

However, the Strong Hanani-Tutte Theorem has only been generalised for the projective plane.

Theorem 10 ([26]) *A graph G has a drawing on the projective plane with no crossings if it has a drawing η on the projective plane such that any two vertex-disjoint edges cross an even number of times in η.*

In fact, Fulek and Kynčl [12] proved that the Strong Hanani-Tutte Theorem cannot be generalised to the orientable surface of genus four.

There are numerous other versions of the Hanani-Tutte Theorem (see for example, [13, 20, 28]).

The notion of thrackle was defined by John Conway as he conjectured the following:

Conjecture 1 (Conway's Thrackle Conjecture, [4, 31]) *For a thracklable graph $G = (V, E)$, $|E| \leq |V|$.*

Despite considerable effort, Conway's thrackle conjecture is still open.

Lovasz, Pach and Szegedy proved that every bipartite thracklable graph is planar [21] and hence the number of edges of a bipartite thrackle with n vertices is at most $2n-4$ (assuming $n \geq 3$). This bound later was improved to $(3n-3)/2$ by Cairns and Nikolayevsky [7] and then to $\frac{167}{117}n \approx 1.428n$ by Fulek and Pach [14].

Fulek and Pach proved an upper bound of $1.3984n$ edges for a thrackle with n vertices [15]. There are numerous other papers trying to tighten the upper bound on the number of edges of thrackles with the tightest one given in [36] (see, for example, [6, 16, 18, 29, 30]).

Assuming that Conway's Thrackle Conjecture is true, Woodall proved that the bound stated in the conjecture is tight since any cycle other than C_4 is a thrackle [35]. Moreover, with the same assumption Woodall characterised all thrackles as follows [35]: a graph is a thrackle if and only if

[2] This paper was written by Hanani before he changed his name.

- it has at most one cycle of odd length, and
- it does not contain C_4, and
- each of its connected components contains at most one cycle.

With this theorem in mind, to prove Conway's Thrackle Conjecture it is enough to verify that a graph that consists of two even cycles with one vertex in common is not a thrackle [35, 30].

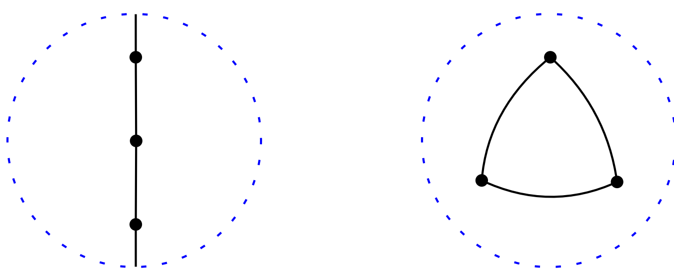
Misereh and Nikolayevsky define annular and pants thrackles as two types of outerthrackles and characterise them in [22]. Other variations of thrackles include: tangles, tangled thrackles, and spherical thrackles. For definitions of and results about these types of thrackles see [5, 24, 25, 32]. For applications of thrackles see [1, 17].

Cairns and Nikolayevsky characterised outerthracklable graphs as follows.

Theorem 11 **([9])** *Let G be an outerthracklable graph such that* $\deg(v) \geq 2$ *for any vertex v in G. Then G is an odd cycle.*

Moreover, they proved that the number of edges of an outerthracklable graph does not exceed the number of vertices of the graph [9].

A simple cycle on the projective plane is *2-sided* if it has a neighbourhood homeomorphic to a cylinder, and *1-sided* if it has a neighbourhood homeomorphic to a Möbius strip (see for example, Figure 4). A *parity embedding* is an embedding of a graph in the projective plane in which a simple cycle C is 1-sided if and only if C is of odd length [8, 3].



(a) A 1-sided cycle

(b) A 2-sided cycle

Figure 4: Examples of 1-sided and 2-sided cycles on the projective plane

Cairns and Nikolayevsky characterised generalised thracklable graphs with respect to the plane as follows.

Theorem 12 **([8])** *A graph is generalised thracklable on the plane if and only if it has a parity embedding in the projective plane.*

An *even edge-subdivision* of an edge e in a graph replaces edge e with a path of an odd length. A *vertex subdivision* at u in a graph G subdivides every edge that is incident to u once. Two graphs G and G' are *parity homeomorphic* if there is a graph H that can be obtained from G and from G' by the operations of even edge-subdivision or vertex subdivision [3].

Superthrackles are defined by Archdeacon and Stor and are characterised as follows.

Theorem 13 **([3])** *A graph is superthracklable if and only if it has a parity embedding in the projective plane.*

A drawing of a graph on a surface Σ is a *1-point superthrackle* if it can be drawn as a superthrackle on Σ such that all the edge crossings occur at a common point[3]. Any graph that can be drawn as a 1-point superthrackle on surface Σ is *1-point superthracklable* with respect to Σ.

Archdeacon and Stor also proved the following.

Theorem 14 ([3]) *The following classes of graphs are equivalent:*

- *superthracklable graphs,*
- *1-point superthracklable graphs,*
- *graphs that have a parity embedding on the projective plane,*
- *graphs without a subgraph that is parity homeomorphic to any graph in Figure 5.*

Cairns and Nikolayevsky characterised generalised thracklable graphs [8] and Archdeacon and Stor characterised superthracklable graphs [3] and the two characterisations are the same (see Theorem 12 and Theorem 13). That is, any generalised thracklable graph is a superthracklable graph.

Theorem 15 ([3]) *A graph is superthracklable if and only if it is generalised superthracklable.*

There are similarities between Theorem 15 and the Weak Hanani-Tutte Theorem since:

- the Weak Hanani-Tutte Theorem can be rephrased as: every graph G with a drawing in which every two edges cross an even number of times has a drawing in which every two edges cross zero times, and
- Theorem 15 can be rephrased as: every graph G with a drawing in which every two edges cross an odd number of times has a drawing in which every two edges cross once.

We provide a simple and direct proof for Theorem 15 (see Theorem 3).

Moreover, given that the Strong Hanani-Tutte Theorem holds for planar graphs one might think that the analogous statement holds for superthrackles, that is, that any weak generalised superthracklable graph is superthracklable. However, we show that the latter statement is false.

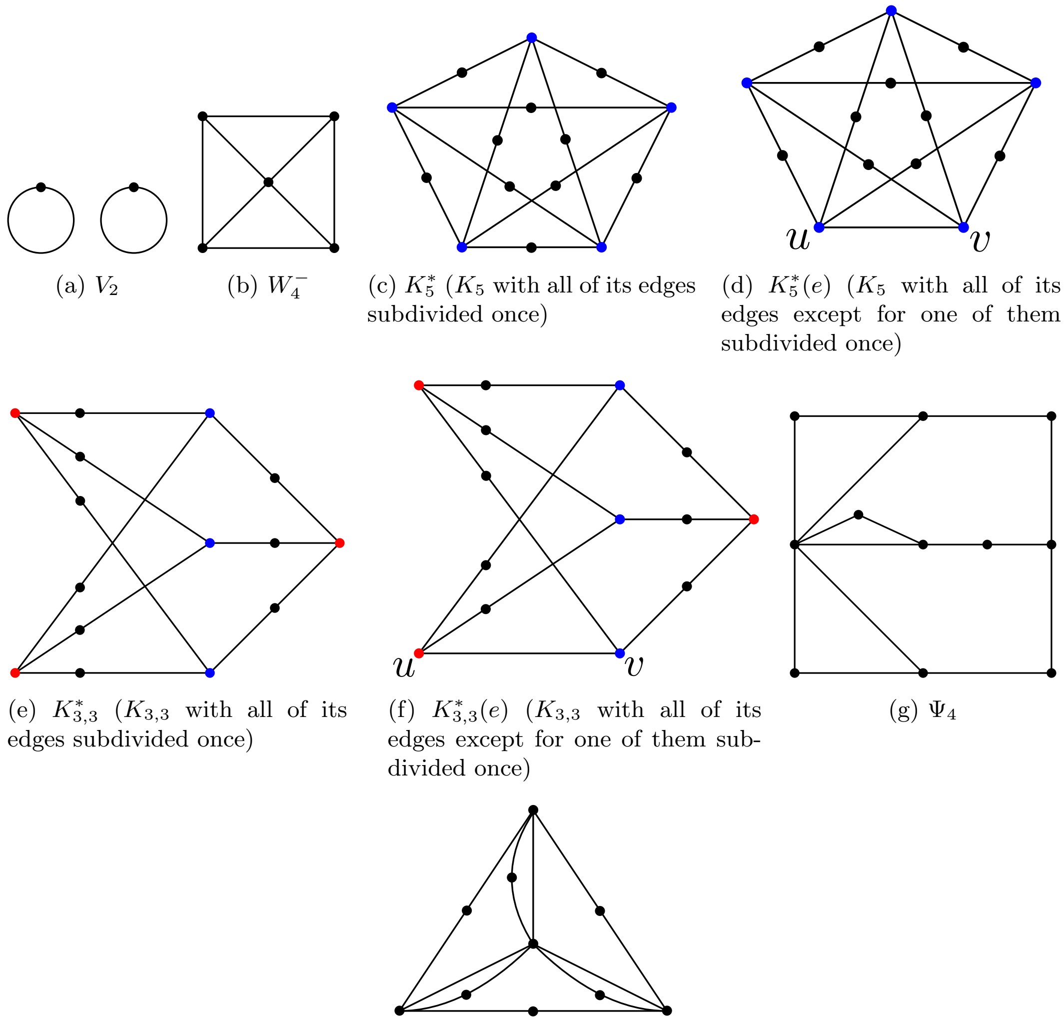
3 Weak Generalised Outerthracklable Graphs

In this section we characterise weak generalised outerthracklable graphs. We start by proving that weak generalised outerthracklable graphs cannot contain the 2-claw graph as a forbidden minor.

Lemma 1 *None of the graphs C_4, $K_2 \cup K_3$ and the 2-claw graph have a weak generalised outerthrackle drawing.*

Proof: Let G be any of the graphs C_4, $K_2 \cup K_3$ or the 2-claw graph. By just placing vertices of G in different cyclic orders around a disk, it is routine to observe that in any given outerdrawing of G there are two edges (u, u') and (v, v') such that vertices u, u', v, v' appear in that order around the boundary of the disc and therefore (u, u') crosses (v, v') an even number of times. Therefore, none of C_4, $K_2 \cup K_3$ or the 2-claw graph has a weak generalised outerthrackle drawing. $\square$

[3] For this specific definition we relax the definition of a drawing so that three or more edges can all cross at a common point.



(a) V_2

(b) W_4^-

(c) K_5^* (K_5 with all of its edges subdivided once)

(d) $K_5^*(e)$ (K_5 with all of its edges except for one of them subdivided once)

(e) $K_{3,3}^*$ ($K_{3,3}$ with all of its edges subdivided once)

(f) $K_{3,3}^*(e)$ ($K_{3,3}$ with all of its edges except for one of them subdivided once)

(g) Ψ_4

(h) Φ_4

Figure 5: The obstruction set for superthrackles

Lemma 2 *Any graph G that contains the 2-claw graph as a minor is not a weak generalised outerthracklable graph.*

Proof: Any graph that contains the 2-claw graph as a minor has the 2-claw graph as a subgraph. So by Lemma 1 any graph G that contains the 2-claw graph as a minor does not have a weak generalised outerthrackle drawing and hence is not a weak generalised outerthracklable graph. □

Let $e = (u, v)$ be an edge of a graph G. Let G' be the graph that is obtained from G by replacing (u, v) with three edges (u, w), (w, x) and (x, v), where $w, x \notin V(G)$. Define the *double topological contraction* operation (or *double contraction* for short) to be the operation that is performed on G' to obtain G (see Figure 6).

(a) before double contraction (G') (b) after double contraction (G)

Figure 6: Double topological contraction operation

Define a graph G^- to be a *double minor* of a graph G if we can obtain G^- from G by some sequence of vertex deletions, edge deletions and double contractions.

Throughout the rest of this paper, we borrow a set of local moves from [11] to manipulate any drawing η of a graph G and obtain any other drawing η' of G.

Proposition 1 ([11]) *Let η and η' be two drawings of a graph G in the plane. Then η and η' are related by a finite sequence of the local moves given in Table 2.*

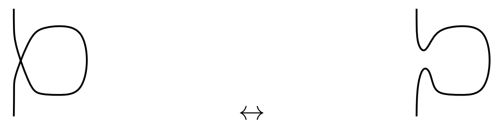
For a vertex v in a drawing η, a *local disk* Σ_v at v is a sufficiently small neighbourhood homeomorphic to an open disk centred on v such that:

- Σ_v does not contain any vertex other than v,
- Σ_v does not contain any crossings,
- for any edge e incident with v, the intersection of the drawing of e with Σ_v is a curve homeomorphic to $(0, 1]$,
- every edge that is not incident with v is disjoint from Σ_v (see, for example, Figure 7).

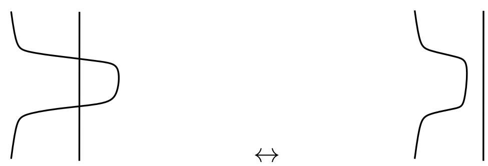
Similarly, for a non-self-intersecting edge $e = (u, v)$ in a drawing η, let a *local disk* Σ_e of the edge e be a sufficiently small region homeomorphic to an open disk that contains e in its interior such that:

- Σ_e does not contain any vertex other than u or v,
- Σ_e does not contain any crossings other than the crossings on e,
- any continuous segment of an edge f that intersects with Σ_e is either a curve homeomorphic to $(0, 1)$ that crosses e once or a curve homeomorphic to $[0, 1)$ that has u or v as one of its endpoints (see, for example, Figure 8).

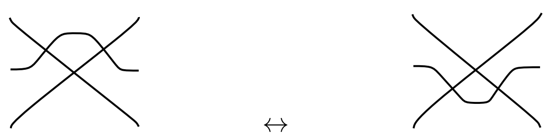
R_I^p



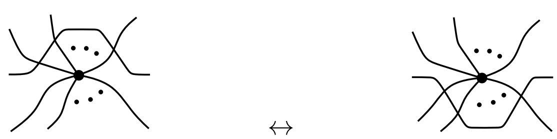
R_{II}^p



R_{III}^p



R_{IV}^p



R_V^p

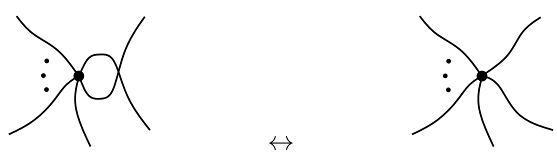


Table 2: Reidemeister moves for plane graphs

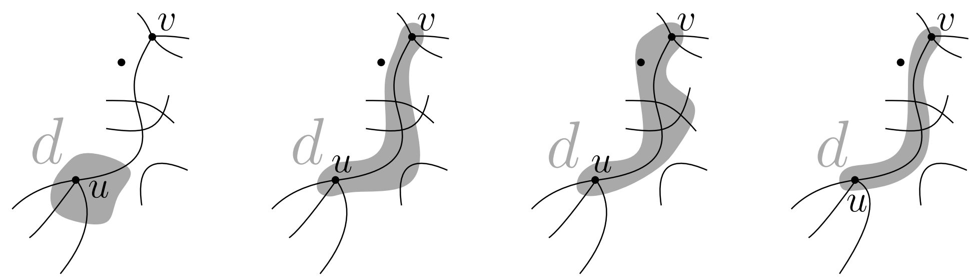
(a) disk d contains a crossing

(b) disk d contains an edge segment that does not have u as its endpoint

(c) disk d contains a segment of an edge that is not adjacent to u

(d) disk d is a local disk for u

Figure 7: (a)–(c) depicts three examples of disks that are not local disks of u. (d) depicts a local disk of u.



(a) disk d does not contain the edge (u, v)

(b) disk d contains an edge segment that does not have u or v as its endpoint and does not cross (u, v)

(c) disk d contains a crossing that is not on (u, v) and a vertex other than u and v

(d) disk d is a local disk for (u, v)

Figure 8: (a)–(c) depicts three examples of disks that are not local disks of (u, v). (d) depicts a local disk of (u, v).

Next we show that weak generalised outerthracklable graphs are closed under the double contraction operation (see Figure 6).

Lemma 3 *Weak generalised outerthracklable graphs are closed under the double contraction operation.*

Proof: Let η be a weak generalised outerthrackle drawing of G. Let $(v_1, v_2), (v_2, v_3), (v_3, v_4)$ be three edges of G such that v_2 and v_3 have degree 2. Let G^- be the graph that is obtained from G by double contracting $(v_1, v_2), (v_2, v_3), (v_3, v_4)$ to (v_1, v_4). Obtain a drawing η^- of G^- from η as follows:

1. remove (v_1, v_4) from η if (v_1, v_4) is an edge in G.

2. add (v_1, v_4) to η such that it follows the path of $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ sufficiently closely so that (v_1, v_4) is drawn within the union of the local disks of (v_1, v_2), (v_2, v_3) and (v_3, v_4) and for every crossing between an edge f and $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ there is only one crossing between f and (v_1, v_4) (see Figure 9a).

3. remove vertices v_2, v_3 and edges $(v_1, v_2), (v_2, v_3), (v_3, v_4)$.

4. remove the self crossings of (v_1, v_4) by the R_I^p move that is shown in Table 2 (see Figure 9b).

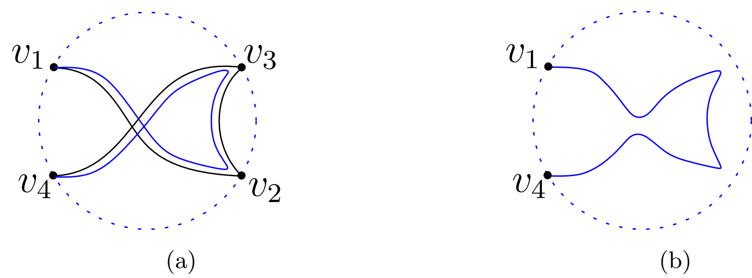


Figure 9: Constructing η^- from η. (For simplicity, the rest of the edges of η or η' are not shown in this Figure.)

Since any two vertex-disjoint edges cross each other an odd number of times in η, any edge that is not incident with v_1, v_2, v_3, v_4 crosses $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ an odd number of times in η. Therefore, any edge in η^- that is not incident with v_1 or v_4 crosses (v_1, v_4) an odd number of times and hence η^- is a drawing of G^- in which any two vertex-disjoint edges cross an odd number of times. □

We use Lemma 3 to show that there cannot be a cycle C of odd size and an edge that is vertex-disjoint from C in any weak generalised outerthrackle.

Lemma 4 *Let G either be a cycle C with an even number of vertices or consist of a cycle C with an odd number of vertices and an edge e that is vertex-disjoint from C. Then G is not a weak generalised outerthracklable graph.*

Proof: We prove this lemma by contradiction. Let us assume that such a G is a weak generalised outerthracklable graph. If G is a cycle C with an even number of vertices then by double contracting C (multiple times if necessary), we obtain a graph G' that is C_4. If G is a cycle C with an odd number of vertices and an edge e that is vertex-disjoint from C then by double contracting C (multiple times if necessary) we obtain a cycle with 3 vertices and an edge that is vertex disjoint from it or in other words a $K_2 \cup K_3$. Now if G is a weak generalised outerthracklable graph then, by Lemma 3, G' is a weak generalised outerthracklable graph. But this is a contradiction by Lemma 1. □

For any vertex u in a graph G, we denote the set of neighbours of u by $N(u)$.

Now we prove Theorem 1.

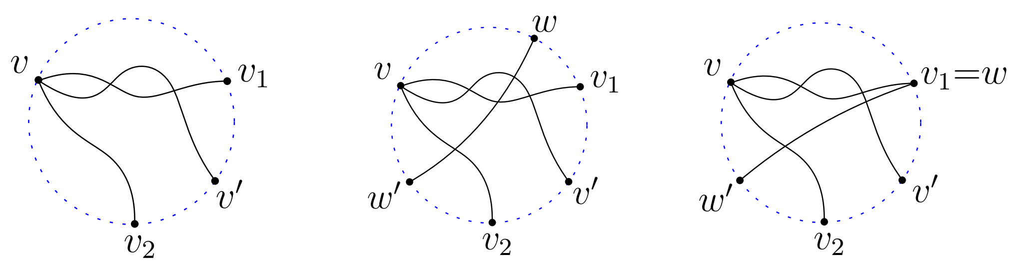
Proof: [Proof of Theorem 1] It is easy to see that any graph that is obtained by adding an isolated vertex to a weak generalised outerthracklable graph is also a weak generalised outerthracklable graph. So in this proof we assume that graph G does not contain an isolated vertex.

By Lemma 2 and Lemma 4, weak generalised outerthracklable graphs do not have the 2-claw graph as minor or C_4 as a double minor. Moreover since $K_2 \cup K_3$ contains an edge and a cycle of odd length that are vertex-disjoint, by Lemma 4 weak generalised outerthracklable graphs do not contain $K_2 \cup K_3$ as a double minor either. Therefore we only need to show that if a graph G does not have the 2-claw graph as minor, $K_2 \cup K_3$ as a double minor or C_4 as a double minor, then G is a weak generalised outerthracklable graph.

We prove this by induction on the number of vertices. In the base case G has one or two or three vertices and the lemma holds trivially. We proceed to the inductive case. We have two cases:

Case 1. There is a vertex v in G, with $\deg(v) \geq 3$. Since G does not contain C_4 or the 2-claw graph or $K_3 \cup K_2$ as a minor, for any vertex v in G with $\deg(v) \geq 3$, there is a vertex v' adjacent to v in G such that $\deg(v') = 1$. Let v_1 and v_2 be two vertices (other than v') that are adjacent to v. Let G^- be the graph that is obtained from G by deleting v' and (v, v') from G.

By induction G^- has a weak generalised outerthrackle drawing η^-. Let d be the disc on which η^- is drawn. To obtain a drawing η of G from η^-, choose the location of v' on $\partial(d)$ such that the order of v, v', v_1, v_2 (clockwise or anticlockwise) on $\partial(d)$ is v, v_1, v', v_2 and let (v, v') be represented by an arbitrary curve from v to v' (see, for example, Figure 10a).



(a) $(v, v_1), (v, v')$ and (v, v_2) in η

(b) (w, w') crosses (v, v') an odd number of times

(c) (w, w') crosses (v, v') an odd number of times

Figure 10: vertex-disjoint edges cross each other an odd number of times in η.

We need to show that (v, v') crosses any other vertex-disjoint edge an odd number of times.

Let (w, w') be an arbitrary edge of G that is vertex-disjoint from (v, v'), if such an edge exists.

If (w, w') is vertex-disjoint from both (v, v_1) and (v, v_2), then since any two vertex-disjoint edges in η^- cross an odd number of times, (w, w') crosses both (v, v_1) and (v, v_2) an odd number of times. Therefore the order of v, v', v_1, v_2, w, w' on $\partial(d)$ is either v, w, v_1, v', v_2, w' or v, w', v_1, v', v_2, w (see Figure 10b). In both of these cases, by the order of the vertices on $\partial(d)$, (w, w') crosses (v, v') an odd number of times.

Now let (w, w') be vertex-disjoint from either (v, v_1) or (v, v_2) (say (v, v_2)). That is, either $w = v_1$ or $w' = v_1$. Without loss of generality let $w = v_1$. The edge (v_1, w') (i.e., (w, w')) crosses (v, v_2) an odd number of times and therefore the order of $v, v', v_1 = w, v_2, w'$ on $\partial(d)$ (clockwise or anticlockwise) is $v, v_1 = w, v', v_2, w'$ (see Figure 10c). In this case, (w, w') crosses (v, v') an odd number of times as well.

Therefore any edge that is vertex-disjoint from (v, v') crosses (v, v') an odd number of times and hence any two vertex-disjoint edges in η cross an odd number of times. That is, η is a weak generalised outerthrackle.

Case 2. There is no vertex v in G with $\deg(v) \geq 3$. Since the degree of any vertex in G is less than 3, G consists of a number of isolated vertices or paths and cycles. By condition 2 of the theorem, there is no edge in G that is vertex-disjoint from a cycle in G. Therefore G either consists of a cycle and a number of isolated vertices or a number of paths and a number of isolated vertices.

Since we can easily add or remove isolated vertices to weak generalised outerthrackles and obtain another weak generalised outerthrackle, we assume that there are no isolated vertices in G. So let G be a cycle. By condition 3 of the theorem, G cannot contain any cycle of even length.

Therefore G is a cycle of odd length. Let $G = C_{2n+1}$, where $n \geq 2$ (if $n = 1$, double contraction operation is not applicable). Let v_1, v_2, v_3, v_4 be four consecutive vertices of G. Let G^- be the graph that is obtained from G by double-contracting $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ to (v_1, v_4).

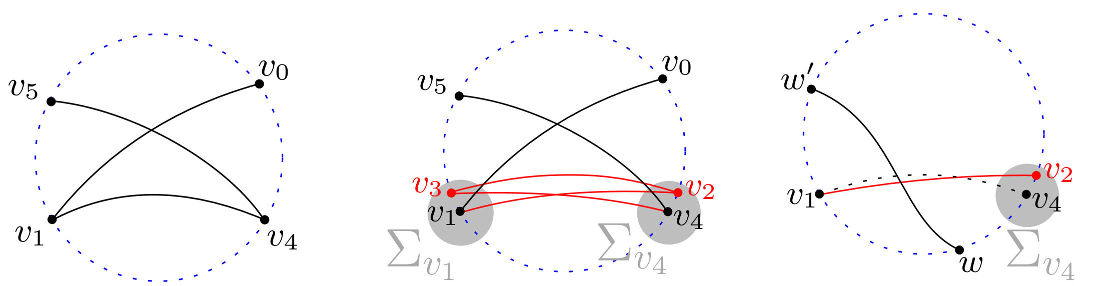
By induction, G^- has a weak generalised outerthrackle drawing η^-. Let d be the disc on which η^- is drawn. Let v_0 be the vertex of G^- that is in $N(v_1) \setminus v_4$ and let v_5 be the vertex of G^- that is in $N(v_4) \setminus v_1$ (v_0 may be equal to v_5).

The cyclic order in which vertices v_0, v_1, v_4, v_5 appear on $\partial(d)$ (clockwise or anticlockwise) is v_1, v_4, v_0, v_5 (see for example Figure 11a), else (v_0, v_1) crosses (v_4, v_5) an even number of times. Obtain a drawing η of G from η^- as follows:

1. Let Σ_{v_1} and Σ_{v_4} be local disks of v_1 and v_4, respectively, in η^-. Insert v_2 and v_3 on $\partial(d)$ such that the order of $v_0, v_1, v_2, v_3, v_4, v_5$ (clockwise or anticlockwise) on $\partial(d)$ is $v_0, v_5, v_3, v_1, v_4, v_2$ and v_2 is located in Σ_{v_4} and v_3 is located in Σ_{v_1}.

2. Represent $(v_1, v_2), (v_2, v_3)$ and (v_3, v_4) by three arbitrary curves (subject to the restrictions of curves used in drawings) between the corresponding vertices.

3. Delete (v_1, v_4) (compare η^- and η in Figures 11a and 11b).



(a) $(v_0, v_1), (v_1, v_4)$ and (v_4, v_5) in η^-

(b) η is obtained by adding the red vertices and red edges (i.e., v_2, v_3, and their incident edges) to η^- and removing (v_1, v_4) from it. Notice that Σ_{v_1} and Σ_{v_4} are the local disks of v_1 and v_4 in η^- (before we add v_3 and v_2 to the drawing).

(c) order of w, w', v_1, v_2 and v_4 on $\partial(d)$. The dashed black line represents the path of (v_1, v_4) in η^-.

Figure 11: vertex-disjoint edges cross each other an odd number of times in η.

We claim that any two vertex-disjoint edges in η cross each other an odd number of times. Since any two vertex-disjoint edges in η^- cross each other an odd number of times, we only need to show that $(v_1, v_2), (v_2, v_3)$ and (v_3, v_4) each cross any other edge that is vertex-disjoint edge an odd number of times.

Let E' be the set of edges in G^- that are vertex-disjoint from (v_1, v_4). The set of edges that are vertex-disjoint from (v_1, v_2) in η is $E' \cup (v_3, v_4)$. Let (w, w') be an arbitrary edge in E'. Since (w, w') crosses (v_1, v_4) in η^- an odd number of times, the order of w, w', v_1, v_4 on $\partial(d)$ (clockwise or anticlockwise) is either v_1, w, v_4, w' or v_1, w', v_4, w. Since we insert v_2 in Σ_{v_4} in η, it follows that the order of w, w', v_1, v_2 on $\partial(d)$ is either v_1, w, v_2, w' or v_1, w', v_2, w (see Figure 11c). Hence (w, w') crosses (v_1, v_2) an odd number of times. Moreover, as the order of v_1, v_2, v_3, v_4 (clockwise or anticlockwise) on $\partial(d)$ is v_3, v_1, v_4, v_2 (v_1 and v_2 are separated by v_3 and v_4 on the boundary), (v_1, v_2) crosses (v_3, v_4) an odd number of times. That is, any edge that is vertex-disjoint from (v_1, v_2) crosses (v_1, v_2) an odd number of times.

By a similar argument, any edge that is vertex-disjoint from (v_3, v_4) crosses (v_3, v_4) an odd number of times.

The set of edges that are vertex-disjoint from (v_2, v_3) in η is E'. Any edge in E' crosses (v_1, v_4) an odd number of times. Therefore, after we insert v_2 in Σ_{v_4} and v_3 in Σ_{v_1}, any edge in E' also crosses (v_2, v_3) an odd number of times. That is, η is a weak generalised outerthrackle.

If G is the union of a number of paths then we obtain a graph G^+ by repeatedly adding pairs of edges (and maybe a vertex) to G to link all these paths up into a cycle of odd length (so that we do not violate condition 3 in the theorem). We construct a weak generalised outerthrackle drawing of G^+ and then we delete the extra edges and vertices of G^+ to obtain a weak generalised outerthrackle drawing of G. □

Corollary 2 *G is weak generalised outerthracklable if and only if it is either*

1. *a disjoint union of caterpillars, or*

2. *an odd wreath together with a number (possibly zero) of isolated vertices.*

Proof: If G is either a union of caterpillars or an odd wreath with some isolated vertices, then it is clear that it does not contain any of the three types of minors or subgraphs forbidden by the conditions of Theorem 1, which then implies that it is weak generalised outerthracklable.

Now suppose G is weak generalised outerthracklable. By Theorem 1, it has no 2-claw minor, no $K_2 \cup K_3$ minor, and no even cycle with at least four vertices. Consider any component H of G that contains no cycle, so must be a tree. Let v be any vertex of H of degree at least three. All but at most two edges incident with v must be leaves, else H has a 2-claw minor. It follows that H consists of a path together with some leaves incident with vertices of the path. So H is a caterpillar.

Now consider any component H of G that contains a cycle C. By the third condition of the Theorem 1, C must have odd length. Furthermore, no edge of H can be disjoint from C, else G has a $K_2 \cup K_3$-minor. Therefore every edge of H that is not in C is incident with some vertex of C.

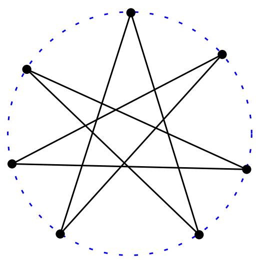
Let e be any such edge. If e is incident with two vertices of C (i.e., it is a chord of C), then C must have at least five edges (since its length is not even, and a triangle can have no chords). But then the chord enables construction of a $K_2 \cup K_3$-minor, or indeed of an even cycle since the length of C is odd. So e can only be incident with one vertex of C. Suppose the other vertex w of e has degree ≥ 2. Let f be any other edge incident with w. Its other vertex x must be on C, as we have already seen. There are two paths from v to x in C, and one of them must have even length, since C has odd length. But taking this even path, together with edges e and f, gives an even cycle in H, violating the third condition of Theorem 1. Therefore the vertex of e that is not in C must have degree 1, so that e is a leaf. Since this holds for any edge e of H that is not in C, we have established that H is an odd wreath.

In this case, G can have no edges in any other components of H, since such an edge, together with C, realises $K_2 \cup K_3$ as a minor of G. So all the other components of H must just be isolated vertices. □

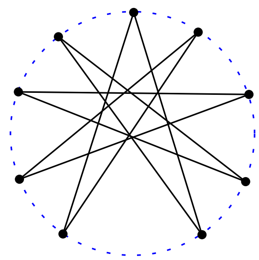
Now we are ready to prove Corollary 1.

Proof: [Proof of Corollary 1] This follows easily from Corollary 2, since both of the graph types used there are subgraphs of odd wreaths, and every subgraph of an odd wreath is of one of those two forms. □

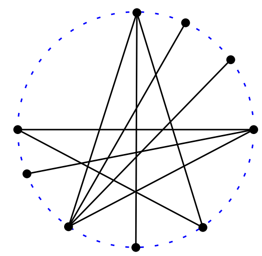
Figures 12a and 12b depict the thrackled outerdrawings of C_7 and C_9 respectively according to the algorithm embedded in the proof of Theorem 1. Figure 12c depicts an outerdrawing of a graph G that has four edges more than a cycle drawn using the same algorithm.



(a) C_7



(b) C_9



(c) C_5 with four extra edges.

Figure 12: Three weak generalised outerthrackles

4 Relationship between different types of superthrackles

Let (v, v_1) and (v, v_2) be two edges of a graph G and let G' be the graph that is obtained from G by identifying v_1 and v_2 and then deleting any loops formed. Define the *folding* operation to be the operation that is performed on G to obtain G'. More specifically, we say that we obtain G' from G by folding (v, v_1) onto (v, v_2).

Lemma 5 *Let $e = (v, v_1)$ and $e' = (v, v_2)$ be two incident edges that appear consecutively in $\pi_\eta(v)$ where η is a generalised superthrackle drawing of a graph. Then there is a non-self-intersecting curve ζ from v_1 to v_2 that crosses each edge of η an even number of times.*

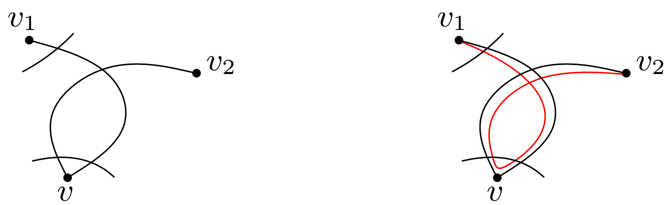
Proof: Let ζ' be a curve from v_1 to v_2 such that:

1. ζ' and (v, v_1) are located consecutively in the circular order of the edges and ζ' around v_1 in η.

2. ζ' and (v, v_2) are located consecutively in the circular order of the edges and ζ' around v_1 in η.

3. ζ' follows the paths of (v, v_1) and (v, v_2) sufficiently closely so that it is drawn within local disks of (v, v_1) and (v, v_2) and for any crossing between an edge e and (v, v_1), there is only one crossing between e and ζ' and for any crossing between an edge e and (v, v_2), there is only one crossing between e and ζ' (see Figure 13).

Since η is a generalised superthrackle, any two edges cross an odd number of times in η and therefore ζ' crosses all the edges other than (v, v_1) and (v, v_2) an even number of times in η. Now obtain ζ from ζ' by:



(a) Local area of (v, v_1) and (v, v_2) in η before adding ζ'

(b) Local area of (v, v_1) and (v, v_2) in η after adding ζ'

Figure 13: Adding ζ' to η

1. removing the self-crossings on ζ' using R_I^p.
2. using R_V^p, if necessary, to make sure that ζ crosses both (v, v_1) and (v, v_2) an even number of times.

The curve ζ crosses each edge of η an even number of times. □

Next we show that generalised superthracklable graphs are closed under the folding operation under certain circumstances.

Theorem 16 *Generalised superthracklable graphs are closed under folding of any two edges $e = (v, v_1)$ and $e' = (v, v_2)$ that appear consecutively at v in some generalised superthrackle drawing of the graph.*

Proof: Let η be a generalised superthrackle drawing of a graph G. Let (v, v_1) and (v, v_2) be two edges that appear consecutively in $\pi_\eta(v)$. By Lemma 5 there is a curve ζ from v_1 to v_2 that crosses each edge of η an even number of times.

Let G^+ be the graph that is obtained by adding (v_1, v_2) to G (if it is not already in G). Let η^+ be a drawing of G^+ that is obtained from η by deleting the edge (v_1, v_2) from η (if (v_1, v_2) is already in η) and adding (v_1, v_2) back to η such that it is routed along ζ.

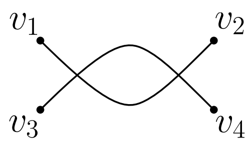
Remove all the crossings on (v_1, v_2) by pushing the crossings over v_2 using R_{IV}^p. Let $\tilde{\eta}$ be the drawing of G^+ so formed.

Let G^- be the graph that is obtained from G^+ by contracting (v_1, v_2) to a vertex x. Then obtain a drawing η^- from $\tilde{\eta}$ by identifying all the points of (v_1, v_2) (i.e., all points on ζ) such that the rotational order of the edges around the vertices is preserved. (This is just contraction of edge (v_1, v_2), as a surface minor operation.)

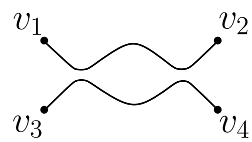
By definition, G^- is obtained from G by topological folding of (v, v_1) onto (v, v_2). Now, any edge in η^+ crosses (v_1, v_2) an even number of times, and R_{IV}^p does not change the parity of the number of crossings between any pair of edges. So the parity of the number of crossings between any pair of edges is the same in η^- as in η. Therefore η^- is a generalised superthrackle drawing of G^-. The theorem follows. □

Now we are ready to prove that any weak generalised outerthracklable graph is an outersuperthracklable graph. In an outerdrawing η, we denote the rotational order of the vertices around the boundary of the disk by $\rho(\eta')$.

Theorem 17 *Let η be a weak generalised outerthrackle drawing of a graph G. Then there is an outersuperthrackle drawing $\tilde{\eta}$ of G with $\rho(\tilde{\eta}) = \rho(\eta)$.*



(a) before



(b) after

Figure 14: Removing two crossings on two edges that cross each other an even number of times.

Proof:

Since R_I^p does not change the parity of the number of crossings between vertex-disjoint edges, we use R_I^p to remove self-crossings. So, throughout this proof we assume that no edge crosses itself.

We prove this theorem in two steps. As the first step, we construct an outerthrackle drawing η_{temp} of G. Then, as the second step, we use η_{temp} to construct an outersuperthrackle drawing $\tilde{\eta}$ of G.

Step 1.

Let η be a weak generalised outerthrackle drawing of a graph G.

Claim: There is an outerthrackle drawing η_{temp} of G with $\rho(\eta_{temp}) = \rho(\eta)$.

We prove the claim by induction on the number of crossings in the drawing.

In the base case, any two vertex-disjoint edges cross once and there are no crossings between any two edges that are incident at the same vertex. That is, η is already an outerthrackle drawing of G and we are done.

We proceed to the inductive stage. Assume that η is a weak generalised outerthrackle drawing of G with k crossings, and that the claim is true for any weak generalised outerthrackle drawing of G with $< k$ crossings. We may assume that *either* there are two edges (which may be vertex-disjoint or not) that cross more than once *or* some pair of incident edges cross exactly once, else we are in the base case.

Case 1, there are two edges $e = (v_1, v_2)$ and $e' = (v_3, v_4)$ in η that cross more than once. Let x_1 and x_2 be two crossings on e and e' that are consecutive on e.

Crossings x_1 and x_2 divide e into three segments: the part from v_1 to either x_1 or x_2 (whichever crossing that we reach first as we move along the curve of (v_1, v_2) from v_1 to v_2), say x_1, the part from x_1 to x_2, and the part from v_2 to either x_1 or x_2 (whichever crossing that we reach first as we move along the curve of (v_1, v_2) from v_2 to v_1), say x_2. Similarly x_1 and x_2 divide e' into three segments: the part from v_3 to x_1, the part from x_1 to x_2 and the part from x_2 to v_4 (without loss of generality).

We reroute e or e' in η to obtain a weak generalised outerthrackle drawing of G with a smaller number of crossings as follows. Let $\bar{l}$ denote the part of e that goes from x_1 to x_2 and let $\overline{l'}$ denote the part of e' that goes from x_1 to x_2. Let C denote the simple cycle (a closed curve that does not cross itself) formed by $\bar{l}$ and $\overline{l'}$.

Since η is an outerdrawing, all the vertices of G are located outside C and therefore the parity of the number of crossings of any arbitrary edge e'' and $\bar{l}$ is equal to the parity of the number of crossings of e'' and $\overline{l'}$. Therefore if we remove x_1 and x_2 as is shown in Figure 14, we obtain a drawing η^- of G with a smaller number of crossings than η such that the parities of the numbers of times any two independent edges cross are all the same in η and $\rho(\eta) = \rho(\eta^-)$. Therefore, η^- is a weak generalised outerthrackle drawing with a smaller number of crossings than η. Hence, by the inductive hypothesis, G has an outerthrackle drawing η_{temp} in which the edges that are incident with a vertex do not cross and $\rho(\eta_{temp}) = \rho(\eta^-) = \rho(\eta)$.

(a) before (b) after

Figure 15: Removing crossings on two adjacent edges.

Case 2, there are two incident edges e and e' in η that cross exactly once. Obtain an outerdrawing η_2 of G from η by removing any crossings between two edges that are incident at the same vertex using the move that is shown in Figure 15.

By the same argument as in Case 1, this move does not change the parity of the number of crossings between vertex-disjoint edges in outerdrawings. Therefore, η_2 is an outerthrackle drawing of G with fewer crossings than η and $\rho(\eta_2) = \rho(\eta)$. So by the inductive hypothesis, G has an outerthrackle drawing η_{temp} in which incident edges do not cross and $\rho(\eta_{temp}) = \rho(\eta_2) = \rho(\eta)$.

Therefore, by induction, the claim is true for every possible number k of crossings in η.

We now show that our outerthrackle drawing η_{temp}, provided by the claim, can be converted to an outersuperthrackle drawing $\tilde{\eta}$ such that $\rho(\eta_{temp}) = \rho(\tilde{\eta})$.

Step 2.

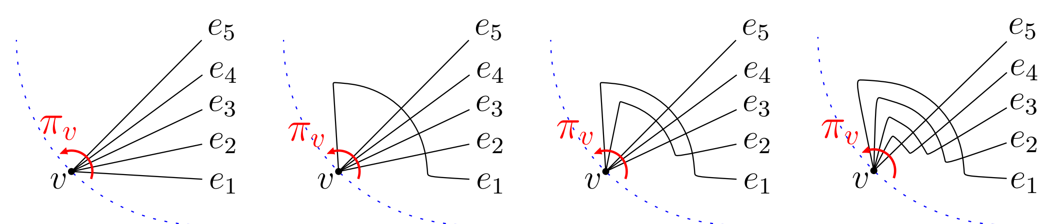
In this step we use R_V^p to reverse the order of the edges adjacent to any vertex v of η_{temp} to obtain another drawing $\tilde{\eta}$ of G. That is, if $\pi(v)$ in η_{temp} is $e_1, e_2, \ldots, e_i$ then change $\pi(v)$ to $e_i, e_{i-1}, \ldots, e_1$ in $\tilde{\eta}$ through the following series of steps:

1. $\pi_1(v) = e_2, e_3, \ldots, e_i, e_1$ (see, for example, Figure 16b).

2. $\pi_2(v) = e_i, e_{i-1}, e_1, e_2, \ldots, e_{i-2}$ (see, for example, Figure 16c).

$\vdots$

$i-1$. $\pi_{\tilde{\eta}}(v) = \pi_{i-1}(v) = e_i, e_{i-1}, \ldots, e_1$ (see, for example, Figure 16d).



(a) (b) (c) (d)

Figure 16: Constructing $\tilde{\eta}$ from η_{temp} by reversing the order of the edges around the vertices.

R_V^p does not change the parity of the number of crossings between vertex-disjoint edges. However, it changes the parity of the number of crossings between the edges that are incident with v. Since η_{temp} is an outerthrackle, any pair of edges that are incident to one vertex did not cross each other in η_{temp}. Therefore, any pair of edges that are incident to one vertex cross each other once in $\tilde{\eta}$ and $\tilde{\eta}$ is a drawing of G in which any two edges cross once. The theorem follows. $\square$

We conclude this section by establishing the relationship between these different types of outerthracklable graphs.

Theorem 18 *The following four classes of graphs are equivalent:*

1. *outersuperthracklable graphs*
2. *weak outerthracklable graphs*
3. *generalised outersuperthracklable graphs*
4. *weak generalised outerthracklable graphs.*

Proof: By definition, any outersuperthrackle is both a weak outerthrackle and a generalised outersuperthrackle. Moreover any weak outerthrackle or generalised outersuperthrackle is a weak generalised outerthrackle. Therefore, by definition, the class of weak generalised outerthracklable graphs includes all the weak outerthracklable graphs and all generalised outersuperthracklable graphs, and both of the classes of weak outerthracklable graphs and the generalised outersuperthracklable graphs include outersuperthracklable graphs (see Figure 17).

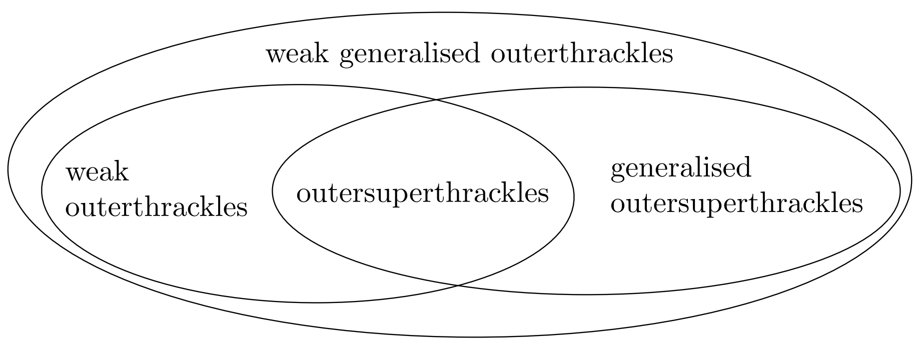


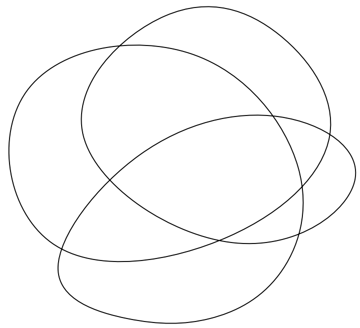
Figure 17: Relationship between different types of outerthracklable graphs

Therefore, to prove that all the aforementioned classes of graphs are equal, we only need to show that any weak generalised outerthracklable graph is an outersuperthracklable graph. So, the theorem follows by Theorem 17. □

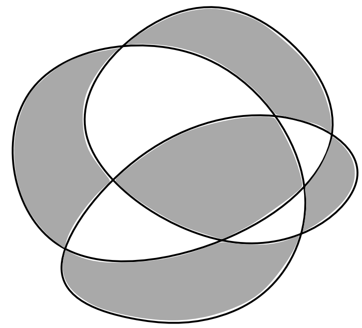
5 Generalised Superthrackles and Superthrackles

A drawing η of a graph G in $\mathbb{R}^2$ partitions all the points of $\mathbb{R}^2 \setminus \eta$ (where η also denotes the set of all points in the drawing η) into a set of *regions*, denoted by *regions*(η), such that any two points p and q are in the same region $r \in$ *regions*(η) if there is a curve from p to q that does not cross any vertex or edge of η.

For any drawing η of a cycle, define a *black-and-white colouring* of the plane with respect to η to be a colouring of each region of *regions*(η) either black or white such that no two adjacent regions are coloured in the same colour (see, for example, Figure 18). The existence of such a 2-colouring is well-known, however to make the paper self-contained we give a short proof.



(a) A drawing η of cycle C



(b) Colouring regions of the plane black and white based on η

Figure 18

Lemma 6 *Let η be a drawing of a cycle on the plane. Then there is a black-and-white colouring for η.*

Proof: Let G be a graph with a vertex for each region of η and an edge (u, v) for any two adjacent regions r_1 and r_2 of η where r_1 is represented by vertex u of G and r_2 is represented by vertex v of G.

The graph G is a dual of an Eulerian planar graph and therefore it is bipartite. □

Let η be a drawing of a graph G and let $G^- = G[S]$, where $S \subseteq V(G)$. Then $\eta[G^-]$ is the drawing of G^- obtained from η as follows:

- for each vertex v in G^-, let v be represented by the same point that represents v in η, and
- for each edge e in G^-, let e be represented by the same curve that represents e in η.

Lemma 7 *Let η be a generalised superthrackle drawing of a multigraph with only two vertices u and v and no loops. Then $\pi_\eta(u) = \pi_\eta(v)$.*

Proof: We prove this lemma by induction on the number m of edges in the drawing. In the base case η does not have any edges and the lemma holds trivially. We proceed to the inductive case.

Let $e_1, e_2, \ldots, e_m$ be the edges in η, named so that $\pi(u) = e_1, e_2, \ldots, e_m$. Let η^- be the drawing obtained by deleting e_m from η. Any two edges in η^- cross each other an odd number of times and therefore η^- is a generalised superthrackle as well. Hence, by induction, $\pi_{\eta^-}(u) = \pi_{\eta^-}(v) = e_1, e_2, \ldots, e_{m-1}$.

Now to reach a contradiction suppose that $\pi_\eta(u) \neq \pi_\eta(v)$. Since $\pi_{\eta^-}(u) = \pi_{\eta^-}(v)$, this means that e_m is not located between e_{m-1} and e_1 in $\pi_\eta(v)$.

Without loss of generality, let us assume that $\pi_\eta(v) = e_1, e_2, \ldots, e_i, e_m, e_{i+1}, e_{i+2}, \ldots, e_{m-1}$, where $1 \leq i \leq m-2$, as shown in Figure 19. Let C be the cycle that is defined by the two edges e_{m-1} and e_1.

By Lemma 6, we can colour all the regions of the plane with respect to $\eta[C]$ either black or white, such that any curve that is routed from a point p_1 to a point p_2 crosses C

- an even number of times, if p_1 and p_2 are located in regions with the same colour, or

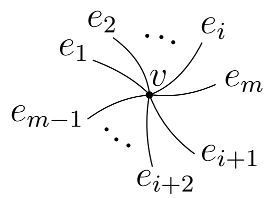


Figure 19: $\pi_\eta(v)$

- an odd number of times, otherwise.

Now, let us consider the colouring of the regions in the neighbourhoods of vertices u and v. Such a colouring can be in the form of one of the four cases shown in Figure 20.

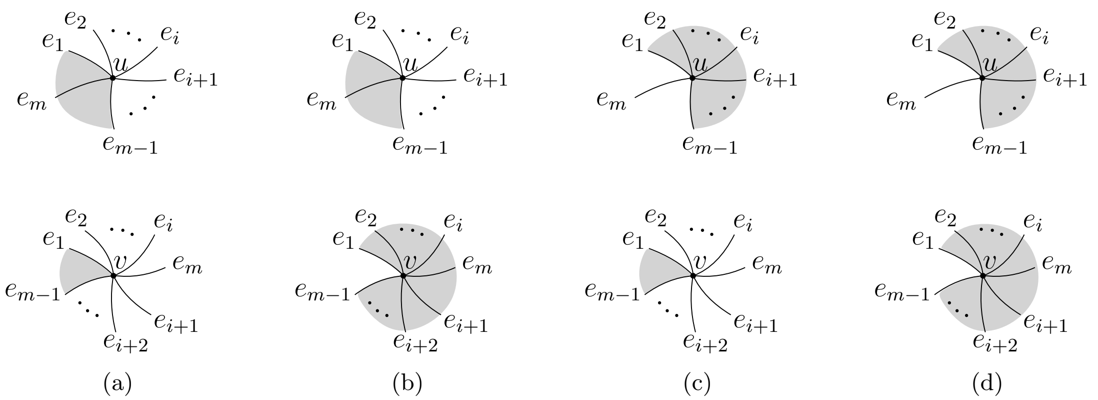


Figure 20: Four different forms of black-and-white colouring of neighbourhood of u and v in η based on C

Since η is a generalised superthrackle, any edge of η other than e_1 and e_{m-1} crosses C an even number of times. Consider e_2. The colouring of the regions of the plane in the neighbourhoods of u and v cannot be as in Figure 20b or Figure 20c. (The edge e_2 should start and end in two isochromatic regions.) However, this would lead to a contradiction, since then the initial and final portions of e_m, which crosses C an even number of times, lie in two regions that are not coloured the same. $\square$

Now we deal with the case of a graph with two vertices with loops.

Lemma 8 *Let η be a generalised superthrackle drawing of a multigraph G with two vertices. There is a superthrackle drawing η' of G such that $\Pi(\eta') = \Pi(\eta)$.*

Proof: Let u and v be the vertices of G. Then the following three cases are forbidden in a generalised superthrackle drawing η of G:

1. Let e_1 and e_2 be two loops in G where e_1 is incident with u and e_2 is incident with v. In this case, in any drawing of G, e_1 and e_2 cross each other an even number of times and therefore G is not a generalised superthracklable graph. (See, for example, Figure 21a.)

2. Let e_1 and e_2 be two loops that are both incident with the same vertex, say u, then the restriction of $\pi_\eta(u)$ to e_1 and e_2 cannot be e_1, e_1, e_2, e_2 since in that case e_1 and e_2 cross each other an even number of times. (See, for example, Figure 21b.)

3. Let e_1 and e_2 be two parallel edges between u and v, let e_3 be a loop that is incident with u, and let the two occurrences of e_3 in $\pi_\eta(u)$ be separated from each other by e_1 and e_2. (See, for example, Figure 21c.) In this case, e_3 crosses the cycle defined by the two edges e_1 and e_2 an odd number of times since it starts and ends on opposite sides of the cycle. Therefore e_3 crosses either e_1 or e_2 an even number of times.

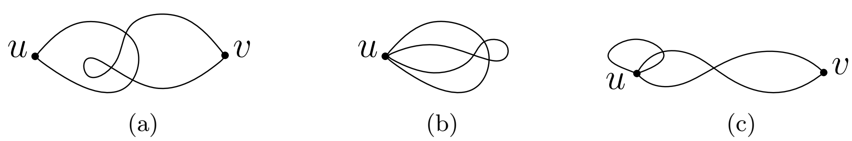


Figure 21: Three forbidden cases in a superthrackle drawing of a multigraph with two vertices.

Using forbidden case 1, from this point on, we assume that if there is any loop in G it is incident with u and not with v.

We prove this lemma by induction on the number of edges of G. Let u and v be the two vertices of G. In the base case, there is at most one loop and one edge (u, v) in G and it is easy to see that there is a superthrackle drawing of G. We proceed to the inductive case.

We have the following two cases:

Case 1. There are at least two parallel edges between u and v in G. By forbidden case 3, we know that all the endpoints of the edges that are not loops in G appear consecutively in $\pi_\eta(u)$. Therefore, by Lemma 7, there are two parallel edges $e_1 = (u, v)$ and $e_2 = (u, v)$ in G such that both of their endpoints appear consecutively and in the same order in both $\pi_\eta(u)$ and $\pi_\eta(v)$. (See, for example, Figure 22a.)

Let G^- be the graph obtained by deleting e_2 from G and let η^- be the drawing obtained by deleting e_2 from η. By the inductive hypothesis, G^- has a superthrackle drawing η_1^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

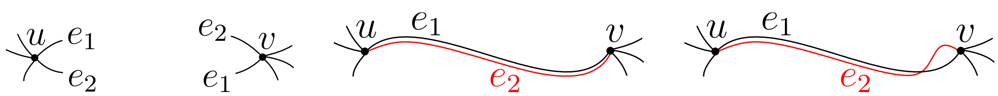
Now we obtain a drawing η_1 of G such that $\Pi(\eta_1) = \Pi(\eta)$ by adding the edge e_2 back to η_1^- using the following two steps:

1. add e_2 to the drawing such that e_2 follows e_1 sufficiently closely so that it is drawn in a local disk of e_1 and does not meet e_1 and $\pi_{\eta_1}(u)$ is the same as $\pi_\eta(u)$. (See, for example, Figure 22b.)

2. use the R_I^p move to switch the rotational order of e_1 and e_2 around v. (See, for example, Figure 22c.)

Since η_1^- is a superthrackle, any pair of edges in η_1 that does not contain e_2 cross each other once. Moreover since e_2 follows e_1 sufficiently closely, e_2 also crosses any edge other than e_1 in η_1 once. Lastly, with the R_I^p move in step 2, we guarantee that e_2 crosses e_1 once as well. Hence any two edges in η_1 cross each other once and therefore η_1 is a superthrackle.

Case 2. There are at least two loops in G and there is at most one edge incident with both u and v. By forbidden case 1, all the loops in G are incident with one vertex. Let $e_1, e_2, \ldots, e_i$ be the loops that are incident with u and let e' be the edge that is incident with both u and v.



(a) order of e_1 and e_2 in $\pi_\eta(u)$ and $\pi_\eta(v)$ (b) e_2 closely follows e_1 (c) using R_1 to ensure that e_2 crosses e_1

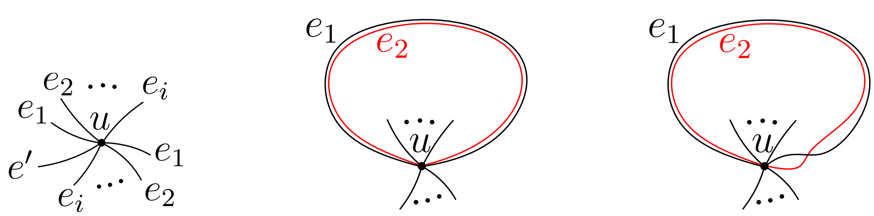
Figure 22: Obtaining η_1 by adding the edge e_2 to η_1^- in case 1.

By forbidden cases 2 and 3, it is easy to see that the loops can be named so that $\pi_\eta(u) = e_1, e_2, \ldots, e_i,\ e_1, e_2, \ldots, e_i, e'$ (see Figure 23a). Let G^- be the graph obtained by deleting e_2 from G and let η^- be the drawing that is obtained by deleting e_2 from η. By the inductive hypothesis, G^- has a superthrackle drawing η_1^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

Now we obtain a drawing η_1 of G such that $\Pi(\eta_1) = \Pi(\eta)$ by adding e_2 back to G as in the following two steps:

1. add e_2 to the drawing such that e_2 follows e_1 sufficiently closely and does not cross it (see for example Figure 23b).

2. use the R_I^p move to switch the rotational order of e_1 and e_2 around u such that $\Pi(\eta_1) = \Pi(\eta)$ (see for example Figure 23c).



(a) $\pi(u)$ (b) e_2 closely follows e_1 (c) using R_1 to ensure that e_2 crosses e_1

Figure 23: Obtaining η_1 by adding the edge e_2 to η_1^- in case 2.

By similar reasoning to case 1, any two edges in η_1 cross each other once and therefore η_1 is a superthrackle. □

We use the above lemma as the base case of the proof of Theorem 19. Note that by proving Theorem 19 we have proved Theorem 3.

Theorem 19 *Let η be a generalised superthrackle drawing of a connected multigraph G. Then there is a superthrackle drawing η' of G such that $\Pi(\eta') = \Pi(\eta)$.*

Proof: It is easy to prove the lemma if G has only one vertex. So let us assume that G is connected and has at least two vertices.

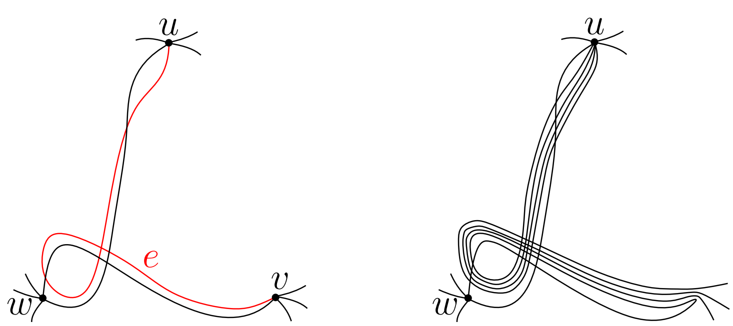
We prove this theorem by induction on the number of vertices in G. In the base case, G has two vertices and by Lemma 8, we know that the theorem holds. We proceed to the inductive case where there are at least three vertices in G.

Let u and v be two distinct vertices of G such that there are two edges (u, w) and (v, w) in G where (u, w) and (v, w) appear consecutively in $\pi_\eta(w)$ and $w \neq u, v$. Since G is connected and G has at least three vertices, such u and v exist.

By Lemma 5, there is a curve c from u to v that crosses each edge of G an even number of times. Let G^+ be the graph obtained by adding an edge $e = (u, v)$ to G (we add an extra (u, v) edge to G if such an edge already exists in G) and let η^+ be a drawing of G^+ obtained by adding e to η such that e is routed along the curve c.

Let G^- be the graph that is obtained by contracting e in G^+. G^- has one vertex fewer than G. Obtain a drawing η^- of G^- by contracting e in η^+ such that:

- u remains in the same position, and v is identified with it;
- any edge e' incident with v in G^+ follows the route of e sufficiently closely until it reaches u without crossing any other edge incident with v;
- for any new crossing introduced between e' and another edge e'' in η^- there is a crossing between e and e'' on η^+ (see for example Figure 24).



(a) Edge e crosses all the other edges in η^+ an even number of times.

(b) Contracting e such that all the edges incident with v follow e sufficiently closely.

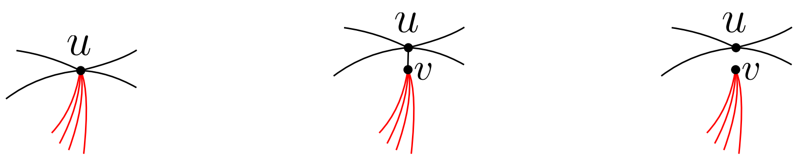
Figure 24: Obtaining η by contracting e in η^+

The edge e crosses all other edges of η^+ an even number of times and since η is a generalised thrackle, all the edges in η^+ except e cross each other an odd number of times. Therefore, any two edges in η^- cross each other an odd number of times. In other words, η^- is a generalised thrackle as well.

Since G^- has one vertex fewer than G, by the inductive hypothesis, there is a superthrackle drawing η_1^- of G^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

All the edges that were incident with v in η^+ appear consecutively in $\pi_{\eta^-}(u)$. Therefore, since $\Pi(\eta_1^-) = \Pi(\eta^-)$, those edges appear consecutively in $\pi_{\eta_1^-}(u)$ as well (see for example Figure 25a). Hence it is easy to decontract e to obtain a drawing η_1^+ of G^+ such that all the edges of η_1^+ except for e cross each other once and e does not cross any other edges (see for example Figure 25b).

Now we can delete e from η_1^+ to obtain a superthrackle drawing η' of G (see for example Figure 25c). □



(a) $\pi_{\eta^-}(u)$. Edges that were incident to v in η^+ are depicted in red.

(b) Obtaining η_1^+ from η^- by decontracting e

(c) Obtaining η' from η_1^+ by deleting e

Figure 25: Obtaining η'

6 The Hanani-Tutte Theorem and Superthrackles

In this section we examine the relationship between the Hanani-Tutte Theorem and superthrackles. The two subsections are about the connections between superthrackles and the weak and the strong Hananni-Tutte Theorems respectively.

6.1 The Weak Hanani-Tutte Theorem and Superthrackles

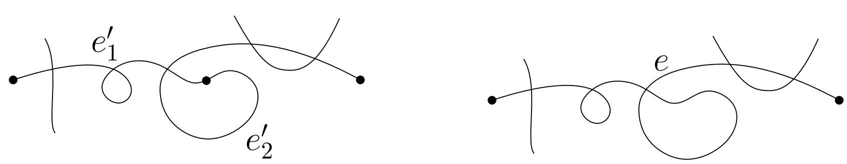
Archdeacon and Stor characterised superthrackles in terms of eight forbidden configurations [3] (see Figure 5). Four of these configurations are closely related to $K_{3,3}$ and K_5 which are the forbidden graphs in Kuratowski's Theorem. Next we will explain why there is such a close relationship between these two theorems.

Now we are ready to prove Theorem 6.

Proof: [Proof of Theorem 6]

Any edge e in G is replaced by a path in G'. Let us denote that path by $P(e)$ and the length of that path by $l(e)$. Denote the edges of $P(e)$ by $e'_1, e'_2, \ldots, e'_{l(e)}$.

Let η' be a drawing of G'. Obtain a drawing η of G from η' as follows. For each edge e in G, replace edges $e'_1, e'_2, \ldots, e'_{l(e)}$ in η' by a curve, so that e is routed exactly on the curve along which the edges $e'_1, e'_2, \ldots, e'_{l(e)}$ are routed in Σ (see for example, Figure 26).



(a) Edges e'_1 and e'_2 of path $P(e)$ in drawing η' of graph G'

(b) Edge e in drawing η of graph G

Figure 26: Replacing e^*_1 and e^*_2 with e, where $i(e)$ is 2

By the theorem's assumption, in any drawing η of G there are two edges that cross each other an odd number of times. Let e and f be two edges that cross an odd number of times in η. For any edge (u, v) in G with the exception of one of the edges, say e, there is a (u, v) path with even length in G'. Therefore, f is replaced by a path of even length to obtain G' from G.

Let us denote the number of crossings between two paths P_1 and P_2 with $\chi(P_1, P_2)$. Since $\chi(e, f)$ is odd in η, $\chi(P(e), P(f))$ is odd in η'. But $\chi(P(e), P(f))$ is obtained by summing up

$\chi(e_1, e_2)$ for all the pairs e_1, e_2 of edges where e_1 is an edge of $P(e)$ and e_2 is an edge of $P(f)$. Since $l(f)$ is even, there is an even number of such pairs of edges. To reach a contradiction, assume that all such pairs of edges cross an odd number of times. Then we have an even number of odd integers that sum up to an odd integer, which is a contradiction. Hence there is an edge e_1 in $P(e)$ and an edge f_2 in $P(f)$ such that e_1 crosses f_2 an even number of times (see for example, Figure 27).

□

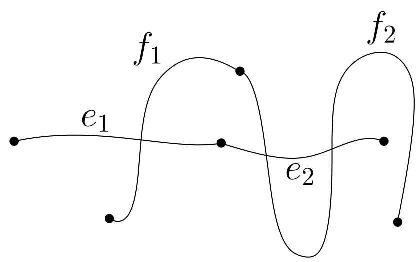


Figure 27: $\pi_\eta(v)$

An implication of the Weak Hanani-Tutte Theorem together with Kuratowski's Theorem is that, in any drawing of $K_{3,3}$ or K_5 or any subdivision of them, there are two edges that cross each other an odd number of times. This fact, along with Theorem 6, proves that K_5^*, $K_5^*(e)$, $K_{3,3}^*$ and $K_{3,3}^*(e)$ (depicted in Figure 5) have the property that, in any drawing of them in the plane, there are two edges that cross an even number of times. Therefore, by definition, these graphs are not generalised superthracklable and therefore not superthracklable either.

The Weak Hanani-Tutte Theorem can be generalised to all surfaces [26, 27]. That is, if a graph G does not have a drawing that can be drawn on a surface Σ without crossings, then there are two edges that cross each other an odd number of times in any drawing of G on Σ and hence, by Theorem 6, any graph in $\mathcal{G}(G, E \setminus \{e\})$ is not superthracklable on Σ.

Theorem 20 *Let G be a graph that is in the set of forbidden minors for graphs embeddable on a surface Σ. Let $H = (V, E)$ be a subdivision of G. Then any graph that contains a graph in $\mathcal{G}(H, E \setminus \{e\})$, where $e \in E$, is neither a generalised superthracklable graph nor a superthracklable graph with respect to Σ.*

Proof: Since H is a subdivision of G, it is not embeddable on Σ. Therefore, by the Weak Hanani-Tutte Theorem for all surfaces [26, 27], in any drawing of H on Σ there are two edges that cross each other an odd number of times.

Therefore, by Theorem 6, in any drawing of a graph $H' \in \mathcal{G}(H, E \setminus \{e\})$ on Σ, there are two edges that cross each other an even number of times. Hence if a graph H'' contains H' it cannot be drawn on Σ such that any two edges of H'' cross an odd number of times. Hence, by Theorem 19, H'' is neither a generalised superthracklable graph nor a superthracklable graph. □

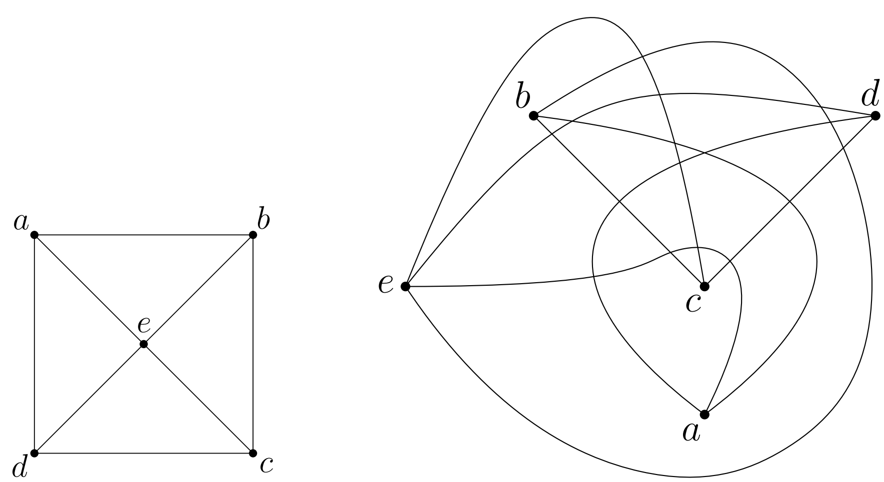
This theorem explains why there are four forbidden configurations in the Archdeacon and Stor characterisation of superthrackles ([3], see Figure 5) that are closely related to $K_{3,3}$ and K_5 (forbidden graphs in Kuratowski's Theorem).

6.2 The Strong Hanani-Tutte Theorem and Superthrackles

In Section 5, we proved that any generalised superthracklable graph is a superthracklable graph. In other words, if there is a drawing of a graph G in which any two edges cross each other an odd number of times, then there is a drawing of G in which any two edges cross once. Notice the similarities between this theorem and the Weak Hanani-Tutte Theorem which states that if there is a drawing of a graph G in which any two edges cross each other an even number of times, then there is a drawing of G in which any two edges cross zero times.

A natural question that arises the above observation is whether we can prove a theorem similar to the Strong Hanani-Tutte Theorem for superthrackles. Recall that the Strong Hanani-Tutte Theorem states that, if there is a drawing of a graph G in which any two vertex-disjoint edges cross an even number of times, then there is a drawing for G such that any two edges cross zero times. We can ask a similar question for superthrackles as follows. Let G be any graph that has a drawing in which any two vertex-disjoint edges cross an odd number of times. Does G always have a drawing in which any two edges cross once?

The answer to the above question is no. Figure 28a depicts a planar embedding of a graph G that is not superthracklable [3]. Figure 28b depicts a drawing of G in which any two vertex-disjoint edges cross each other an odd number of times.



(a) A planar embedding of a non-superthracklable graph G

(b) A drawing of G in which any two vertex-disjoint edges cross an odd number of times.

Figure 28: A non-superthracklable graph G and a drawing of G in which any two vertex-disjoint edges cross an odd number of times.

7 Conclusion

This paper studied variations of outersuperthrackles. We also proved that any generalised superthrackle is a superthrackle and we examined the relationship between the Hanani-Tutte Theorem, generalised superthrackles and superthrackles.

Archdeacon and Stor proved that a graph is superthracklable if and only if it does not contain a subgraph that is parity hemeomorphic to any graph in Figure 5 (see Theorem 14). Although superthracklable graphs are very well studied, we still do not know if we can characterise superthracklable graphs for all surfaces in terms of graphs without a subgraph that is parity homeomorphic to a finite set of graphs.

Moreover, we do not know about characterisations of superthrackles on surfaces other than the plane. For example, what are the superthracklable graphs with respect to the projective plane or any surface other than the plane?

Lastly, in Section 6, we have shown that if we have the set of forbidden minors for the graphs that are embeddable on a surface Σ then we can find families of graphs that are not superthracklable with respect to Σ. This suggests the following question. Assuming that we have a characterisation of graphs that are superthracklable with respect to a surface Σ, can we determine forbidden minors for graphs that are embeddable on Σ?

Acknowledgments

The authors thank Peter Eades and David Wood for their helpful discussions and suggestions. We are also very grateful to the referees for their detailed comments which have resulted in many improvements to the paper.

References

[1] E. Ackerman, J. Fox, J. Pach, and A. Suk. On grids in topological graphs. In *Proceedings of the Twenty-Fifth Annual Symposium on Computational Geometry*, pages 403–412. ACM, 2009. doi:10.1145/1542362.1542430.

[2] D. Archdeacon. A Kuratowski theorem for the projective plane. *Journal of Graph Theory*, 5(3):243–246, 1981. doi:10.1002/JGT.3190050305.

[3] D. Archdeacon and K. Stor. Superthrackles. *Australasian Journal of Combinatorics*, 67(2):145–158, 2017.

[4] P. Brass, W. O. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer, New York, 2005.

[5] G. Cairns, T. J. Koussas, and Y. Nikolayevsky. Great-circle spherical thrackles. *Discrete Mathematics*, 338(12):2507–2513, 2015. doi:10.1016/J.DISC.2015.06.023.

[6] G. Cairns, M. McIntyre, and Y. Nikolayevsky. The thrackle conjecture for K_5 and $K_{3,3}$. In J. Pach, editor, *Towards a theory of geometric graphs*, volume 342 of *Contemporary Mathematics*, pages 35–54. Providence, RI; American Mathematical Society, 2004.

[7] G. Cairns and Y. Nikolayevsky. Bounds for generalized thrackles. *Discrete & Computational Geometry*, 23(2):191–206, 2000. doi:10.1007/PL00009495.

[8] G. Cairns and Y. Nikolayevsky. Generalized thrackle drawings of non-bipartite graphs. *Discrete & Computational Geometry*, 41(1):119–134, 2009. doi:10.1007/S00454-008-9095-5.

[9] G. Cairns and Y. Nikolayevsky. Outerplanar thrackles. *Graphs and Combinatorics*, 28(1):85–96, 2012. doi:10.1007/S00373-010-1010-1.

[10] C. Chojnacki. Über wesentlich unplättbare Kurven im dreidimensionalen Raume. *Fundamenta Mathematicae*, 23(1):135–142, 1934.

[11] H. R. Dehkordi and G. Farr. Non-separating planar graphs. *The Electronic Journal of Combinatorics*, 28(1):P1.11, 2021. doi:10.37236/8816.

[12] R. Fulek and J. Kynčl. Counterexample to an extension of the Hanani-Tutte Theorem on the surface of genus 4. *Combinatorica*, 39(6):1267–1279, 2019. doi:10.1007/S00493-019-3905-7.

[13] R. Fulek, J. Kynčl, and D. Pálvölgyi. Unified Hanani-Tutte theorem. *Electronic Journal of Combinatorics*, 24(3):P3.18, 2017. doi:10.37236/6663.

[14] R. Fulek and J. Pach. A computational approach to Conway's Thrackle Conjecture. *Computational Geometry*, 44(6-7):345–355, 2011. doi:10.1016/J.COMGEO.2011.02.001.

[15] R. Fulek and J. Pach. Thrackles: An improved upper bound. *Discrete Applied Mathematics*, 259:226–231, 2019. doi:10.1016/J.DAM.2018.12.025.

[16] L. A. Goddyn and Y. Xu. On the bounds of Conway's thrackles. *Discrete & Computational Geometry*, 58:410–416, 2017. doi:10.1007/S00454-017-9877-8.

[17] R. L. Graham. The largest small hexagon. *Journal of Combinatorial Theory, Series A*, 18(2):165–170, 1975. doi:10.1016/0097-3165(75)90004-7.

[18] J. Green and R. D. Ringeisen. Combinatorial drawings and thrackle surfaces. In Y. Alavi and A. Schwenk, editors, *Graph Theory, Combinatorics, and Algorithms*, volume 2, pages 999–1009. Wiley-Interscience Kalamazoo, MI, 1995.

[19] C. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15(1):271–283, 1930.

[20] M. Loebl and G. Masbaum. On the optimality of the Arf invariant formula for graph polynomials. *Advances in Mathematics*, 226(1):332 – 349, 2011.

[21] L. Lovász, J. Pach, and M. Szegedy. On Conway's thrackle conjecture. *Discrete & Computational Geometry*, 18(4):369–376, 1997. doi:10.1007/PL00009322.

[22] G. Misereh and Y. Nikolayevsky. Annular and pants thrackles. *Discrete Mathematics & Theoretical Computer Science*, 20(1):#16, 2018. doi:10.23638/DMTCS-20-1-16.

[23] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, MD, 2001.

[24] J. Pach, R. Radoičić, and G. Tóth. Tangled thrackles. In A. Márquez, P. Ramos, and J. Urrutia, editors, *Spanish Meeting on Computational Geometry*, pages 45–53. Springer, 2011. doi:10.1007/978-3-642-34191-5_4.

[25] J. Pach and E. Sterling. Conway's conjecture for monotone thrackles. *The American Mathematical Monthly*, 118:544 – 548, 2011. doi:10.4169/AMER.MATH.MONTHLY.118.06.544.

[26] M. J. Pelsmajer, M. Schaefer, and D. Stasi. Strong Hanani–Tutte on the projective plane. *SIAM Journal on Discrete Mathematics*, 23(3):1317–1323, 2009. doi:10.1137/08072485X.

[27] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings. *Journal of Combinatorial Theory, Series B*, 97(4):489 – 500, 2007. doi:10.1016/J.JCTB.2006.08.001.

[28] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings on surfaces. *European Journal of Combinatorics*, 30(7):1704–1717, 2009. doi:10.1016/J.EJC.2009.03.002.

[29] A. Perlstein and R. Pinchasi. Generalized thrackles and geometric graphs in $\mathbb{R}^3$ with no pair of strongly avoiding edges. *Graphs and Combinatorics*, 24(4):373–389, 2008. doi:10.1007/S00373-008-0796-6.

[30] B. L. Piazza, R. D. Ringeisen, and S. K. Stueckle. Subthrackleable graphs and four cycles. *Discrete Mathematics*, 127(1-3):265–276, 1994. doi:10.1016/0012-365X(92)00484-9.

[31] R. Ringeisen. Two old extremal graph drawing conjectures: Progress and perspectives. *Congressus Numerantium*, pages 91–104, 1996.

[32] A. J. Ruiz-Vargas, A. Suk, and C. D. Tóth. Disjoint edges in topological graphs and the tangled-thrackle conjecture. *European Journal of Combinatorics*, 51:398–406, 2016. doi:10.1016/J.EJC.2015.07.004.

[33] W. T. Tutte. Toward a theory of crossing numbers. *Journal of Combinatorial Theory*, 8(1):45–53, 1970.

[34] K. Wagner. Bemerkungen zum vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.

[35] D. R. Woodall. Thrackles and deadlock. In D. J. A. Welsh, editor, *Combinatorial Mathematics and Its Applications*, pages 335–348. Academic Press, New York, 1971.

[36] Y. Xu. A new upper bound for Conway's thrackles. *Applied Mathematics and Computation*, 389:125573, 2021. doi:10.1016/J.AMC.2020.125573.