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Min-k-planar Drawings of Graphs

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Abstract. The study of nonplanar drawings of graphs with restricted crossing configurations is a well-established topic in graph drawing, often referred to as *beyond-planar graph drawing*. One of the most studied types of drawings in this area are the *k-planar drawings* ($k \geq 1$), where each edge cannot cross more than k times. We generalize k-planar drawings, by introducing the new family of *min-k-planar drawings*. In a min-k-planar drawing edges can cross an arbitrary number of times, but for any two crossing edges, one of the two must have no more than k crossings. We prove a general upper bound on the number of edges of min-k-planar drawings, a finer upper bound for $k = 3$, and tight upper bounds for $k = 1, 2$. Also, we study the inclusion relations between min-k-planar graphs (i.e., graphs admitting min-k-planar drawings) and k-planar graphs. In our setting, we only allow *simple* drawings, that is, any two edges cross at most once, no two adjacent edges cross, and no three edges intersect at a common point.

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1 Introduction

Beyond planarity [20, 26] is a recent area of focus in graph drawing and topological graph theory, having its foundations established in the 1970s and 1980s. It comprises works on graphs that go beyond planar graphs in the sense that several, mostly local, crossing configurations are forbidden. The simplest are *1-planar* graphs, where at most one crossing per edge is allowed [28, 32], and their generalization k*-planar* graphs, where at most $k \geq 1$ crossings per edge are tolerated [13, 20, 24, 30, 31]. Other prominent examples of graph classes are *fan-planar* graphs [11, 12, 15, 16, 27], where several edges might cross the same edge but they should be adjacent to the same vertex, and k*-gap-planar* graphs ($k \geq 1$) [8, 9, 10], where for each pair of crossing edges one of the two edges contains a small gap through which the other edge can pass, and at most k gaps per edge are allowed. Another popular family is the one of k*-quasiplanar* graphs, which forbids k mutually crossing edges [2, 3, 4, 5, 23]. Mostly, edge density and inclusion relations of different beyond-planar graph classes have been studied [5, 20, 26].

In this paper we introduce a new graph family that generalizes k-planar graphs by permitting certain edges to have more than k crossings. Namely, for each two crossing edges we require that at least one of them contains at most k crossings. Formally, this graph family is defined as follows:

Definition 1 *A graph G is* min-k-planar ($k \geq 1$) *if it admits a drawing on the plane, called* min-k-planar *drawing, such that for any two crossing edges e and e' it holds* $\min\{\mathrm{cr}(e), \mathrm{cr}(e')\} \leq k$ *, where* $\mathrm{cr}(e)$ *and* $\mathrm{cr}(e')$ *are the number of crossings of e and e', respectively.*

Clearly, every k-planar drawing Γ is also min-k-planar, but not vice versa. An edge of the graph that crosses in Γ is a *heavy* edge if it crosses more than k times, otherwise it is a *light* edge. There are two main motivations behind the study of min-k-planar graphs:

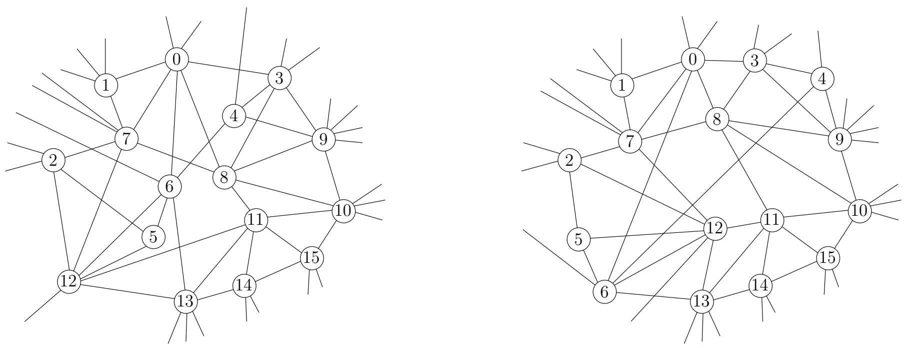
(i) From a theoretical perspective, when a graph is not k-planar we may want to draw it by allowing some heavy edges, whose removal yields a k-planar drawing. To this regard, if m is the total number of edges in the graph, we will prove that the number of heavy edges in a min-k-planar drawing is at most $\frac{k}{2k+1} \cdot m$, whose value varies in the interval $[\frac{m}{3}, \frac{m}{2})$ for $k \geq 1$.

(ii) From a practical perspective, even if a graph is k-planar, allowing (few) pairwise-independent heavy edges may reduce the visual complexity of the layout, even when the total number of crossings grows. For example, Figure 1 shows two drawings of the same portion of a graph. Despite the drawing in Figure 1(a) being 2-planar and having fewer crossings in total, the one in Figure 1(b) appears more readable; it is not 2-planar, but it is min-2-planar.

Min-k-planar graphs are also implicitly studied in [33, 34], proving that the underlying graph of a convex min-k-planar drawing has treewidth $3k + 11$.

Contribution. We study the edge density of min-k-planar graphs (Section 3) and their inclusion relations with k-planar graphs (Section 4). In our setting, we only allow *simple* drawings, that is, any two edges cross at most once, no two adjacent edges cross, and no three edges intersect at a common point. After giving general bounds on edge and crossing numbers, we focus on $k \in \{1, 2, 3\}$:

− We provide tight upper bounds on the maximum number of edges of min-1-planar and min-2-planar graphs. Namely, we prove that n-vertex min-1-planar graphs and min-2-planar graphs have at most $4n - 8$ edges and at most $5n - 10$ edges, respectively, as for 1-planar and 2-planar graphs. For min-3-planar graphs we give an upper bound of $6n - 12$ and show min-3-planar graphs with $5.6n - O(1)$ edges, hence having density higher than the one of every 3-planar graph.



(a) 2-planar drawing

(b) min-2-planar drawing

Figure 1: Two drawings of the same portion of a graph: (a) is 2-planar and has 10 crossings; (b) is min-2-planar, is not 2-planar, and has 12 crossings; it contains two "heavy" edges incident to vertex 6, each with several crossings.

− Despite the maximum density of min-k-planar graphs for $k = 1, 2$ equals the one of k-planar graphs, we show that 1-planar and 2-planar graphs are proper sub-classes of min-1-planar and min-2-planar graphs (as for $k = 3$). However, the min-1-planar graphs that can reach the maximum density of $4n-8$ are also 1-planar (i.e., the two classes coincide), while this is not true for $k = 2, 3$.

Section 2 introduces notation and terminology; final remarks and open problems are in Section 5.

2 Basic Definitions

We only deal with connected graphs. A graph is *simple* if it does not contain multiple edges and self-loops. A graph with multiple edges but not self-loops is also called a *multi-graph.* Let G be any (not necessarily simple) graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. A *drawing* Γ of G maps each vertex $v \in V(G)$ to a distinct point in the plane and each edge $uv \in E(G)$ to a simple Jordan arc between the points corresponding to u and v. We always assume that Γ is a *simple* drawing, that is: (i) two *adjacent* edges (i.e., edges that share a vertex) do not intersect, except at their common endpoint (in particular, no edge is self-crossing); (ii) two *independent* (i.e. non-adjacent) edges intersect at most in one of their interior points, called a *crossing point*; (iii) no three edges intersect at a common crossing point.

Let Γ be a drawing of G. A *vertex of* Γ is either a point corresponding to a vertex of G, called a *real-vertex*, or a point corresponding to a crossing point, called a *crossing-vertex* or simply a *crossing.* We remark that in the literature a plane graph obtained by replacing crossing points with dummy vertices is often referred to as a *planarization* [19]. We denote by $V(\Gamma)$ the set of vertices of Γ. An *edge* of Γ is a curve connecting two vertices of Γ. We denote by $E(\Gamma)$ the set of edges of Γ. An edge $e \in E(\Gamma)$ is a portion of an edge in $E(G)$, which we denote by $\overline{e}$; if both the endpoints of e are real-vertices, then e and $\overline{e}$ coincide.

Drawing Γ subdivides the plane into topologically connected regions, called *faces.* The boundary of a face consists of a cyclical sequence of vertices (real- or crossing-vertices) and edges of Γ. We denote by $F(\Gamma)$ the set of faces of Γ. Exactly one face in $F(\Gamma)$ corresponds to an infinite region of the plane, called the *external face* of Γ; the other faces are the *internal faces* of Γ. If the boundary of a face f of Γ contains a vertex v (or an edge e), we say that f *contains* v (or e).

In the following, if not specified, we denote by $n = |V(G)|$ and $m = |E(G)|$ the number of vertices and the number of edges of G, respectively.

Degree of vertices and faces. For a vertex $v \in V(G)$, denote by $\deg_G(v)$ the *degree of* v *in* G, i.e., the number of edges incident to v. Analogously, for a vertex $v \in V(\Gamma)$, denote by $\deg_\Gamma(v)$ the *degree of* v *in* Γ. Note that, if $v \in V(G)$ then $\deg_\Gamma(v) = \deg_G(v)$, while if v is a crossing-vertex then $\deg_\Gamma(v) = 4$. For a face $f \in F(\Gamma)$, denote by $\deg_\Gamma(f)$ the *degree of* f, i.e., the number of times we traverse vertices (either real- or crossing-vertices) while walking on the boundary of f clockwise. Each vertex contributes to $\deg_\Gamma(f)$ the number of times we traverse it (possibly more than once if the boundary of f is not a simple cycle). Also, denote by $\deg^r_\Gamma(f)$ the *real-vertex degree of* f, i.e., the number of times we traverse a real-vertex of Γ while walking on the boundary of f clockwise. Again, each real-vertex contributes to $\deg^r_\Gamma(f)$ the number of times we traverse it. Finally, $\deg^c_\Gamma(f)$ denotes the number of times we traverse a crossing-vertex of Γ while walking on the boundary of f clockwise. Clearly, $\deg_\Gamma(f) = \deg^r_\Gamma(f) + \deg^c_\Gamma(f)$.

We say that a face $f \in F(\Gamma)$ is an *h-real face*, for $h \geq 0$, if $\deg^r_\Gamma(f) = h$. An h-real face of degree d is called an *h-real d-gon*. For $k = 2, 3, 4, 5, 6$, a face that is an h-real k-gon, is also called an *h-real bigon* ($k = 2$), an *h-real triangle* ($k = 3$), an *h-real quadrilateral* ($k = 4$), an *h-real pentagon* ($k = 5$), and an *h-real hexagon* ($k = 6$), respectively. An edge $e = uv \in E(\Gamma)$ is an *h-real edge* ($h \in \{0, 1, 2\}$) if $|\{u, v\} \cap V(G)| = h$, i.e., e contains h real-vertices.

Beyond-planar graphs. A family $\mathcal{F}$ of *beyond-planar graphs* is a set of (nonplanar) graphs that admit drawings with desired or forbidden edge-crossing configurations [20]. The *edge density* of a graph $G \in \mathcal{F}$ is the ratio between its number m of edges and its number n of vertices. Graph G is *maximally dense* if it has the maximum edge density over all graphs of $\mathcal{F}$ with n vertices. Graph G is *optimal* if it has the maximum edge density over all graphs in $\mathcal{F}$. Note that $\mathcal{F}$ might not contain optimal graphs for all values of n (see, e.g., [20]).

3 Edge Density of Min-k-planar Graphs

We start by proving some general bounds on the number of crossings in a min-k-planar drawing and on the number of edges of min-k-planar graphs.

Property 1 *Any min-k-planar drawing Γ of a graph G (with $k \geq 1$) has at most $k \cdot \ell$ crossings, where ℓ is the number of light edges of G in Γ.*

Proof: Two heavy edges cannot cross, thus each crossing in Γ belongs to at least one light edge. Since each light edge has at most k crossings, the bound follows. □

Property 2 *Let Γ be a min-k-planar drawing of an m-edge graph G (with $k \geq 1$). The number of heavy edges of G in Γ is at most $\frac{k}{2k+1} \cdot m$.*

Proof: Let h and ℓ be the number of heavy edges and the number of light edges of G in Γ, respectively. Observe that $m \geq h + \ell$. By definition, each heavy edge contains at least $(k + 1)$ crossings, and two heavy edges do not cross. Hence, the number of crossings in Γ is at least $h \cdot (k + 1)$. By Property 1, we have $h \cdot (k + 1) \leq k \cdot \ell \leq k \cdot m - k \cdot h$, which implies $h \leq \frac{k}{2k+1} \cdot m$. □

We now give a general bound on the edge density of min-k-planar simple graphs, for any $k \geq 2$. Finer bounds for $k = 1, 2, 3$ are given in the next sections.

Theorem 1 *For any min-k-planar simple graph G with n vertices and m edges it holds $m \leq \min\{5.39\sqrt{k} \cdot n, (3.81\sqrt{k}+3) \cdot n\}$ when $k \geq 2$.*

Proof: Let $\mu = \min\{5.39\sqrt{k} \cdot n, (3.81\sqrt{k}+3) \cdot n\}$. Note that $\mu = 5.39\sqrt{k} \cdot n$ when $2 \leq k \leq 3$, while $\mu = (3.81\sqrt{k}+3) \cdot n$ when $k \geq 4$. Hence, we prove that for $2 \leq k \leq 3$ we have $m \leq 5.39\sqrt{k} \cdot n$, while for $k \geq 4$ we have $m \leq (3.81\sqrt{k}+3) \cdot n$.

Suppose first that $2 \leq k \leq 3$. If $m < 6.95n$, the relation $m \leq 5.39\sqrt{k} \cdot n$ trivially holds, as $5.39\sqrt{k} \cdot n \geq 7.63n$. If $m \geq 6.95n$, let $\mathrm{cr}(G)$ be the minimum number of crossings required by any min-k-planar drawing Γ of G. The improved version by Ackerman of the popular Crossing Lemma (Theorem 6 in [1]) implies that $\mathrm{cr}(G) \geq \frac{1}{29}\frac{m^3}{n^2}$. If ℓ is the number of light edges of G in Γ, by Property 1 we have $\mathrm{cr}(G) \leq k \cdot \ell \leq k \cdot m$. Hence $\frac{1}{29}\frac{m^3}{n^2} \leq k \cdot m$, which yields $m \leq 5.39\sqrt{k} \cdot n$.

Suppose now that $k \geq 4$ and let Γ be any min-k-planar drawing of G with ℓ light edges. Since no two heavy edges cross, the subgraph of G consisting of all heavy and crossing-free edges in Γ has at most $3n-6$ edges, hence $m \leq \ell + 3n - 6$. Let G' be the subgraph of G consisting of the ℓ light edges of G only, and let Γ' be the restriction of G' in Γ. We show that $\ell \leq 3.81\sqrt{k} \cdot n$. The relation trivially holds when $\ell < 6.95n$, as $k \geq 4$. If $\ell \geq 6.95n$, using Ackerman's version of the Crossing Lemma applied to G', we have $\mathrm{cr}(G') \geq \frac{1}{29}\frac{\ell^3}{n^2}$. Also, Γ' has at most $\frac{k \cdot \ell}{2}$ crossings, because each light edge has at most k crossings and each crossing is shared by two edges of G'. It follows that $\frac{1}{29}\frac{\ell^3}{n^2} \leq \frac{k \cdot \ell}{2}$, which still implies $\ell \leq 3.81\sqrt{k} \cdot n$. Therefore, $m \leq \ell + 3n - 6 \leq \ell + 3n \leq (3.81\sqrt{k}+3) \cdot n$. □

3.1 Density of Min-1-planar Graphs

Let Γ be a min-1-planar drawing of a graph G. We color each edge of $E(G)$ either red or green with the following rule: (i) edges that are crossing-free in Γ are colored red; (ii) if $\{e_1, e_2\} \in E(G)$ is a pair of edges that cross in Γ, with $\mathrm{cr}(e_1) \geq \mathrm{cr}(e_2)$, we color e_1 as green and e_2 as red (if $\mathrm{cr}(e_1) = \mathrm{cr}(e_2) = 1$, the red edge is chosen arbitrarily). Note that, since Γ is a min-1-planar drawing, each red edge is crossed at most once, hence the above coloring rule is well-defined. In particular, heavy edges are always colored green, while if two light edges cross, one is colored green and the other is colored red. Hence, the subgraph induced by the red edges is a plane graph, called the *red subgraph of G defined by* Γ, or simply the *red subgraph of* Γ.

Lemma 1 *Let G be a simple graph and let Γ be a min-1-planar drawing of G. We can always augment Γ with edges in such a way that the new drawing is still min-1-planar and all faces of its red subgraph have degree three.*

Proof: Let Γ_r be the red subgraph of Γ. Since G is simple, every face of Γ has degree greater than two. Suppose that Γ_r has at least one face f such that $\deg_{\Gamma_r}(f) \geq 4$. We augment Γ with new red edges in two steps, described below. The augmentation may introduce multiple edges, but it will guarantee that all faces of the new red subgraph have degree three.

Step 1. Suppose that there exists a face f of Γ_r with $\deg_{\Gamma_r}(f) \geq 4$ and containing two vertices u and v that can be connected by an edge uv that splits f without crossing other edges of Γ. We add edge uv and color it as red (as it is crossing-free); we also say that this operation *augments* f. We repeat this procedure until no such a face f exists. The obtained drawing is still min-1-planar, since we added only crossing-free edges.

Step 2. Suppose that Γ_r still contains a face f with $\deg_{\Gamma_r}(f) \geq 4$. Observe that:

(i) Face f is traversed by a green edge in Γ, otherwise it would have been augmented in Step 1; see Figure 2(a).

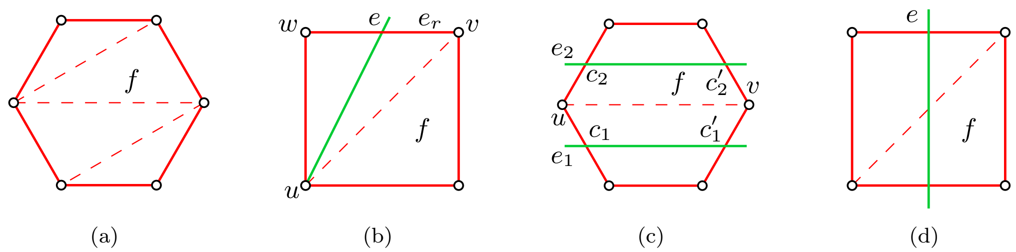


Figure 2: Augmentation of Γ as described in Step 2 of the proof of Lemma 1. Edges that can be added to Γ are represented as dashed red segments.

(ii) Every green edge e that traverses f is not incident to any vertex u of f. Namely, suppose for contradiction that e is incident to a vertex u of f, and let $e_r = vw$ be the red edge of f crossed by e. Since $\deg_{\Gamma_r}(f) \geq 4$, at least one among v and w, say for example v, is not adjacent to u. However this implies that either f can be augmented by adding a red edge uv, which contradicts that we completed Step 1, or there is another green edge that crosses e_r, which contradicts that e_r is crossed at most once; see Figure 2(b).
(iii) Face f cannot be traversed by two distinct green edges e_1 and e_2 (refer to Figure 2(c)). More precisely, if this is the case, these edges cannot cross each other and, by property (ii), each of e_1 and e_2 crosses two distinct red edges of f. Also, since each red edge is crossed at most once, e_1 and e_2 cross two disjoint pairs of red edges of f. Denote by c_1 and c'_1 (resp. c_2 and c'_2) the two crossing points of e_1 (resp. e_2) with the boundary of f. Assume that c_1, c_2, c'_2, c'_1 occur in this clockwise order on the boundary of f. This implies that, while moving clockwise on the boundary of f, there is at least one vertex u of f between c_1 and c_2, and at least one vertex v of f between c'_2 and c'_1. Hence, we can augment f with a red edge uv, which contradicts that we completed Step 1.
By properties (i), (ii), and (iii), f is traversed by exactly one green edge e; however this edge cannot leave on the same side two vertices of f that are not consecutive on its boundary, as otherwise they would have been connected in Step 1. Hence f is a quadrilateral and e splits f into two equal parts (see Figure 2(d)). We can then augment f by adding a diagonal red edge in the quadrilateral face. We repeat this procedure until Γ_r contains no face f such that $\deg_{\Gamma_r}(f) \geq 4$.

□

We now prove a tight bound on the edge density of min-1-planar graphs.

Theorem 2 *Any n-vertex min-1-planar simple graph has at most $4n-8$ edges, and this bound is tight.*

Proof: Let Γ be a min-1-planar drawing of a simple graph G with n vertices. By Lemma 1, we can augment Γ (and hence G) with new edges, in such a way that the new drawing Γ' (and the corresponding graph G') is min-1-planar and its red subgraph Γ'_r is a triangulated planar graph. Hence, Γ'_r has exactly $3n-6$ edges and $2n-4$ faces. Every green edge of G' (which is also a green edge of G) traverses at least two faces of Γ'_r. Also, since Γ' is a min-1-planar drawing and the red subgraph has only triangular faces, each face of the red subgraph is crossed by at most one green edge. Hence the number of green edges is at most $\frac{2n-4}{2} = n-2$, and therefore G' has at most

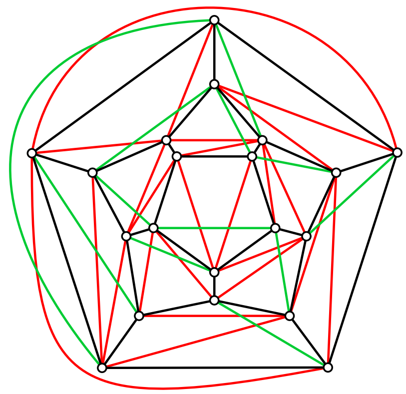


Figure 3: Construction for the lower bound of Theorem 3.

$(3n-6)+(n-2) = 4n-8$ edges in total. Since G is a subgraph of G', also G has at most $4n-8$ edges.

About the tightness of the bound, we recall that optimal 1-planar graphs with n vertices (which are also min-1-planar) have $4n-8$ edges [17, 31, 32]. □

Plugging the bound of Theorem 2 into the bound of Property 2, we immediately get that a min-1-planar simple graph has at most $\frac{4}{3}n - \frac{8}{3}$ heavy edges in any of its min-1-planar drawings. We considerably improve this bound in the next theorem.

Theorem 3 *Let G be an n-vertex min-1-planar simple graph and let Γ be a min-1-planar drawing of G. There are at most $\frac{2}{3}n - 1$ heavy edges of G in Γ. Further, there exist min-1-planar drawings that contain $\frac{2}{3}n - O(1)$ heavy edges.*

Proof: Let Γ be a min-1-planar drawing of a simple graph G with n vertices. As in the proof of Theorem 2, by Lemma 1 we can augment Γ with new red edges, in such a way that the new drawing Γ' is min-1-planar and its red subgraph Γ'_r has all faces of degree three. Hence, Γ'_r has exactly $3n-6$ edges and $2n-4$ faces. Clearly, the number of heavy edges of G in Γ' is not smaller than the number of heavy edges of G in Γ. By definition, every heavy edge of G in Γ' is crossed at least twice, hence it traverses at least three faces of Γ'_r. As before, each face of the red subgraph is crossed by at most one heavy edge. Hence the number of heavy edges is at most $\frac{2n-4}{3} \leq \frac{2}{3}n - 1$.

For the lower bound, consider a min-1-planar drawing constructed as follows. Start from a pentangulation P on n vertices, that is, an n-vertex planar drawing with all faces of degree five. Then, in each face of P, add two light edges incident to the same vertex and one heavy edge that crosses these two edges. Refer to Figure 3. By Euler's formula, P contains $\frac{2}{3}n - \frac{4}{3}$ faces, and therefore $\frac{2}{3}n - \frac{4}{3}$ heavy edges. □

3.2 Density of Min-2-planar Graphs

Proving a tight bound on the edge density of min-2-planar graphs is more challenging than for min-1-planar graphs. Observe that there are min-2-planar simple graphs with $5n-10$ edges, namely the optimal 2-planar graphs [13]. Each optimal 2-planar drawing consists of a subset of planar edges forming faces of size five (i.e., pentagons), and each face is filled up with five more edges that cross each other twice. In the following we prove that $5n-10$ is also an upper bound to the number of edges of min-2-planar graphs. To this aim, for any $k \geq 1$, we introduce a class of multi-graphs that generalize min-k-planar simple graphs.

Let G be a (multi-)graph (without self-loops) and let Γ be a (simple) drawing of G. A set of parallel edges of G between the same pair of vertices is called a *bundle* of G. We say that Γ is *bundle-proper* if for every bundle in G: (i) at most one of the edges of the bundle is involved in a crossing; and (ii) Γ has no face bounded only by two edges of the bundle (i.e., no face of Γ is a 2-real bigon). We remark that, in the literature, two parallel edges that form a face of degree two are called *homotopic*. Hence, property (ii) is equivalent to saying that a *bundle-proper* drawing does not contain homotopic parallel edges.

Graph G is *bundle-proper min-k-planar* if it admits a (simple) drawing Γ that is both min-k-planar and bundle-proper. If G has n vertices and has the maximum number of edges over all bundle-proper min-k-planar n-vertex graphs, then we say that G is a *maximally-dense* bundle-proper min-k-planar graph. Consider a pair (G, Γ), where G is an n-vertex bundle-proper min-k-planar graph and Γ is a bundle-proper min-k-planar drawing of G. We say that (G, Γ) is a *maximally-dense crossing-minimal bundle-proper min-k-planar pair* if G is maximally-dense and Γ has the minimum number of crossings over all bundle-proper min-k-planar drawings of maximally-dense bundle-proper min-k-planar n-vertex graphs.

Lemma 2 *Let (G, Γ) be a maximally-dense crossing-minimal bundle-proper min-k-planar pair. These properties hold:* (a) *If a face f of Γ contains two distinct real-vertices u and v, then f contains an edge uv.* (b) *For each face f of Γ, $\deg_\Gamma(f) \geq 3$.* (c) *A face f of Γ with $\deg^r_\Gamma(f) \geq 3$ is a 3-real triangle.*

Proof: We prove the three properties separately.

(a) Suppose for contradiction that f does not contain an edge uv. Then we can add (another copy of) edge uv to Γ (and therefore to G) in the interior of f, without introducing any additional crossings or creating a 2-real bigon. Since the resulting drawing is bundle-proper min-k-planar, this contradicts the hypothesis that G is maximally-dense.

(b) Let f be a face of Γ. Since G has no self-loops and Γ is a simple drawing, $\deg_\Gamma(f) > 1$. Also, since Γ is simple, f is neither a 0-real bigon nor a 1-real bigon. Finally, since Γ is also bundle-proper, f cannot be a 2-real bigon. It follows that $\deg_\Gamma(f) > 2$.

(c) Suppose $\deg^r_\Gamma(f) \geq 3$. If $\deg_\Gamma(f) \geq 4$ then there would be two non-consecutive real vertices on the boundary of f that are not connected by an edge, which is impossible by (a). Then f is necessarily a 3-real triangle.

□

To prove the upper bound we use *discharging* techniques. See [1, 2, 14, 21] for previous works that use this tool. Define a *charging function* $\mathrm{ch} : F(\Gamma) \to \mathbb{R}$ such that, for each $f \in F(\Gamma)$:

$$\mathrm{ch}(f) = \deg_\Gamma(f) + \deg^r_\Gamma(f) - 4 = 2\deg^r_\Gamma(f) + \deg^c_\Gamma(f) - 4 \tag{1}$$

The value $\mathrm{ch}(f)$ is called the *initial charge* of f. Using Euler's formula, it is not difficult to see that the following equality holds (refer to [2] for details):

$$\sum_{f \in F(\Gamma)} \mathrm{ch}(f) = 4n - 8 \tag{2}$$

The goal of a discharging technique is to derive from the initial charging function $\mathrm{ch}(\cdot)$ a new function $\mathrm{ch}'(\cdot)$ that satisfies two properties: (C1) $\mathrm{ch}'(f) \geq \alpha \deg^r_\Gamma(f)$, for some real number $\alpha > 0$; and (C2) $\sum_{f\in F(\Gamma)} \mathrm{ch}'(f) \leq \sum_{f\in F(\Gamma)} \mathrm{ch}(f)$.

If $\alpha > 0$ is a number for which a function $\mathrm{ch}'(\cdot)$ satisfies (C1) and (C2), by Eq. (2) we get: $4n - 8 = \sum_{f\in F(\Gamma)} \mathrm{ch}(f) \geq \sum_{f\in F(\Gamma)} \mathrm{ch}'(f) \geq \alpha \sum_{f\in F(\Gamma)} \deg^r_\Gamma(f)$. Also, since $\sum_{f\in F(\Gamma)} \deg^r_\Gamma(f) = \sum_{v\in V(G)} \deg_G(v) = 2m$, we get the following:

$$m \leq \frac{2}{\alpha}(n-2) \tag{3}$$

Thus, Eq. (3) can be exploited to prove upper bounds on the edge density of a graph for specific values of α, whenever we find a charging function $\mathrm{ch}'(\cdot)$ that fulfills (C1) and (C2). We prove the following.

Theorem 4 *Any n-vertex min-2-planar simple graph has at most $5n - 10$ edges, and this bound is tight.*

Proof: We already observed at the beginning of this section that there exist min-2-planar simple graphs with $5n-10$ edges (e.g., the optimal 2-planar). It remains to prove that min-2-planar simple graphs cannot have more than $5n-10$ edges. Since any simple graph is also a bundle-proper graph, we can show that the upper bound holds more in general for multi-graphs that are bundle-proper min-2-planar. Also, since we want to find an upper bound on the number of edges, we can restrict our attention to maximally-dense bundle-proper min-2-planar graphs, and in particular to those having the minimum number of crossings. Let (G, Γ) be any maximally-dense crossing-minimal bundle-proper min-2-planar pair, with $|V(G)| = n$. We show the existence of a charging function $\mathrm{ch}'(\cdot)$ that satisfies (C1) and (C2) for $\alpha = \frac{2}{5}$, so the result will follow from Eq. (3).

Consider the initial charging function $\mathrm{ch}(\cdot)$ defined in Eq. (1). For each type of triangle t we analyze the value of $\mathrm{ch}(t)$ and the deficit/excess w.r.t. $\frac{2}{5}\deg^r_\Gamma(t)$.

- If t is a 0-real triangle, $\mathrm{ch}(t) = -1 < 0 = \frac{2}{5}\deg^r_\Gamma(t)$, thus t has a deficit of 1.
- If t is a 1-real triangle, $\mathrm{ch}(t) = 0 < \frac{2}{5} = \frac{2}{5}\deg^r_\Gamma(t)$, thus t has a deficit of $\frac{2}{5}$.
- If t is a 2-real triangle, $\mathrm{ch}(t) = 1 > \frac{4}{5} = \frac{2}{5}\deg^r_\Gamma(t)$, thus t has an excess of $\frac{1}{5}$.
- If t is a 3-real triangle, $\mathrm{ch}(t) = 2 > \frac{6}{5} = \frac{2}{5}\deg^r_\Gamma(t)$, thus t has an excess of $\frac{4}{5}$.

Also, if f is any face of Γ with $\deg_\Gamma(f) \geq 4$, then $\mathrm{ch}(f) = 2\deg^r_\Gamma(f) + \deg^c_\Gamma(f) - 4 = \deg_\Gamma(f) - 4 + \deg^r_\Gamma(f) \geq \deg^r_\Gamma(f) \geq \frac{2}{5}\deg^r_\Gamma(f)$.

Therefore $\mathrm{ch}(\cdot)$ only fails to satisfy (C1) at 0-real and 1-real triangles. We begin by setting $\mathrm{ch}'(f) = \mathrm{ch}(f)$ for each face f of Γ and we explain how to modify $\mathrm{ch}'(\cdot)$ in such a way that $\mathrm{ch}'(f) \geq \frac{2}{5}\deg^r_\Gamma(f)$ for each face $f \in F(\Gamma)$, thus satisfying (C1), and the total charge remains the same, thus satisfying (C2).

Fixing 0-real triangles. Let t be a 0-real triangle in Γ with edges e_1, e_2, and e_3. Refer to Figure 4. The edges $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$ are three pairwise crossing edges of G. Since Γ is a simple drawing, $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$ are independent edges of G (i.e., their six end-vertices are all distinct). Also, since Γ is min-2-planar, at least two of these three edges, say $\overline{e}_2$ and $\overline{e}_3$, do not cross other edges of G in Γ. This implies that each of the two end-vertices of $\overline{e}_2$ shares a face with an end-vertex of $\overline{e}_3$. Hence, by Lemma 2(a), the four vertices of $\overline{e}_2$ and $\overline{e}_3$ form a 4-cycle $e'\overline{e}_2e''\overline{e}_3$ in G

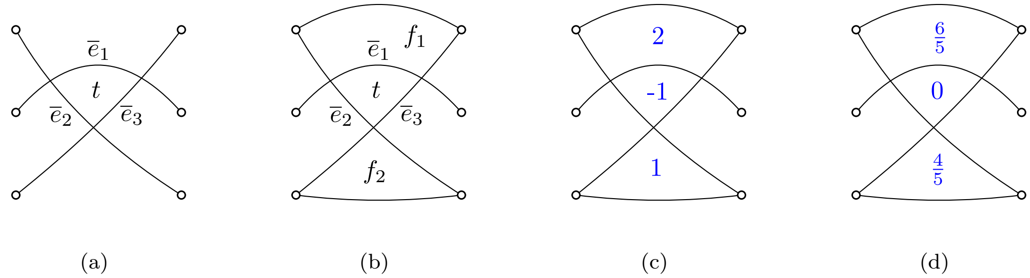


Figure 4: (a) A 0-real triangle t. (b) A 2-real quadrilateral f_1 and a 2-real triangle f_2 neighboring t. (c) The initial charges. (d) The charges after a redistribution.

and Γ contains a 2-real quadrilateral f_1 bounded by portions of e'', $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$, and a 2-real triangle f_2 bounded by portions of e', $\overline{e}_2$, $\overline{e}_3$.

The charge of f_1 is $\mathrm{ch}'(f_1) = 2$, with an excess of $\frac{6}{5}$ w.r.t. $\frac{2}{5}\deg^r_\Gamma(f_1) = \frac{4}{5}$. The charge of f_2 is $\mathrm{ch}'(f_2) = 1$, with an excess of $\frac{1}{5}$ w.r.t. $\frac{2}{5}\deg^r_\Gamma(f_2) = \frac{4}{5}$. We reduce $\mathrm{ch}'(f_1)$ by $\frac{4}{5}$, reduce $\mathrm{ch}'(f_2)$ by $\frac{1}{5}$, and increase $\mathrm{ch}'(t)$ by 1. After that, the total charge is unchanged and all the three faces t, f_1, and f_2 satisfy (C1). Namely, $\mathrm{ch}'(t) = 0$ (it has no deficit/excess), $\mathrm{ch}'(f_1) = \frac{6}{5}$ (it has an excess of $\frac{2}{5}$), and $\mathrm{ch}'(f_2) = \frac{4}{5}$ (it has no deficit/excess). In the remainder of the proof, since we need a way to keep track of the 2-real triangles and 2-real quadrilaterals whose charge has been modified as described above, we call each of the faces f_1 and f_2 a *0-real triangle-neighboring face*. Each 0-real triangle-neighboring face that is a 2-real triangle (as f_2) shares its unique crossing-vertex with a 0-real triangle; each 0-real triangle-neighboring face that is a 2-real quadrilateral (as f_1) shares its unique 0-real edge with a 0-real triangle.

Fixing 1-real triangles. Let t be a 1-real triangle, with real-vertex v_1 and crossing-vertices v_2 and v_3. Refer to Figure 5 for an illustration. Let $e_0 = v_2v_3$ be the 0-real edge of t, and let f_1 be the face of Γ that shares e_0 with t. If f_1 is a 0-real quadrilateral, denote by e_1 the 0-real edge of f_1 not adjacent to e_0, and by f_2 the face of Γ that shares e_1 with f_1. If f_2 is a 0-real quadrilateral, denote by e_2 the 0-real edge of f_2 not adjacent to e_1, and by f_3 the face of Γ that shares e_2 with f_2. We continue in this way until we encounter a face f_p ($p \geq 1$) that is not a 0-real quadrilateral. This procedure determines a sequence of faces $f_0, f_1, f_2, \dots f_p$, and a sequence of 0-real edges $e_0, e_1, \dots, e_{p-1}$ such that $f_0 = t$, f_i is a 0-real quadrilateral for each $i \in \{1, \dots, p-1\}$, f_p is not a 0-real quadrilateral, and the faces f_i and f_{i-1} share edge e_{i-1} ($i \in \{1, \dots, p\}$).

Note that $\deg_\Gamma(f_p) \geq 4$. Namely, let $e = v_1v_2$ and $e' = v_1v_3$, and let $\overline{e} = v_1u$ and $\overline{e'} = v_1w$ be the edges of G that contain e and e'. Since f_p has at least two crossing-vertices, if f_p were a triangle then it would be either a 0-real triangle or a 1-real triangle. If f_p were a 0-real triangle then $\overline{e}$ and $\overline{e'}$ would cross in Γ, which is impossible as $\overline{e}$ and $\overline{e'}$ are adjacent edges and Γ is a simple drawing. If f_p were a 1-real triangle then $u = w$, i.e., $\overline{e}$ and $\overline{e'}$ would be parallel edges both involved in a crossing, which is impossible as Γ is bundle-proper.

Therefore, $\deg_\Gamma(f_p) \geq 4$ and, as already observed at the beginning of this proof, $\mathrm{ch}'(f_p) \geq \frac{2}{5}\deg^r_\Gamma(f_p)$. Also, the charge excess of f_p is larger than $\frac{2}{5}$. Namely, the charge excess of f_p is $x = 2\deg^r_\Gamma(f_p) + \deg^c_\Gamma(f_p) - 4 - \frac{2}{5}\deg^r_\Gamma(f_p) = \deg_\Gamma(f_p) + \frac{3}{5}\deg^r_\Gamma(f_p) - 4$. If f_p has no real-vertices, then it must have at least five crossing-vertices (because f_p is not a 0-real quadrilateral), which implies $x \geq 1 > \frac{2}{5}$. If f_p has at least one real-vertex then $x \geq \frac{3}{5} > \frac{2}{5}$.

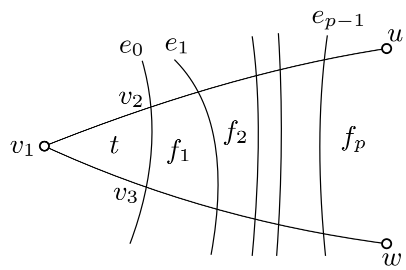


Figure 5: The demand path for a 1-real triangle t, ending at a face f_p.

Hence, since the charge excess of f_p is larger than $\frac{2}{5}$, the idea is to fill the $\frac{2}{5}$ charge deficit of t by moving an equivalent amount of charge from f_p to t. We say that t *demands from* f_p *through edge* e_{p-1} a charge amount of $\frac{2}{5}$. We call $f_0, \dots, f_p$ (which is a path in the dual of Γ) the *demand path* for t. Therefore, for each 1-real triangle t of Γ whose demand path ends at a face $f = f_p$, we decrease $\mathrm{ch}'(f)$ by $\frac{2}{5}$ and increase $\mathrm{ch}'(t)$ from 0 to $\frac{2}{5}$. Note that f cannot be a 0-real triangle-neighboring face. Indeed, f is not a triangle, and if f is a 2-real quadrilateral then its 0-real edge is shared either with a 0-real quadrilateral or directly with the 1-real triangle t. It follows that the set of faces whose charge is affected by fixing 1-real triangles does not intersect with the set of faces whose charge is affected by fixing 0-real triangles.

Due to the considerations above, after we have fixed all 1-real triangles, we may have problems only if multiple 1-real triangles demanded from the same face f. In this case, f might no longer satisfy (C1). In the remainder of the proof, we analyze which types of faces may be in this situation and, if so, we prove how to fix their charge.

Fixing faces that received multiple demands from 1-real triangles. Let f be a face of Γ of degree larger than three that received multiple demands from 1-real triangles. This is possible only if f has more than one 0-real edge, hence we can exclude that f is a 2-real quadrilateral. Note that, by Lemma 2(c), each face of Γ contains at most three real-vertices. If $\deg_\Gamma(f) \geq 7$ then f still satisfies (C1) even if it received a demand through each of its $\deg_\Gamma(f)$ edges when fixing 1-real triangles. Indeed, in the worst case, the new charge of f is $\mathrm{ch}'(f) = \deg_\Gamma(f) + \deg^r_\Gamma(f) - 4 - \frac{2}{5}\deg_\Gamma(f) = \frac{3}{5}\deg_\Gamma(f) + \deg^r_\Gamma(f) - 4 \geq \frac{2}{5}\deg^r_\Gamma(f)$ (because $\deg_\Gamma(f) \geq 7$). The same happens if $\deg_\Gamma(f) \geq 5$ and $\deg^r_\Gamma(f) \geq 1$ (i.e., f has at least one real-vertex). Indeed, in this case, the number of 0-real edges of f is at most $\deg^c_\Gamma(f) - 1 = \deg_\Gamma(f) - \deg^r_\Gamma - 1$, so f received at most this number of demands from 1-real triangles. Hence, in the worst case, the new charge of f is $\mathrm{ch}'(f) = \deg_\Gamma(f) + \deg^r_\Gamma(f) - 4 - \frac{2}{5}(\deg_\Gamma(f) - \deg^r_\Gamma(f) - 1) \geq \frac{3}{5}\deg_\Gamma(f) + \frac{7}{5}\deg^r_\Gamma(f) - \frac{18}{5} \geq \frac{7}{5}\deg^r_\Gamma(f) - \frac{3}{5} \geq \frac{2}{5}\deg^r_\Gamma(f) + \deg^r_\Gamma(f) - \frac{3}{5} \geq \frac{2}{5}\deg^r_\Gamma(f)$. It follows that the only faces that may have received multiple demands from 1-real triangles and that (after we have fixed all 1-real triangles) no longer satisfy (C1) are the 1-real quadrilaterals, the 0-real pentagons, and the 0-real hexagons. Each face f of one of these types has at least two adjacent 0-real edges. If f no longer satisfies (C1), we show how to find extra charges that can be moved from some suitable faces with charge excess towards f, so to compensate the charge deficit of f. To this aim, we first prove the following claim; refer to Figure 6.

Claim 1 *Let f be a face of Γ and let e_1, e_2, e_3, e_4 be consecutive edges on the boundary of f for which a demand is made through both e_2 and e_3. Let t_1 be the 1-real triangle that demanded from*

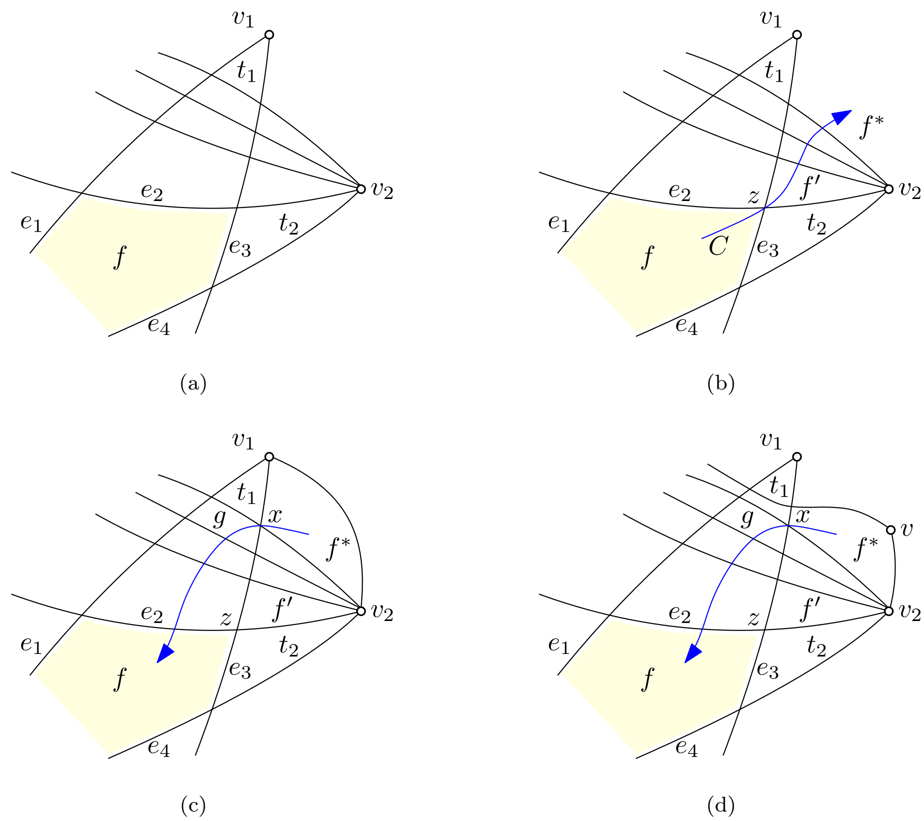


Figure 6: A supporting face f^* for a face f that receives charge demands through two consecutive 0-real edges of its boundary (e_2 and e_3 in the figure).

f through e_2 and let $v_1 = \overline{e}_1 \cap \overline{e}_3$ be the real-vertex of t_1. Analogously, let t_2 be the 1-real triangle that demanded from f through e_3 and let $v_2 = \overline{e}_2 \cap \overline{e}_4$ be the real-vertex of t_2. Then there is a curve C that begins in f, leaves f through the crossing-vertex common to e_2 and e_3, passes through a sequence of zero or more 1-real triangles and 1-real edges, and ends in a face f^* that is either a 2-real triangle containing v_1 and v_2 or a 2-real quadrilateral containing only one of v_1 and v_2.

Proof: Observe that the closed region Δ_{123} bounded by $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$ does not contain any vertex of G other than v_1 since if it did, t_1 would be making a demand from some face other than f. Similarly, the closed region Δ_{234} bounded by $\overline{e}_2$, $\overline{e}_3$, $\overline{e}_4$ contains no vertices of G.

We can construct a curve C that begins in f, passes through the crossing-vertex z of Γ common to e_2 and e_3, and then enters the face f' opposite f at z. From the interior of f' the curve C then crosses a sequence of zero or more 1-real edges incident to v_2 and passes through zero or more 1-real triangles that contain v_2 until reaching some face f^* that contains v_2 and is not a 1-real triangle. One of the following must occur:

1. The face f^* contains v_1 (see Figure 6(c)). In this case Lemma 2(a) implies that f^* also contains the 2-real edge v_1v_2. The crossing-minimality of Γ implies that f^* is a 2-real triangle that contains v_1 and v_2. (Otherwise $\overline{e}_3$ has more than one crossing on the boundary of f^*

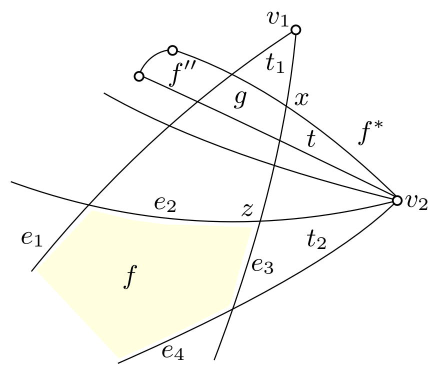


Figure 7: A supporting face f^* supports at most one face f.

and could be rerouted to avoid all but one of these crossings.)

2. The face f^* has degree larger than three (see Figure 6(d)). Then f^* contains an edge e of Γ that is not incident to v_1 and $\overline{e}$ crosses $\overline{e}_3$. No endpoint of $\overline{e}$ is in Δ_{123}, $\overline{e}$ does not cross e_2, and since Γ is simple, $\overline{e}$ has only one crossing with $\overline{e}_3$, so $\overline{e}$ must cross $\overline{e}_1$. Since e_3 already crosses e_2 and e_4, this implies that $\overline{e}$ has no additional crossings. Therefore one end-vertex v of $\overline{e}$ belongs to f^*. By Lemma 2(a), the edge v_2v is on f^*. The crossing minimality of Γ then implies that f^* is a 2-real quadrilateral that contains v_2.

This completes the proof of the claim. ■

Let f be a 1-real quadrilateral, a 0-real pentagon, or 0-real hexagon with edges e_1, e_2, e_3, e_4 that satisfy the conditions of the claim and let f^* be the face whose existence is established by the claim. In each such case, we move a charge of $\frac{1}{5}$ from f^* to f. Based on the claim, there are two cases to consider:

1. The face f^* is a 2-real triangle that contains v_1 and v_2 (see Figure 6(c)). Let x be the crossing-vertex of f^*. Then the face g that shares x with f^* but has no edge in common with f^* is either a 0-real quadrilateral or it coincides with f (because g is one of the faces of the demand path for t_1 ending at f). In particular, g is not a 0-real triangle, which implies that $\mathrm{ch}'(f^*)$ was not modified when fixing 0-real triangles. Therefore $\mathrm{ch}'(f^*) = 1$ immediately after fixing 0-real triangles. The charge on 2-real triangles is never modified when fixing 1-real triangles, hence $\mathrm{ch}'(f^*) = 1$ even after fixing all 1-triangles. Since we reduce the charge of f^* by $\frac{1}{5}$ and increase the charge of f by $\frac{1}{5}$, we can think of this charge travelling along the suffix of the demand path $t_1 \rightsquigarrow f$ that begins at g; we also say that the charge *leaks out of* f^* *through* x.

   We show that charge leaks out of f^* through x at most once. This is obviously the case if f^* and f share the vertex $x = z$ (i.e., $g = f$). The only other possibility is that the charge leaks out of f^* into the 0-real quadrilateral g that is part of another demand path $t \rightsquigarrow f''$, with $t \neq t_1$; refer to Figure 7. Let e and e' be the two edges of g other than e_1 and e_3. Then t is the 1-real triangle that contains v_2 and whose 1-real edges are portions of $\overline{e}$ and $\overline{e}'$. Each of $\overline{e}$ and $\overline{e}'$ crosses e_1 and e_3. Possibly $e_2 \in \{e, e'\}$ but we can assume without loss of generality that $e \neq e_2$. Therefore the edge $\overline{e}_3$ crosses $\overline{e}_2$, $\overline{e}_4$, and $\overline{e}$, for a total of at least 3 crossings. Hence, neither e nor e' is involved in any additional crossings, which implies that the face next to g on the demand path $t \rightsquigarrow f''$ contains end-vertices of $\overline{e}$ and $\overline{e}'$, i.e., this face coincides with f''. Hence, since Γ is simple (which excludes that $\overline{e}$ and $\overline{e}'$ cross), $\deg^r_\Gamma(f'') \geq 2$. It follows that f'' is neither a 1-real quadrilateral, nor a 0-real pentagon, nor a 0-real hexagon. Since the charge that leaks out of f^* through x is always left at a 1-real quadrilateral, or at a 0-real pentagon, or at a 0-real hexagon, we conclude that charge leaks out of f^* at x at most once.

2. The face f^* is a 2-real quadrilateral (see Figure 6(d)). In this case, f^* has only one 0-real edge, shared with a 0-real quadrilateral. Again, this implies that $\mathrm{ch}'(f^*)$ was not modified when fixing 0-real triangles. Hence, immediately after fixing 0-real triangles we have $\mathrm{ch}'(f^*) = 2$. Since we reduce $\mathrm{ch}'(f^*)$ by $\frac{1}{5}$ and increase $\mathrm{ch}'(f)$ by $\frac{1}{5}$, we can again think of this as a charge of $\frac{1}{5}$ leaking from f^* through a crossing-vertex x of f^* and then travelling down a suffix of the demand path $t_1 \rightsquigarrow f$. With the same reasoning as above, this can happen at most once for each of the two crossing-vertices of f^*. Therefore, the total charge that leaves through these two vertices of f^* is at most $\frac{2}{5}$.

To summarize the discussion above, each face f^* can give a charge of $\frac{1}{5}$ for each of its crossing-vertices, and after that it still satisfies (C1). In the following we call f^* a *supporting face*. To complete the proof, we have to show that if f is either a 1-real quadrilateral, or a 0-real pentagon, or a 0-real hexagon, and if f received multiple demands from 1-real triangles, then f always finds a suitable number of supporting faces to satisfy (C1). We analyze separately the three different types of categories for f, and assume that f received more than one demand from some 1-real triangles.

- If f is a 1-real quadrilateral, then it received exactly two demands of $\frac{2}{5}$, through its two consecutive 0-real edges. In this case, a charge of $\frac{1}{5}$ leaks into f from one supporting face. Hence we have $\mathrm{ch}'(f) \geq 1 + \frac{1}{5} - 2 \cdot \frac{2}{5} = \frac{2}{5} = \frac{2}{5}\deg^r_\Gamma(f)$, that is f satisfies (C1).

- If f is a 0-real pentagon, the following cases are possible:

  – f received exactly two demands. We have $\mathrm{ch}'(f) = 1 - 2 \cdot \frac{2}{5} = \frac{1}{5} > 0 = \frac{2}{5}\deg^r_\Gamma(f)$, thus f satisfies (C1) without needing supporting faces.

  – f received exactly three demands. Two of these demands necessarily occur through two consecutive edges of f, so a charge of at least $\frac{1}{5}$ leaks into f from a supporting face. Therefore $\mathrm{ch}'(f) = 1 + \frac{1}{5} - 3 \cdot \frac{2}{5} = 0 = \frac{2}{5}\deg^r_\Gamma(f)$, that is f satisfies (C1).

  – f received exactly four demands. There are three pairs of consecutive edges of f at which the demands occur, so a total charge of $\frac{3}{5}$ leaks into f from three supporting faces. Therefore $\mathrm{ch}'(f) = 1 + \frac{3}{5} - 4 \cdot \frac{2}{5} = 0 = \frac{2}{5}\deg^r_\Gamma(f)$, that is f satisfies (C1).

  – f received exactly five demands. There are five pairs of consecutive edges at which the demands occur, so a total charge of $\frac{5}{5} = 1$ leaks into f from five supporting faces. Therefore $\mathrm{ch}'(f) = 1 + 1 - 5 \cdot \frac{2}{5} = 0 = \frac{2}{5}\deg^r_\Gamma(f)$, that is f satisfies (C1).

- If f is a 0-real hexagon, we have two cases. If f received at most five demands, then $\mathrm{ch}'(f) \geq 2 - 5 \cdot \frac{2}{5} = 0 = \frac{2}{5}\deg^r_\Gamma(f)$. Otherwise, a total charge of at least $\frac{6}{5}$ leaks into f from six supporting faces, thus we have $\mathrm{ch}'(f) = 2 + \frac{6}{5} - 6 \cdot \frac{2}{5} = \frac{4}{5} > 0 = \frac{2}{5}\deg^r_\Gamma(f)$. Hence, in both cases f satisfies (C1).

In conclusion, at the end of the discharging process, the new function $\mathrm{ch}'(\cdot)$ satisfies (C1) for all faces of Γ, and the total charge is the same as the initial total charge, that is, $\mathrm{ch}(\cdot)$ satisfies (C2). This completes the proof. $\square$

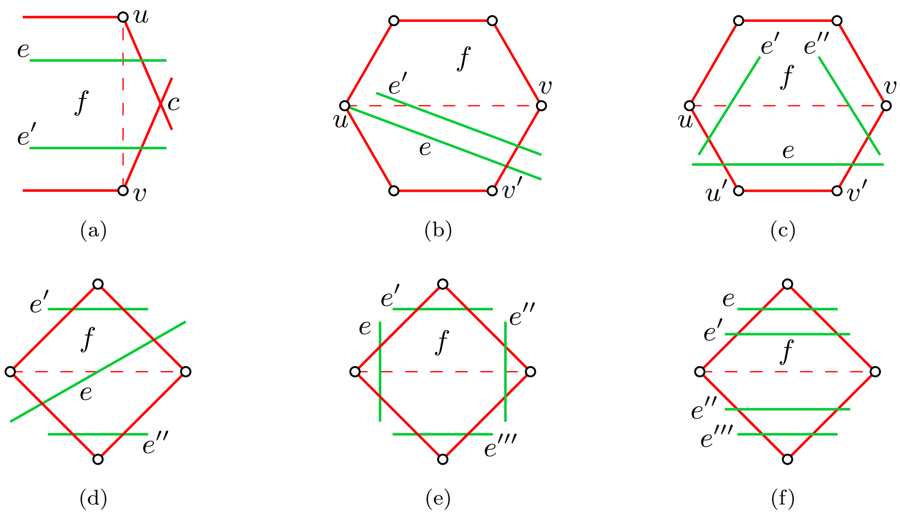


Figure 8: Illustration for the proof of Claim 2. Green edges are heavy edges, solid red edges are light edges of Γ', and dashed edges are light edges that can be added to Γ' to get Γ^+.

Combining Theorem 4 with Property 2 we immediately get that any min-2-planar drawing has at most $2n-4$ heavy edges. The next theorem considerably improves this bound by exploiting discharging techniques.

Theorem 5 *Let G be an n-vertex min-2-planar simple graph and let Γ be any min-2-planar drawing of G. There are at most $\frac{6}{5}(n-2)$ heavy edges of G in Γ. Also, for infinitely many integers $n \geq 2$, there exist min-2-planar drawings on n-vertex simple graphs with at least $n-4$ heavy edges.*

Proof: By definition, all heavy edges of G in Γ have at least three crossings with light edges. From Γ, we derive a graph G^- and a corresponding drawing Γ^- by removing all the heavy edges of G in Γ. Then we remove all light edges that have two crossings in Γ^- and we call Γ' the resulting drawing. Note that the light edges that originally crossed the heavy edges are not removed in this phase. Drawing Γ' is 1-planar. All the faces of Γ' describe cyclic sequences of real-vertices and crossing-vertices, and in each sequence we do not have two subsequent crossing-vertices, as this would mean two crossings on a light edge.

Claim 2 *Γ' can be augmented by adding light edges to get a 1-planar drawing Γ^+ whose faces are all triangles. Also, the drawing Γ^* consisting of Γ^+ plus the heavy edges of G in Γ is still min-2-planar.*

Proof: The augmentation of Γ' with the new light edges is done by considering the heavy edges of Γ. If u and v are vertices of two distinct 1-real edges with a crossing c, then we can add an edge between u and v that will have at most two crossings with heavy edges e and e' in Γ^*; see Figure 8(a). In this way, we can guarantee that the remaining faces that are not triangles have only real vertices. We now iteratively show how to split each non-triangular face f to complete the construction of Γ^+. If f is not traversed by a heavy edge, we can triangulate it instantly. If f is traversed by a heavy edge e that is incident to a vertex u of f and crosses the boundary of f

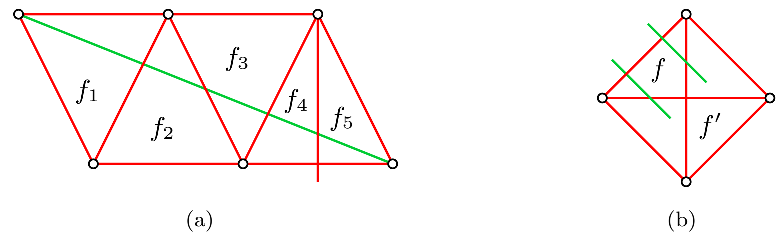


Figure 9: Heavy edges crossing 2-real and 3-real triangles. (a) f_1 and f_5 are start faces for the heavy edge e. (b) If f is traversed by two heavy edges, then f' is traversed by no heavy edges.

through the edge vv', we can add an edge uv that has at most one crossing with a heavy edge e' in Γ^*; see Figure 8(b). Note that instead of uv we can choose uv', if u and v are already adjacent. Assume now that no heavy edge e traversing f is incident to any vertex of f. Let u, u', v, v' be vertices of f and uu', vv' the edges that are crossed by e. Observe that uu' and vv' might have a common vertex (e.g., $u' \equiv v'$). Then we can add the edge uv as it has at most two crossings with heavy edges e', e'' in Γ^*; see Figure 8(c). Note again that instead of uv we may choose $u'v'$, if u and v are already adjacent. If u' and v' are also adjacent or they coincide, then $\deg_{\Gamma^+}(f) = 4$. In this case, since Γ is min-2-planar, there are at most four heavy edges traversing f and every possible configuration of these heavy edges allow adding a light edge between two vertices of f traversing at most two of these heavy edges in Γ^*; see Figures 8(d) to 8(f).

Finally, observe that the edges added in Γ' to achieve Γ^+ can only cross heavy edges of G in Γ, hence Γ^+ is 1-planar. ■

Since Γ^+ is 1-planar, it contains neither 0-real triangles nor 1-real triangles. On the other hand it may contain 2-real and 3-real triangles. Also, observe that a heavy edge in Γ^* can cross several light edges in Γ^+ (see for example Figure 9(a)). The next two claims show some properties of 2-real and 3-real triangles of Γ^+ with respect to the heavy edges reinserted in Γ^*.

Claim 3 *Let f be a 2-real triangle in Γ^+. The following properties hold: (i) If f is a start face for a heavy edge e (i.e., e is incident to a real-vertex of f), then e is the only heavy edge traversing f in Γ^*. (ii) At most two heavy edges traverse f in Γ^*. (iii) If two heavy edges traverse f, then the face f' that shares only the crossing-vertex with f is not traversed by a heavy edge in Γ^*.*

Proof: (i) This is clear as one edge of f has already two crossings and a heavy edge through the other edges of f would imply two crossing heavy edges. (ii) Because of (i), we know that this is true if a heavy edge starts at f. Otherwise, three or more traversing heavy edges would imply at least six extra crossings with the border of f and this contradicts that Γ^* is min-2-planar. (iii) Both 1-real edges of f belong to edges of the graph that also contain 1-real edges of f'. Because of (i), both heavy edges do not start at f and therefore the border of f is crossed four times. So the 1-real edges of f already have each two crossings and no heavy edge can traverse f' (see Figure 9(b)). ■

Claim 4 *Let f be a 3-real triangle in Γ^+. The following properties hold: (i) If f is a start face for a heavy edge e, then there is at most one other heavy edge traversing f in Γ^*. (ii) At most three heavy edges traverse f in Γ^*.*

Proof: (i) Let e' be the edge of f that is crossed by e. Since no two heavy edges can cross, each other heavy edge traversing f must cross e'. It follows that, since Γ^* is min-2-planar and since e' is

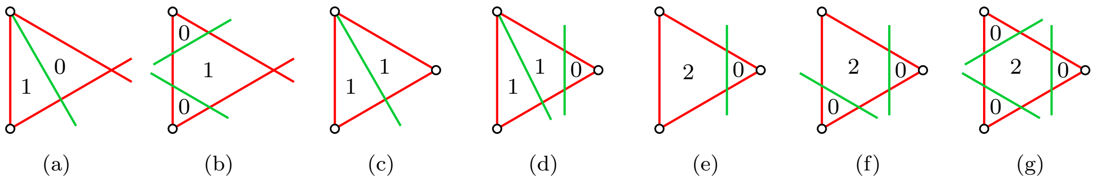


Figure 10: Illustration for the proof of Theorem 5: Different faces of Γ^*. Light edges are red and heavy edges are green. Each face is labeled with its initial charge.

a light edge, there can be at most one heavy edge other than e that traverses f. (ii) The previous case implies that at most two heavy edges can traverse f if one of them has f as a starting face. On the other hand, if a heavy edge traverses f and does not have f as a starting face, it causes two crossings along the boundary of f. Hence, since Γ^* is min-2-planar, there can be at most three heavy edges of this type. ■

Thanks to the claims above, each 2-real triangle of Γ^+ that is traversed by some heavy edges in Γ^* is partitioned into either two or three faces of Γ^*. Also each 3-real triangle of Γ^+ that is traversed by some heavy edges in Γ^* is partitioned into two, three, or four faces of Γ^*. Refer to Figure 10 for an illustration of all the cases. To count the maximum number of heavy edges in Γ^* (and therefore in Γ), we apply a variant of the discharging technique given in Theorem 4. This variant is based on assigning some of the initial charge of the faces of Γ^*, as defined in Eq. (1), to the edges of Γ^*, without changing the overall charge, which is equal to $4n-8$ based on Eq. (2). Recall that a heavy edge of G in Γ^* is partitioned into several portions (which correspond to edges of Γ^*). In particular, since a heavy edge e has at least three crossings, it is formed by at least four portions: two of these portions are 1-real edges of Γ^*, which we call *end-segments* of e, and the others are 0-real edges of Γ^*, which we call *intermediate-segments* of e. For example, the heavy edge in Figure 9(a) has two end-segments (those splitting the faces f_1 and f_5) and three intermediate-segments (those splitting the faces f_2, f_3, and f_4). For each heavy edge e, we assign charge 1 to each of its end-segments, by subtracting the same amount of charge from one of the faces of Γ^* incident to the end-segment. Note that for each end-segment, there always exists a face of Γ^* incident to it that has an initial charge equal to 1 (see Figures 10(a), 10(c) and 10(d)); hence, this subtraction of charge does not cause a negative charge in a face of Γ^*. Also, for each intermediate-segment of e, we assign to this segment a charge of $\frac{2}{3}$, by subtracting the same amount of charge from one of the faces of Γ^* incident to the intermediate-segment. Note that in all cases except for the case of 2-real triangles crossed by two heavy edges (see Figure 10(b)) each face of Γ^* has always a sufficient amount of initial charge to support all the intermediate-segments of a heavy edge that are incident to it (see Figures 10(e) to 10(g)). In the last unsolved case of a 2-real triangle f that is crossed by two heavy edges, we apply Claim 3*(iii)* to the face f and therefore use the 1 charge of the face f' which uniquely corresponds to f (see Figure 9(b)). Hence, again, this subtraction of charge does not cause a negative charge in a face of Γ^*. At the end of this assignment process, each heavy edge is assigned a charge of at least $1+1+\frac{2}{3}+\frac{2}{3}=\frac{10}{3}$ and no faces of Γ^* have a negative charge. Hence, denoted by h the number of heavy edges in Γ^*, we have $h\frac{10}{3} \leq 4n-8$, which implies $h \leq \frac{6}{5}(n-2)$.

For the lower bound, consider a planar drawing consisting of $\frac{n-2}{2}$ hexagons as shown in Figure 11(a). Without creating multiple edges, seven edges including two heavy edges can be added in each hexagon except the middle and the external face (see Figures 11(b) and 11(c)). For these two

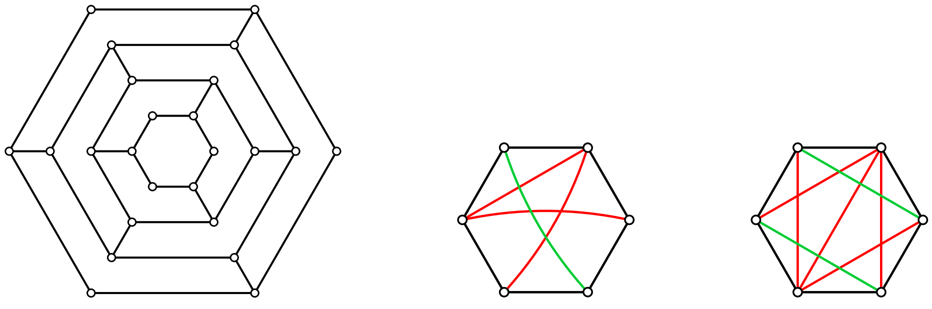


Figure 11: (a) Construction for the lower bound of Theorem 5. (b) Addition of edges in the middle and the external face and (c) in all other faces of the drawing of Figure 11(a).

hexagons we can have four edges, including one heavy edge each. Therefore the resulting drawing contains $\frac{n-2}{2} \cdot 2 - 2 = n - 4$ heavy edges. □

3.3 Density of Min-3-planar Graphs

For the family of min-3-planar graphs we consider graphs that can contain non-homotopic parallel edges. Indeed, it is known that n-vertex 3-planar graphs that are simple have at most $5.5n - 15$ edges [31], but this bound is not tight. On the other hand, a tight upper bound is known for 3-planar graphs that can contain non-homotopic multiple edges, namely $5.5n - 11$ [29]. Note that later we will present a class of min-3-planar graphs with $\frac{17}{3}(n-2)$ edges, cf. Theorem 10, which exceeds the upper bound for 3-planar graphs. Here, we give corresponding upper bounds on the edge density and on the density of the heavy edges in min-3-planar graphs. The proofs still exploit discharging techniques.

Theorem 6 *Any min-3-planar graph with n vertices has at most $6n - 12$ edges.*

Proof: As in the proof of Theorem 4 we can assume a restriction to maximally-dense crossings-minimal bundle-proper min-3-planar pairs (G, Γ) with $|V(G)| = n$ vertices, and we will use a discharging technique to prove the statement. Our discharging function $\mathrm{ch}'(\cdot)$ is similar to that in [1], where the same bound for 4-planar graphs was proven, but the details differ, as here edges with more than four crossings can exist. We can assume that Γ is 2-connected and hence that the boundary of each face f of Γ is a simple cycle. Indeed, Ackerman proved that if Γ is not 2-connected it always has no more than $6n - 12$ edges [1, Proposition 2.1]. Although Ackerman concentrates on 4-planar graphs, he does not use this hypothesis to show this fact, thus his argument works also in our case.

We introduce the discharging steps of $\mathrm{ch}'(\cdot)$ and show that all faces satisfy (C1) and (C2) for $\alpha = \frac{1}{3}$, which implies with Eq. (3) the desired bound on the edge density. This is relatively difficult to see for 0-real pentagons, so the major part of the proof is reserved for these. For that we distinguish different cases depending on the structure of the graph near a 0-real pentagon.

Before we can write down the discharging function, we introduce some definitions. Let x be a crossing-vertex in a drawing Γ. We call the faces f and f' *vertex-neighbors*, if both their boundaries contain x but not a common edge $e \in E(\Gamma)$. Recall the definition of *demand path* in the proof of Theorem 4 for 1-real triangles. We naturally extend the definition of demand path for arbitrary faces f_0 through each of their 0-real edges. If f is the end of a demand path for f_0, then we say that f and f_0 are *demand-path-neighbors* (even if no demand is made). Note that in Section 3.2 $\deg_\Gamma(f) \geq 4$ was already shown for the case that f_0 is a 1-real triangle and this holds also for 0-real triangles, still using the fact that Γ is a simple drawing and two edges cannot cross more than once.

We assign to every face $f \in F(\Gamma)$ the initial charge $\mathrm{ch}'_0(f) = \mathrm{ch}(f)$ as defined in Eq. (1) and modify the charges by the following steps $i \in \{1, 2, 3, 4\}$ to get $\mathrm{ch}'_i(f)$. The final charge is $\mathrm{ch}'(f) = \mathrm{ch}'_4(f)$. The idea is as in Section 3.2 to fix first the charge of the 0-real and then the 1-real triangles. The last step fixes 0-real pentagons, which contribute to multiple triangles in the first step.

Step 1. Every 0-real triangle receives $\frac{1}{3}$ from each of its demand-path-neighbors.

Step 2. If f is a face with positive charge that is not a 1-real quadrilateral, then f gives $\frac{1}{6}$ to each 1-real triangle that shares a 1-real edge with f. However, if f is a 2-real triangle that shares only one of its two 1-real edges with a 1-real triangle t, then f gives $\frac{1}{3}$ to t.

Step 3. Let t be a 1-real triangle with $\mathrm{ch}'_2(t) < \frac{1}{3}$ charge. Then t receives $\frac{1}{3} - \mathrm{ch}'_2(t)$ from its demand-path-neighbor.

Step 4. Every face f distributes its (positive) excess $\mathrm{ch}'_3(f) - \frac{1}{3}\deg^r_\Gamma(f)$ equally over all 0-real pentagons that are vertex-neighbors of f.

Since charge is only moved, (C2) holds for $\mathrm{ch}'(f)$. For the four steps we have:

Proposition 1 *In any maximally-dense crossing-minimal bundle-proper min-3-planar pair (G, Γ) the following holds for the above charging function $\mathrm{ch}'(f)$:*

(a) In Step 1 and Step 3 charge gets only moved through 0-real edges.

(b) In Step 2 charge gets only moved through 1-real edges.

(c) Charge gets never moved through 2-real edges.

(d) In Step 3 each 1-real triangle receives most $\frac{1}{3}$ charge.

(e) Let f be a 1-real triangle and e a 1-real edge of the boundary of f. If $\overline{e}$ has more than three crossings, then f receives at least $\frac{1}{6}$ charge in Step 2.

Proof: (a) follows by the definition of a demand-path-neighbor and (b) directly by the definition of Step 2 of $\mathrm{ch}'(\cdot)$. Because of (a) and (b) charge gets never moved through 2-real edges in Step 1-3. Since in Step 4 charge is distributed only over 0-real pentagons, no 2-real edge is involved. Claim (c) holds also for that step. Each 1-real triangle f has $\mathrm{ch}'_0(f) = 0$ and contributes no charge in Step 1-2. Thus $\frac{1}{3} - \mathrm{ch}'_2(f) \leq \frac{1}{3}$ and so (d) is true. For (e) we consider the 0-real edge e' of the boundary of the 1-real triangle f. Since $\overline{e}$ has more than three crossings, $\overline{e'}$ has at most three crossings. Therefore there is a 2-real triangle f' adjacent to f with two real-vertices. One of them is an end-point of $\overline{e'}$ and the other one is the common vertex with f. Note that by maximal edge-density the edge between the two real-vertices does exist. 2-real triangles have no 0-real edges and therefore by applying (a) we have $\mathrm{ch}'_1(f) = \mathrm{ch}'_0(f) = \frac{1}{3} > 0$. So f' contributes at least $\frac{1}{6}$ in Step 2 to f. ■

We analyze the final charges $\mathrm{ch}'(f)$ for all faces $f \in F(\Gamma)$. By Lemma 2(c) we have to consider only h-real faces for $h \leq 2$ and 3-real triangles, since there are no other faces under the assumption of maximal density. Note that a face contributes in Step 1-3 through each edge at most once. Also a face can not get a deficit in Step 4, so it is enough to show $\mathrm{ch}'_3(f) \geq \frac{1}{3}\deg^r_\Gamma(f)$. We use Proposition 1$(a)$-$(d)$ to receive the following results:

- Each 0-real triangle f has $\mathrm{ch}'_0(f) = -1$, receives $3 \cdot \frac{1}{3}$ in Step 1 and does not contribute or receive charge in Step 2-3. So $\mathrm{ch}'_3(f) = 0 = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 1-real triangle f has either already $\mathrm{ch}'_2(f) \geq \frac{1}{3}$ and then does not contribute any charge in Step 3 or otherwise, by Step 3, we have $\mathrm{ch}'_3(f) = \mathrm{ch}'_2(f) + (\frac{1}{3} - \mathrm{ch}'_2(f)) = \frac{1}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 2-real triangle f has $\mathrm{ch}'_0(f) = 1$ and contributes through one or two edges in total at most $\frac{1}{3}$ in Step 2. So $\mathrm{ch}'_3(f) \geq \frac{2}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 3-real triangle f has $\mathrm{ch}'_0(f) = 2$ and contributes or receives no charge, so $\mathrm{ch}'_3(f) = 2 > 1 = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 0-real quadrilateral f has $\mathrm{ch}'_0(f) = 0$ and contributes or receives no charge, so $\mathrm{ch}'_3(f) = 0 = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 1-real quadrilateral f has $\mathrm{ch}'_0(f) = 1$ and contributes through each 0-real edge at most once and never through a 1-real edge, so it loses in Step 1-3 in total at most $\frac{2}{3}$. Therefore $\mathrm{ch}'_3(f) \geq \frac{1}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 2-real quadrilateral f has $\mathrm{ch}'_0(f) = 2$ and contributes at most $\frac{1}{3}$ through its only 0-real edge and $2 \cdot \frac{1}{6}$ through the 1-real edges, so it loses in Step 1-3 in total at most $\frac{2}{3}$. Therefore $\mathrm{ch}'_3(f) \geq \frac{4}{3} > \frac{2}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 1-real pentagon f has $\mathrm{ch}'_0(f) = 2$ and contributes through its three 0-real edges at most $3 \cdot \frac{1}{3}$ and through its two 1-real edges at most $2 \cdot \frac{1}{6}$, so $\mathrm{ch}'_3(f) \geq \frac{2}{3} > \frac{1}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each 2-real pentagon f has $\mathrm{ch}'_0(f) = 3$ and contributes through its two 0-real edges at most $2 \cdot \frac{1}{3}$ and through its two 1-real edges at most $2 \cdot \frac{1}{6}$, so $\mathrm{ch}'_3(f) \geq 2 > \frac{2}{3} = \frac{1}{3}\deg^r_\Gamma(f)$.
- Each face f with $\deg_\Gamma(f) \geq 6$ has $\mathrm{ch}'_0(f) = \deg_\Gamma(f) - 4 + \deg^r_\Gamma(f)$ and contributes at most $\frac{1}{3}\deg_\Gamma(f)$ charge through its $\deg_\Gamma(f)$ edges. So $\mathrm{ch}'_3(f) \geq \frac{2}{3}\deg_\Gamma(f) - 4 + \deg^r_\Gamma(f) \geq \deg^r_\Gamma(f) \geq \frac{1}{3}\deg^r_\Gamma(f)$.

It only remains to show $\mathrm{ch}'_4(f) \geq 0$ for all 0-real pentagons f. For this we denote for $i \in \{0,1,2,3,4\}$ by e_i the edges of the boundary of f in clockwise order and by t_i (even if not a triangle) the demand-path-neighbor of f at e_i. Further we denote by f_i the vertex-neighbors of f at the crossing-vertex of e_i and $e_{(i+1) \bmod 5}$. We consider different cases for the demand-path-neighbors of f. Because of rotation- and mirror-symmetry we can shift or negate all indices modulo 5 without loss of generality and to ease notation we will always use fixed indices. We consider three main cases: The number of demand-path-neighbors of f that are 0-real triangles is more than one (cases 1 and 2), exactly one (cases 3 and 4), or zero (cases 5 and 6). Note that $\mathrm{ch}'_0(f) = 1$, so if f receives $\frac{2}{3}$ or loses not more than 1 charge then $\mathrm{ch}'_4(f) \geq 0$. This is for example the case if three or less demand-path-neighbors of f are 0-real or 1-real triangles, so we can assume always the opposite. We use this argument repeatedly in the proof.

Case 1. The demand-path-neighbors t_i and $t_{(i+1) \bmod 5}$ are 0-real triangles. Fix $i = 1$. Assume that $\bar{e}_0$ and $\bar{e}_3$ have exactly three crossings. Then $\bar{e}_0$ and $\bar{e}_3$ both have an end-point adjacent to f_1. If these end-points are the same, then f_1 is a 1-real quadrilateral. Otherwise, by Lemma 2(c), these end-points are connected by an edge and f_1 is a 2-real pentagon. Note that this argument will be used at various places in the proof.

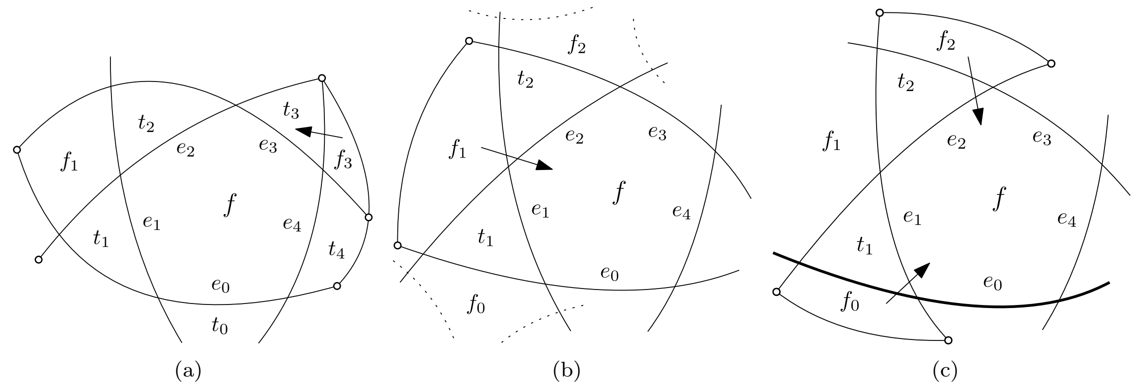


Figure 12: Case 1: If $\overline{e}_0$ and $\overline{e}_3$ have exactly three crossings, then f_1 is a (a) 1-real quadrilateral or (b) a 2-real pentagon. (c) If $\overline{e}_0$ has more than three crossings, then f receives charge from f_0 and f_2. (*Here and in the following figures, edges that exist in some subcases are dotted and edges with more than three crossings are bold.*)

- If f_1 is a 1-real quadrilateral, then t_4 is a 2-real quadrilateral and therefore $\mathrm{ch}'_4(f) \geq 0$, in case one of t_3 and t_5 is not a triangle. If both are triangles, one of them – say without loss of generality t_3 – is a 1-real triangle as otherwise $\overline{e}_1$ and $\overline{e}_2$ would have more than three crossings (see Figure 12(a)). This implies that t_3 receives $\frac{1}{3}$ from the 2-real triangle f_3 in Step 2 and so $\mathrm{ch}'_4(f) \geq 0$.

- If f_1 is a 2-real pentagon, then f_0 and f_2 can both not be 0-real pentagons, because then $\overline{e}_1$ and $\overline{e}_2$ would have more than three crossings (see Figure 12(b)). So f_1 contributes its excess of at least $\frac{4}{3}$ in Step 4 only to f and therefore $\mathrm{ch}'_4(f) \geq 0$.

Assume now that without loss of generality $\overline{e}_0$ has four or more crossings (see Figure 12(c)). Then $\overline{e}_1$ and $\overline{e}_2$ have exactly three crossings and f_0 and f_2 are 2-real quadrilaterals, which contribute each their excess of at least $\frac{2}{3}$ to at most two faces in Step 4. So f receives $2 \cdot \frac{1}{3}$ and therefore $\mathrm{ch}'_4(f) \geq 0$.

Case 2. The demand-path-neighbors t_i and $t_{(i+2) \bmod 5}$ are 0-real triangles. Fix $i = 1$. Then $\overline{e}_2$ has four or more crossings and so $\overline{e}_0$ and $\overline{e}_4$ have exactly three crossings. So f_4 is a 2-real triangle and contributes in Step 2 and Step 4 in total $\frac{1}{3}$ to f and its demand-path-neighbors. If f has four or less demand-path-neighbors that are 0-triangles or 1-triangles, then already $\mathrm{ch}'_4(f) \geq 0$. Assume that f has five such demand-path-neighbors and t_0, t_2, t_4 are 1-real triangles (see Figure 13) as otherwise we can refer to case 1.

If f_1 is a 2-real quadrilateral, then it contributes its excess of at least $\frac{2}{3}$ to f in Step 4 and so $\mathrm{ch}'_4(f) \geq 0$. If f_1 is a 1-real quadrilateral, then it has an excess of $\frac{1}{3}$, because it contributes no charge in Step 1 to f_2. This excess is contributed in Step 4 to only f, so $\mathrm{ch}'_4(f) \geq 0$. For $\deg_\Gamma(f_1) \geq 5$ the excess of f_1 after Step 3 is at least

$$\deg_\Gamma(f_1) - 4 + \frac{2}{3}\deg^r_\Gamma(f_1) - \frac{1}{3}\deg_\Gamma(f_1) + 2 \cdot \frac{1}{3} \geq \frac{2}{3}\deg_\Gamma(f_i) - \frac{8}{3},$$

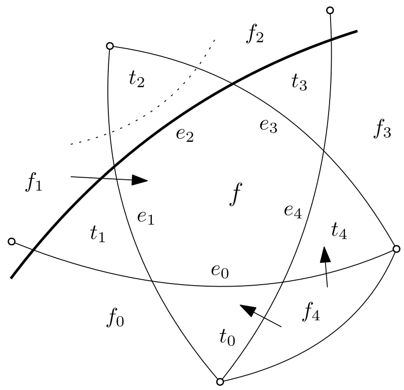


Figure 13: Case 2: If t_1 and t_3 are 0-real triangles, then f_4 is a 2-real triangle. f receives also charge from f_1.

because f_1 contributes through the 1-real edges in total at most $\frac{1}{3}$, it contributes no charge to f_2 and $\deg^r_\Gamma(f_1) \geq 1$. This is distributed over at most $\deg_\Gamma(f_1)-3$ faces as f_1 has at most $\deg_\Gamma(f_1)-1$ vertex-neighbors and does not contribute charge to f_0 and t_2. For $\deg_\Gamma(f_1) \geq 5$ the equation

$$\frac{\frac{2}{3}\deg_\Gamma(f_1) - \frac{8}{3}}{\deg_\Gamma(f_1) - 3} \geq \frac{1}{3}$$

holds. This implies $\mathrm{ch}'_4(f) \geq 0$.

Case 3. The demand-path-neighbor t_i is a 0-real triangle and the demand-path-neighbor $t_j, j \in \{(i+1) \mod 5, (i-1) \mod 5\}$ is a 1-real triangle with no 0-real quadrilaterals in its demand path. Fix $i = 1$ and $j = 2$.

- Assume $\overline{e}_0$ has more than three crossings. So $\overline{e}_2$ has exactly three crossings. Then f_2 is a 2-real triangle contributing $\frac{1}{3}$ to f and its demand-path-neighbors (see Figure 14(a)).

  So we can assume that all demand-path-neighbors of f are 0-real or 1-real triangles (otherwise $\mathrm{ch}'_4(f) \geq 0$) and t_0, t_2, t_3, t_4 are 1-real triangles (otherwise we can refer to case 1-2). Note that if the demand path of t_4 contains no 0-real quadrilateral, then f_3 is a 2-real triangle and therefore $\mathrm{ch}'_4(f) \geq 0$. If it contains one or more 0-real quadrilaterals, then t_4 receives at least $\frac{1}{6}$ by Proposition 1(e). Also f receives at least $\frac{1}{6}$ from f_0, as we can see with the following argument: If f_0 is a 1-real quadrilateral, then t_0 receives $\frac{1}{3}$ charge from a 2-real triangle in Step 2. If f_0 is a 2-real quadrilateral, then f receives the excess of f_0 in Step 4, which is at least $\frac{2}{3}$. If f_0 has more than four vertices, then it distributes not less than $\deg_\Gamma(f_0) - 3 - \frac{\deg_\Gamma(f_0)}{3} = \frac{2}{3}\deg_\Gamma(f_0) - 3$ over at most $\deg_\Gamma(f_0) - 3$ vertex-neighbors and so f receives at least $\frac{1}{6}$ charge in Step 4. So in total $\mathrm{ch}'_4(f) \geq 0$.

- Assume $\overline{e}_0$ has exactly three crossings. Then f_1 is a 2-real quadrilateral, which contributes $\frac{1}{6}$ in Step 2 to t_2 and has an excess of at least $\frac{2}{3}$ after Step 3, which is distributed over at most two faces in Step 4 (see Figure 14(b)). If there is a demand-path-neighbor of f, which is not a 0-real or 1-real triangle, then $\mathrm{ch}'_4(f) \geq 0$. So we assume that t_0, t_3, t_4 are 1-real triangles (otherwise we can refer to case 1 or 2).

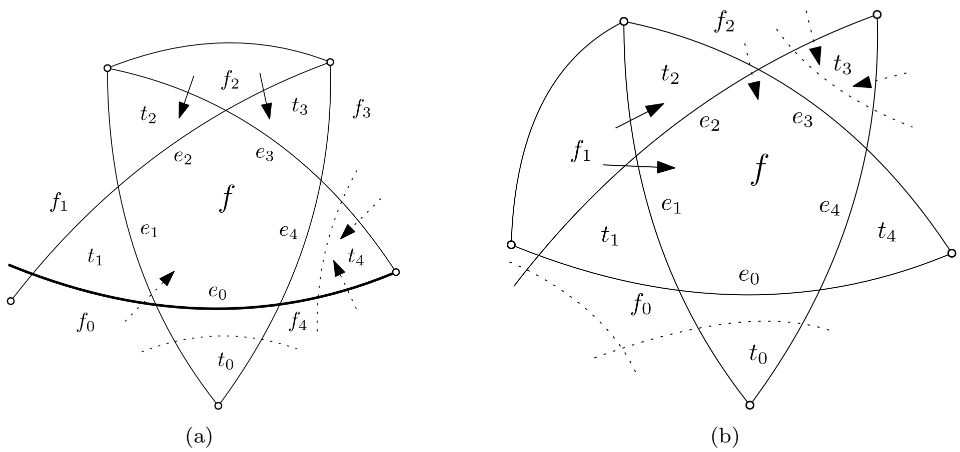


Figure 14: Case 3: (a) If $\overline{e}_0$ has more than three crossings, then f_2 contributes charge to t_2 and t_3. Also charge is moved to t_4 and from f_0 to f. (b) If $\overline{e}_0$ has exactly three crossings, then f_1 contributes charge to f and t_2. If this is not $\frac{2}{3}$, then charge is moved to t_3 or from f_2 to f.

If now f_0 is not a 0-real pentagon, then f_1 contributes its excess in Step 4 only to f and $\mathrm{ch}'_4(f) \geq 0$ follows. If f_0 is a 0-real pentagon, then $\overline{e}_2$ has more than three crossings. So if the demand path of t_3 contains one or more 0-real quadrilaterals, then t_3 receives at least $\frac{1}{6}$ in by Proposition 1(e) and $\mathrm{ch}'_4(f) \geq 0$. If the demand path of t_3 contains no 0-real quadrilaterals, then f_2 is a 2-real triangle that contributes $\frac{1}{6}$ both to t_2 and t_3 and so $\mathrm{ch}'_4(f) \geq 0$.

Case 4. The demand-path-neighbor t_i is a 0-real triangle and the demand-path-neighbor $t_j, j \in \{(i+1) \mod 5, (i-1) \mod 5\}$ is a 1-real triangle with one or more 0-real quadrilaterals in its demand path. Fix $i = 1$ and $j = 2$.

- Assume $\overline{e}_0$ has more than three crossings. So $\overline{e}_1$ and $\overline{e}_2$ have exactly three crossings. It follows that f_0 is a 2-real quadrilateral with an excess of at least $\frac{2}{3}$ after Step 3, so it contributes at least $\frac{1}{3}$ to f in Step 4 (see Figure 15(a)).

  So if four or less demand-path-neighbors of f are 0-real or 1-real triangles, then $\mathrm{ch}'_4(f) \geq 0$. Otherwise we assume that all demand-path-neighbors except t_1 are 1-real triangles as with another 0-real triangle we can refer to case 1 or 2. But then t_0 is a 1-real triangle without a 0-real quadrilateral in its demand path and we can refer to case 3.

- Assume $\overline{e}_0$ has exactly three crossings and t_0 is a 1-real triangle. If the demand path of t_0 has no 0-real quadrilaterals, then we refer to case 3. Otherwise $\overline{e}_1$ has more than three crossings and so by applying Proposition 1(e) both t_0 and t_2 receive each $\frac{1}{6}$ charge in Step 2 (see Figure 15(b)). So if not all demand-path-neighbors of f are 0-real or 1-real triangles, then $\mathrm{ch}'_4(f) \geq 0$. So assume now the opposite and that t_3 and t_4 are 1-triangles (otherwise we can refer to case 1-2). Because $\overline{e}_0$ and $\overline{e}_2$ already have three crossings, t_3 and t_4 have no 0-real quadrilaterals in their demand paths. So f_3 is a 2-real triangle contributing $\frac{1}{6}$ to each t_3 and t_4. Therefore $\mathrm{ch}'_4(f) \geq 0$.

- Assume $\overline{e}_0$ has exactly three crossings and t_0 is not a 1-real triangle. Then we can assume

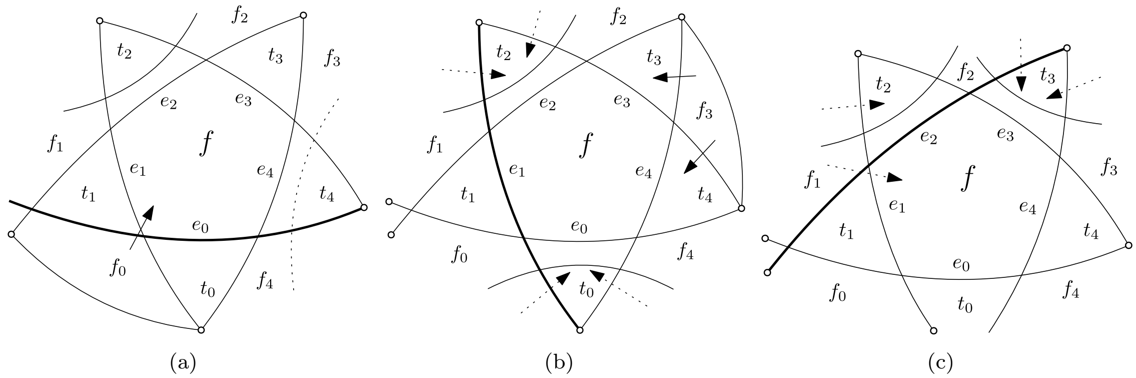


Figure 15: Case 4: (a) If $\overline{e}_0$ has more than three crossings, then f_0 contributes charge to f and we can refer to case 1-3. (b) The situation for the subcase that $\overline{e}_0$ has exactly three crossings and t_0 is a 1-real triangle. (c) Situation that $\overline{e}_0$ has exactly three crossings and t_0 is not a 1-triangle.

that t_0 is also not a 0-real triangle (otherwise we can refer to case 1) and that t_3 and t_4 are 1-real triangles (otherwise we can refer to case 2). If then the demand path of t_3 contains no 0-real quadrilateral, it follows that f_3 is a 2-real triangle contributing $\frac{1}{6}$ each to t_3 and t_4 and so $\mathrm{ch}'_4(f) \geq 0$.
If otherwise the demand path of t_3 contains a 0-real quadrilateral, by Proposition 1(e) t_3 receives $\frac{1}{6}$ charge in Step 2 (see Figure 15(c)). We now consider different cases for f_1. If f_1 is a 1-real quadrilateral, then t_2 receives $\frac{1}{3}$ from a 2-real triangle in Step 2 and so $\mathrm{ch}'_4(f) \geq 0$. If otherwise $\deg_\Gamma(f_1) \geq 5$, then the excess of f_1 after Step 3 is at least $\deg_\Gamma(f) - 3 - \frac{\deg_\Gamma(f)}{3} = \frac{2}{3}\deg_\Gamma(f) - 3$ and this is distributed in Step 4 over at most $\deg_\Gamma(f) - 3$ vertex-neighbors. So f receives at least $\frac{1}{6}$ from f_1 and so $\mathrm{ch}'_4(f_0) \geq 0$.

Case 5. Exactly four demand-path-neighbors of f are 1-real triangles and no demand-path-neighbor is a 0-real triangle. Fix the indices so that $t_i, i \in \{1,2,3,4\}$ are 1-real triangles. If the demand paths of t_2 and t_3 contain no 0-real quadrilaterals, then f_2 is a 2-real triangle contributing $\frac{1}{6}$ each to t_2 and t_3 and this implies $\mathrm{ch}'_4(f) \geq 0$. So assume without loss of generality that the demand path of t_2 contains a 0-real quadrilateral. If the demand path of t_4 contains at least one 0-real quadrilateral, then $\overline{e}_3$ has more than three crossings and it follows by Proposition 1(e) that t_2 and t_4 both receive $\frac{1}{6}$ in Step 2 and therefore $\mathrm{ch}'_4(f) \geq 0$.
If otherwise the demand path of t_4 contains no 0-real quadrilaterals, then we can assume that the demand path of t_3 does so, because otherwise f_3 is a 2-real triangle, what implies $\mathrm{ch}'_4(f) \geq 0$. If now the demand path of t_1 contains a 0-real quadrilateral we know by Proposition 1(e) that t_1 and t_3 both receive $\frac{1}{6}$ in Step 2 and therefore $\mathrm{ch}'_4(f) \geq 0$.
So assume that the demand paths of t_1 and t_4 contain no 0-real quadrilaterals (see Figure 16). If $\overline{e}_1$ has exactly three crossings, then f_0 is 2-real triangle, which contributes $\frac{1}{3}$ to t_1 and so $\mathrm{ch}'_4(f) \geq 0$. If $\overline{e}_4$ has exactly three crossings, then with the same argument f_4 contributes $\frac{1}{3}$ to t_4 and so $\mathrm{ch}'_4(f) \geq 0$. If both $\overline{e}_1$ and $\overline{e}_4$ have more than three crossings, t_2 and t_3 receive by applying Proposition 1(e) each $\frac{1}{6}$ charge in Step 3 and so $\mathrm{ch}'_4(f) \geq 0$.

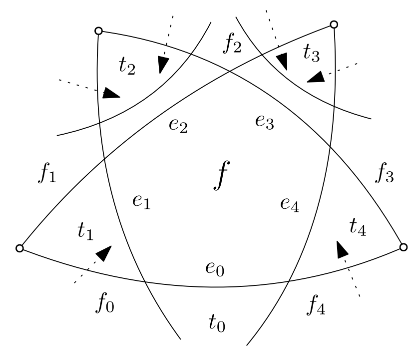


Figure 16: Case 5: The situation if only t_2 and t_3 have a 0-real quadrilateral in their demand paths.

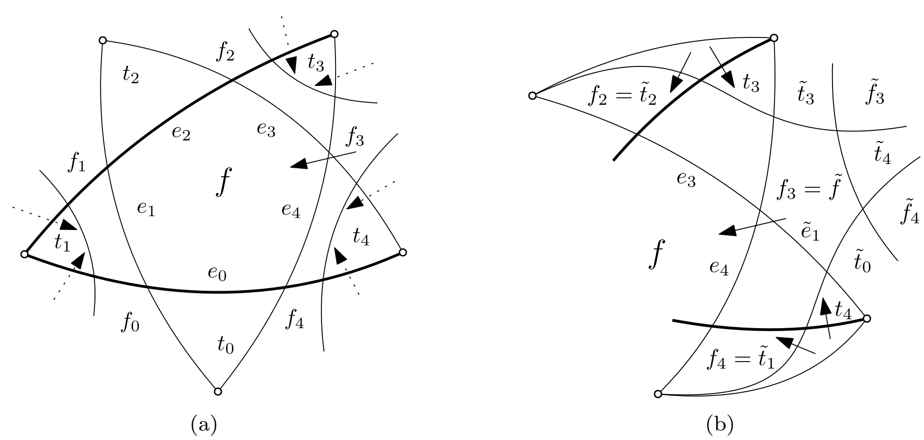


Figure 17: Case 6: (a) The situation if only t_1, t_3 and t_4 have a 0-real quadrilateral in their demand paths. (b) Focus on f_3 and its renamed neighbors with the further assumption that f_2 and f_4 are 1-real triangles. Here we show the example that f_3 is a 0-real pentagon.

Case 6. All demand-path-neighbors of f are 1-real triangles. If the demand paths of only one demand-path-neighbor t_i or two demand-path-neighbors $t_i, t_{(i+1) \mod 5}$ contain 0-real quadrilaterals, then $f_{(i+2) \mod 5}$ and $f_{(i+3) \mod 5}$ are 2-real triangles and so $\mathrm{ch}_4'(f) \geq 0$. So we assume without loss of generality that the demand paths of t_1 and t_3 contain 0-real quadrilaterals.
Then $\overline{e}_2$ has more than three crossings and both t_1 and t_3 receive $\frac{1}{6}$ in Step 2 by applying Proposition 1(e). So if the demand paths of t_0 and t_4 contain no 0-real quadrilaterals and therefore f_4 is a 2-real triangle, it follows $\mathrm{ch}_4'(f) \geq 0$. Thus assume without loss of generality that the demand path of t_4 contains a 0-real quadrilateral (see Figure 17(a)). Then $\overline{e}_0$ has more than three crossings and t_4 receives at least $\frac{1}{6}$ in Step 2 by applying Proposition 1(e).
Note that the demand paths of t_3 and t_4 can not contain more than one 0-real quadrilateral and if one of f_2 and f_4 is a 2-real quadrilateral, it contributes its excess of at least $\frac{2}{3}$ to f in Step 4 and so $\mathrm{ch}_4'(f) \geq 0$. So we can assume the opposite. We consider different cases for f_3 (see Figure 17(b) for the example f_3 is a 0-real pentagon):

- f_3 is a 0-real quadrilateral: Then f_2 and f_4 are 1-real triangles and therefore t_3 and t_4 receive both $\frac{1}{6}$ from a 2-real triangle in Step 2. Further they receive also $\frac{1}{6}$ from a 2-real quadrilateral in Step 2 and so $\text{ch}'_4(f) \geq 0$.
- f_3 is a 1-real quadrilateral: Then t_3 and t_4 receive both $\frac{1}{3}$ charge in Step 2 from a 2-real triangle and so $\text{ch}'_4(f) \geq 0$.
- $\deg_\Gamma(f_3) \geq 5$: We rename for this purpose the faces in the neighborhood of $\tilde{f} := f_3$ so that we denote for $i \in \{0, 1, \ldots, \deg_\Gamma(\tilde{f})\}$ by $\tilde{e}_i$ the edges of the boundary of $\tilde{f}$ in clockwise order and we introduce further $\tilde{e}_1$ so that $\overline{\tilde{e}}_1 = \overline{e}_3$, $\tilde{t}_1 := f_4$ and so on. The demand-path-neighbors $\tilde{t}_1$ and $\tilde{t}_2$ are 1-real triangles, which receive both $\frac{1}{6}$ from a 2-real triangle. If further $\tilde{t}_3$ is a 1-real triangle, then it receives (together with $\tilde{t}_4$, if $\tilde{t}_4$ is a 1-real triangle) $\frac{1}{3}$ charge from a 2-real triangle. The same is true for $\tilde{t}_0$ (together with $\tilde{t}_{\deg_\Gamma(f_3)-1}$). So the excess of f_3 is at least

$$\deg_\Gamma(f_3) - 4 + \frac{2}{3}\deg^r_\Gamma(f_3) - \frac{1}{3}\deg_\Gamma(f_3) + 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} \geq \frac{2}{3}\deg_\Gamma(f_3) - 3$$

  and this is distributed over at at most $\deg_\Gamma(f_3) - 4$ faces. So for $\deg_\Gamma(f_3) \geq 5$ the face f_3 contributes at least $\frac{1}{6}$ to each of its vertex-neighbors and this implies with the other charges $\text{ch}'_4(f) \geq 0$.

Summary Since we have shown for all 0-real pentagons f that we have $\text{ch}'(f) \geq 0$ in all cases, we know that (C1) holds for all faces w.r.t. $\alpha = \frac{1}{3}$. Together with Eq. (3) and the observation that (C2) is true, Theorem 6 follows. □

Theorem 7 *Let G be an n-vertex min-3-planar graph and let Γ be a min-3-planar drawing of G. There are at most $2(n-2)$ heavy edges of G in Γ. Further, there exist min-3-planar drawings that contain $\frac{6}{5}n - \frac{12}{5}$ heavy edges.*

Proof: Denote by h the number of heavy edges of G in Γ and by ℓ the number of light edges of G in Γ. To prove the upper bound, we first prove the following claim.

Claim 5 $h \leq \frac{\ell}{2}$.

Proof: The proof uses a discharging technique. Namely, assume that each light edge is initially assigned charge 1, so that the total charge equals ℓ. We show how to move some charge from light edges to heavy edges, in such a way that: (i) each light edge keeps a non-negative charge; (ii) each heavy edge receives at least 2 charge. If we are able to do that, then the claim holds, as we have $2h \leq \ell$, i.e., $h \leq \frac{\ell}{2}$.

In the first step, we observe that since light edges have at most three crossings, and heavy edges have at least four crossings (with light edges), each heavy edge may receive at least $4 \times \frac{1}{3}$ charge, namely $\frac{1}{3}$ from each crossing with the corresponding light edge.

In a second step, we assign at least $\frac{1}{3}$ charge to each of the two end segments of each heavy edge: Let e be a heavy edge (a, b) with end segments (a, x) and (y, b) where x and y denote the first and last crossings of e. We consider only the first segment, the other can be done analogously. Let $e' = (c, d)$ be the light edge that crosses e at the first crossing x. Clearly, e' might be crossed by at most two other (potentially heavy) edges. We consider the edge (a, c) that starts at a, closely follows along the heavy edge e until the crossing x and then it follows the other edge e' to the

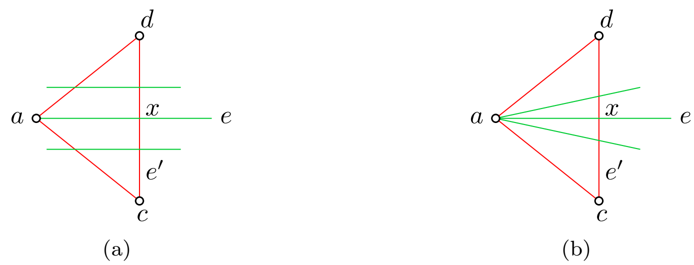


Figure 18: The situation at the end segments of heavy edges in the proof of Theorem 7. (a) In total, at most two edges cross (a, c) and (a, d). (b) If three heavy edges start at a and cross e', then (a, c) and (a, d) are planar.

vertex c. The edge (a, d) has an analogous route. Note that these two edges (a, c) and (a, d) can be crossed by at most two edges in total (see Figure 18(a)), and by the assumption of maximality they do exist.

The edges (a, c) and (a, d) will transfer half of their remaining charge to the (at most three) heavy edges that start at a, that pass through the sector defined by the two edges and that cross e'. The other half of their charge might be used for a potential sector at the other side of the edges. We will distinguish three cases, depending on the number of heavy edges that start at a and will receive charges from (a, c) and (a, d) in this step.

If there is only one such heavy edge, i.e. e, we know that there is at least $\frac{4}{3}$ charge left at the light edges (a, c) and (a, d) and half of that amount can be moved to the heavy edge e. This fulfills properties (i) and (ii).

In the case, that there are two heavy edges that start at a and have their first crossing with edge e', we have at most one other heavy edge that might cross (a, c) or (a, d). Hence those two light edges still have $\frac{5}{3}$ charge left and half of it can be assigned to the two heavy edges with the start segments between a and edge e'. In this case, both such heavy edges receive $\frac{5}{12} > \frac{1}{3}$ charge, still fulfilling (i) and (ii).

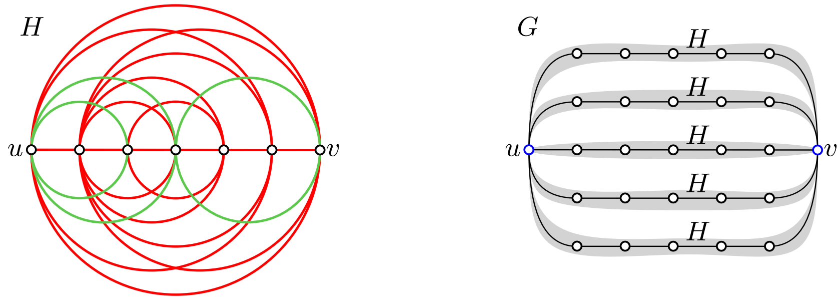
In the last case, where we have three such heavy edges, we conclude that the edges (a, c) and (a, d) are not crossed at all (see Figure 18(b)). Hence we can assign half of their charge to those three heavy edges that start at a. They all receive $\frac{1}{3}$ charge from the edges (a, c) and (a, d), as required by (i) and (ii). ■

By the claim above we have $2h \leq \ell$, and hence $2h + h \leq \ell + h$. By Theorem 6 we have that $h + \ell \leq 6n - 12$. Therefore, $3h \leq 6n - 12$, i.e., $h \leq 2n - 4 = 2(n - 2)$. This concludes the proof of the upper bound.

For the lower bound, consider the graph H in Figure 19(a), which is a min-3-planar with 7 vertices and 6 heavy edges (in green). Construct a graph G obtained by composing in parallel a number p of subgraphs isomorphic to H, with poles u and v, as shown in Figure 19(b). The graph G consists of $n = 5p + 2$ vertices and $h = 6p$ heavy edges. Hence $h = \frac{6}{5}n - \frac{12}{5}$. □

4 Relationships with k-planar Graphs

The next theorem shows that while the family of min-1-planar graphs properly contains the family of 1-planar graphs, the two classes coincide when we restrict to optimal graphs, i.e., those with



(a) (b)

Figure 19: Construction for the lower bound of Theorem 7.

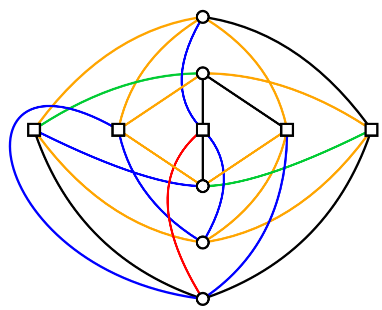


Figure 20: A min-2-planar drawing of the complete bipartite graph $K_{5,5}$.

exactly $4n-8$ edges.

Theorem 8 *1-planar graphs are a proper subset of min-1-planar graphs, while optimal min-1-planar graphs are optimal 1-planar.*

Proof: Any 1-planar graph is min-1-planar. By the NP-hardness of testing whether a given planar graph plus a single edge is 1-planar [18], we know that there are such graphs that are not 1-planar, while any planar graph that is extended by a single edge can be drawn min-1-planar. Hence, 1-planar graphs are a proper subset of min-1-planar graphs. Finally, as in the proof of Theorem 2, in every optimal min-1-planar drawing the red subgraph is maximal planar and each green edge traverses exactly two faces of the red subgraph. Hence, each green edge crosses exactly once, i.e., the drawing is also (optimal) 1-planar. □

Unlike min-1-planar graphs, we show that min-2-planar graphs are a proper superset of the 2-planar graphs even when we restrict to optimal graphs.

Theorem 9 *2-planar graphs are a proper subset of min-2-planar graphs, and there are optimal min-2-planar graphs that are not optimal 2-planar.*

Proof: We first observe that there exist non-optimal min-2-planar graphs that are not 2-planar. For example, $K_{5,5}$ is not 2-planar [6], while Figure 20 illustrates a min-2-planar drawing of $K_{5,5}$,

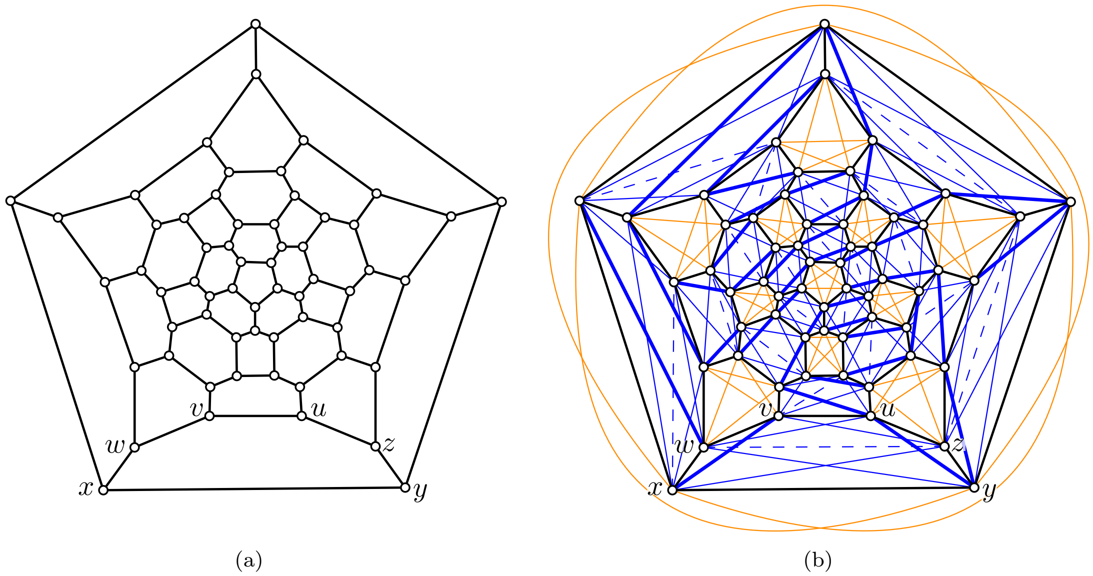
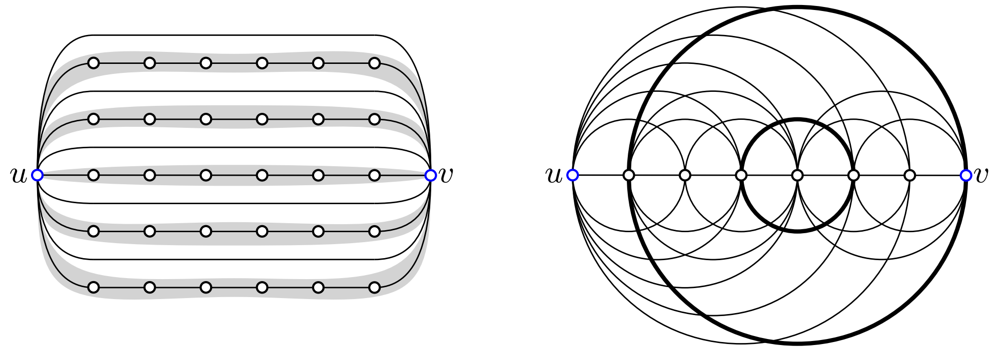


Figure 21: (a) A planar drawing Γ of the truncated icosahedral graph G. (b) A min-2-planar drawing Γ' of the graph G', obtained by adding 5 edges to each pentagonal face and 7 edges to each hexagonal face of Γ.

where black edges have no crossings, orange edges have 1 crossing, blue edges have 2 crossings, green edges have 3 crossings, and the red edge has 4 crossings. In the following, we show how to construct optimal n-vertex min-2-planar graphs that are not 2-planar. Let G be the truncated icosahedral graph and let Γ be a planar drawing of G, as depicted in Figure 21(a). This drawing has 12 pentagonal faces, 20 hexagonal faces, 60 vertices and 90 edges. We enrich Γ by adding 5 edges inside each pentagonal face and 7 edges inside each hexagonal face. Denote the obtained graph and the obtained drawing as G' and Γ', respectively. Γ' is depicted in Figure 21(b), where the edges inside pentagonal faces are colored orange and the edges inside hexagonal faces are colored blue. More precisely, for each pentagonal face we add an edge between each pair of vertices of the face that are not connected. For each hexagonal face f, we add 7 edges as follows; refer to Figure 21(b) for an illustration, where the vertices of f are denoted as u, v, w, x, y, and z . We add an edge between each pair of vertices having distance two on the boundary of f. Additionally, we arbitrarily choose two vertices having the maximum distance on the boundary of f (w and z in Figure 21(b)) and we add an edge between them, which we call the *diagonal of* f. Note that the end-vertices of the diagonal of f have degree 5 in f. All the diagonals are dashed in Figure 21(b).

Observe that: (i) G' has $n = 60$ vertices and $m = 90 + 12 \cdot 5 + 20 \cdot 7 = 290$ edges, thus $m = 5n - 10$; (ii) each edge of G' added inside a pentagonal face of Γ has two crossings in Γ'; (iii) for each hexagonal face f of Γ, the two edges that cross the diagonal (bold in Figure 21(b)) have three crossings each, while the other edges added inside f have two crossings each; (iv) no two edges with three crossings cross each other. This implies that G' is optimal min-2-planar and Γ' is a min-2-planar drawing of G'.

To show that Γ' is not optimal 2-planar, we exploit a property on the degree distribution of optimal 2-planar graphs from [22], which states that the degree of each vertex of an optimal 2-



(a) (b)

Figure 22: Illustration of the construction of Theorem 10. If in the graph of figure (a) we replace each shaded chain with a copy of the graph of figure (b), we get a min-3-planar graph that is not 3-planar. The bold edges are heavy edges.

planar graph is a multiple of three. In what follows, we show that G' contains vertices whose degree is not a multiple of three.

Each vertex of G belongs to the boundary of two hexagonal faces and one pentagonal face, and it has degree 3. We now show that each vertex of G' has degree 9, or 10, or 11, and that not all of them have degree 9. Let u be a vertex of G' whose degree is 9 (see, e.g., vertex u in Figure 21(b)). Since u belongs to the boundary of two hexagonal faces and one pentagonal face in G, and it has degree 9, it cannot be incident to any diagonals (otherwise it would have degree larger than 9). This implies that at least one of the vertices that are adjacent to u in G, call it v, is the end-vertex of at least one diagonal in G'. Two cases are possible: (a) v is the end-vertex of one diagonal; (b) v is the end-vertex of two diagonals. In case (a), v has degree 10 (see, e.g., vertex v in Figure 21(b)); in case (b), v has degree 11 (see, e.g., vertex z in Figure 21(b)). □

In contrast to 1-planar and 2-planar graphs, the maximum densities of 3-planar and min-3-planar graphs differ.

Theorem 10 *There are min-3-planar (non-simple) graphs denser than optimal 3-planar (non-simple) graphs.*

Proof: First, consider a planar graph G, and a corresponding drawing Γ, consisting of h parallel chains ($h \geq 1$), each with 8 vertices, sharing the two end-vertices u and v, and interleaved by h copies of edge uv; refer to Figure 22(a). Then, construct a new graph G', and a corresponding drawing Γ', obtained from G, and from Γ, by replacing each parallel chain with a copy of the graph G'' depicted in Figure 22(b). In the drawing Γ', each copy of G'' has the same edge crossings as the drawing illustrated in Figure 22(b). Graph G'' has 8 vertices and 33 edges, and it is min-3-planar. Indeed, only four edges in the drawing of G'' shown in the figure have more than three crossings and they do not cross each other (see the bold edges in Figure 22(b)). It follows that G' is min-3-planar; also it has $n = 6h + 2$ vertices and $m = 33h + h = 34h$ edges. Since $h = \frac{n-2}{6}$, we have $m = \frac{17}{3}n - \frac{34}{3} = 5.\overline{6}n - 11.\overline{3}$. Therefore, $m > 5.5n - 11$ for every $n > \frac{1}{2}$. Since a 3-planar graph has at most $5.5n - 11$ edges [29], G' is not 3-planar. □

5 Final Remarks and Open Problems

In this paper, we focused on simple drawings. A natural open research direction is to study min-k-planar graphs of non-simple drawings. We remark that some considerations for this setting have recently been presented by P. Hliněný [25].

About edge density, one can ask whether the bound of Theorem 6 for min-3-planar graphs is tight or if it can be further lowered. Providing finer bounds for $k \geq 4$ is also interesting. Another classical research direction is to establish inclusion or incomparability relations between min-k-planar graphs and classes of beyond-planar graphs other than k-planar graphs. In the following, we state two lemmas providing initial results in this direction.

Lemma 3 leaves open to establish the relationship between min-2-planar graphs and 1-gap-planar graphs (which have the same maximum edge density). We recall that in a k-gap-planar drawing ($k \geq 1$) it is possible to assign each crossing to one of the two edges that form it, in such a way that no more than k crossings are assigned to the same edge; see, e.g., [8, 9, 10]. Also, in a k-quasiplanar drawing ($k \geq 3$) there is no k mutually (pairwise) crossing edges; see, e.g., [2, 3, 4, 5, 23].

Lemma 3 *Min-k-planar graphs are a subset of k-gap-planar graphs and of $(k+2)$-quasiplanar graphs, for every $k \geq 1$.*

Proof: Let Γ be a min-k-planar drawing, for any given $k \geq 1$. Each crossing in Γ involves at least one light edge, hence the set of light edges covers all crossings in Γ. Consider any light edge e and assign each crossing of e to e. Then, consider a second light edge e' and assign all unassigned crossings of e' to e'. Iterate this procedure until all crossings have been assigned to some light edge; refer to Figure 23(a) for an example. Since each light edge has at most k crossings, no more than k crossings are assigned to a single edge. Hence Γ is k-gap-planar.

We now prove that Γ is also $(k+2)$-quasi planar. Suppose by contradiction that this is not the case. This means that Γ contains $k+2$ mutually crossing edges. Since no two heavy edges cross, at least $k+1 \geq 2$ of these edges are light edges. But each of them cross $k+1$ times, a contradiction.□

Lemma 4 implies that min-2-planar graphs and fan-planar graphs are incomparable classes, even if they have the same maximum edge density. We recall that in a fan-planar drawing there cannot be two independent edges that cross a third one; see, e.g., [11, 12, 15, 16, 27].

Lemma 4 *For any given $k \geq 2$, fan-planar and min-k-planar graphs are incomparable, i.e., each of the two classes contains graphs that are not in the other.*

Proof: The existence of min-k-planar graphs that are not fan-planar is an immediate consequence of the fact that there exist 2-planar graphs that are not fan-planar [15]. To show the existence of fan-planar graphs that are not min-k-planar, consider the graph $K_{1,3,h}$, for any $h \geq 1$, and let $n = 4+h$ be its number of vertices. It is easy to see that this graph is fan-planar (see, e.g., [15] and Figure 23(b)). Also, it is known that any drawing of $K_{1,3,h}$ has $\Omega(h^2) = \Omega(n^2)$ crossings [7]. On the other hand, by Theorem 1, any min-k-planar drawing with n vertices has at most $c\sqrt{k} \cdot n$ edges (for a constant c) and therefore, by Property 1, it has at most $ck^{1.5}n$ crossings. Hence, $K_{1,3,h}$ is not min-k-planar for sufficiently large values of n. □

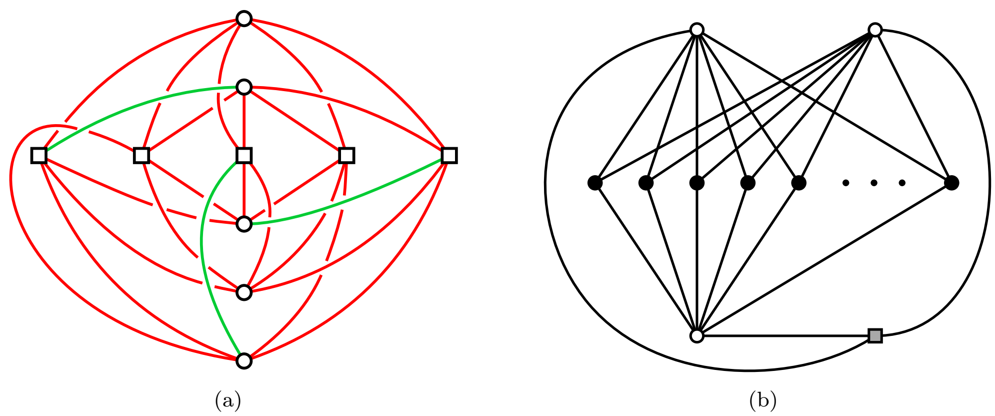


Figure 23: (a) A 2-gap-planar drawing of the min-2-planar drawing of $K_{5,5}$ shown in Figure 20; heavy edges are colored green; each crossing is assigned to a light edge (red edge), and it is represented as a small gap. (b) A fan-planar drawing of $K_{1,3,h}$, which is not min-k-planar as shown in the proof of Lemma 4.

References

[1] E. Ackerman. On topological graphs with at most four crossings per edge. *Comput. Geom.*, 85, 2019. doi:10.1016/j.comgeo.2019.101574.

[2] E. Ackerman and G. Tardos. On the maximum number of edges in quasi-planar graphs. *J. Comb. Theory, Ser. A*, 114(3):563–571, 2007. doi:10.1016/j.jcta.2006.08.002.

[3] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. In F. Brandenburg, editor, *Graph Drawing, GD '95*, volume 1027 of *LNCS*, pages 1–7. Springer, 1995. doi:10.1007/BFb0021784.

[4] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997. doi:10.1007/BF01196127.

[5] P. Angelini, M. A. Bekos, F. J. Brandenburg, G. Da Lozzo, G. Di Battista, W. Didimo, M. Hoffmann, G. Liotta, F. Montecchiani, I. Rutter, and C. D. Tóth. Simple k-planar graphs are simple $(k+1)$-quasiplanar. *J. Comb. Theory, Ser. B*, 142:1–35, 2020. doi:10.1016/j.jctb.2019.08.006.

[6] P. Angelini, M. A. Bekos, Michael Kaufmann, and T. Schneck. Efficient generation of different topological representations of graphs beyond-planarity. *J. Graph Algorithms Appl.*, 24(4):573–601, 2020. doi:10.7155/jgaa.00531.

[7] K. Asano. The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$. *J. Graph Theory*, 10(1):1–8, 1986. doi:10.1002/jgt.3190100102.

[8] C. Bachmaier, I. Rutter, and P. Stumpf. 1-gap planarity of complete bipartite graphs. In T. C. Biedl and A. Kerren, editors, *Graph Drawing and Network Visualization, GD 2018*, volume 11282 of *LNCS*, pages 646–648. Springer, 2018.

[9] S. W. Bae, J. Baffier, J. Chun, P. Eades, K. Eickmeyer, L. Grilli, S. Hong, M. Korman, F. Montecchiani, I. Rutter, and C. D. Tóth. Gap-planar graphs. In F. Frati and K. Ma, editors, *Graph Drawing and Network Visualization, GD 2017*, volume 10692 of *LNCS*, pages 531–545. Springer, 2017. doi:10.1007/978-3-319-73915-1_41.

[10] S. W. Bae, J. Baffier, J. Chun, P. Eades, K. Eickmeyer, L. Grilli, S. Hong, M. Korman, F. Montecchiani, I. Rutter, and C. D. Tóth. Gap-planar graphs. *Theor. Comput. Sci.*, 745:36–52, 2018. doi:10.1016/j.tcs.2018.05.029.

[11] M. A. Bekos, S. Cornelsen, L. Grilli, S. Hong, and M. Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. In C. A. Duncan and A. Symvonis, editors, *Graph Drawing, GD 2014*, volume 8871 of *LNCS*, pages 198–209. Springer, 2014. doi:10.1007/978-3-662-45803-7_17.

[12] M. A. Bekos, S. Cornelsen, L. Grilli, S. Hong, and Michael Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. *Algorithmica*, 79(2):401–427, 2017. doi:10.1007/s00453-016-0200-5.

[13] M. A. Bekos, Michael Kaufmann, and C. N. Raftopoulou. On optimal 2- and 3-planar graphs. In B. Aronov and M. J. Katz, editors, *SoCG*, volume 77 of *LIPIcs*, pages 16:1–16:16. Schloss Dagstuhl, 2017. doi:10.4230/LIPIcs.SoCG.2017.16.

[14] C. Binucci, G. Di Battista, W. Didimo, S. Hong, M. Kaufmann, G. Liotta, P. Morin, and A. Tappini. Nonplanar graph drawings with k vertices per face. In D. Paulusma and B. Ries, editors, *Graph-Theoretic Concepts in Computer Science - 49th International Workshop, WG 2023*, volume 14093 of *LNCS*, pages 86–100. Springer, 2023. doi:10.1007/978-3-031-43380-1_7.

[15] C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, A. Symvonis, and I. G. Tollis. Fan-planarity: Properties and complexity. *Theor. Comput. Sci.*, 589:76–86, 2015. doi:10.1016/j.tcs.2015.04.020.

[16] C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, and I. G. Tollis. Fan-planar graphs: Combinatorial properties and complexity results. In C. A. Duncan and A. Symvonis, editors, *Graph Drawing, GD 2014*, volume 8871 of *LNCS*, pages 186–197. Springer, 2014. doi:10.1007/978-3-662-45803-7_16.

[17] R. Bodendiek, H. Schumacher, and K. Wagner. Über 1-optimale graphen. *Mathematische Nachrichten*, 117:323–339, 1984.

[18] S. Cabello and B. Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013. doi:10.1137/120872310.

[19] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs*. Prentice-Hall, 1999.

[20] W. Didimo, G. Liotta, and F. Montecchiani. A survey on graph drawing beyond planarity. *ACM Comput. Surv.*, 52(1):4:1–4:37, 2019. doi:10.1145/3301281.

[21] V. Dujmovic, J. Gudmundsson, P. Morin, and T. Wolle. Notes on large angle crossing graphs. *Chicago J. Theor. Comput. Sci.*, 2011, 2011. doi:10.4086/cjtcs.2011.004.

[22] H. Förster, M. Kaufmann, and C. N. Raftopoulou. Recognizing and embedding simple optimal 2-planar graphs. In H. C. Purchase and I. Rutter, editors, *Graph Drawing and Network Visualization, GD 2021*, volume 12868 of *LNCS*, pages 87–100. Springer, 2021. doi:10.1007/978-3-030-92931-2_6.

[23] J. Fox, J. Pach, and A. Suk. The number of edges in k-quasi-planar graphs. *SIAM J. Discrete Math.*, 27(1):550–561, 2013. doi:10.1137/110858586.

[24] A. Grigoriev and H. L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007. doi:10.1007/s00453-007-0010-x.

[25] P. Hlinený. Note on min-k-planar drawings of graphs. *CoRR*, abs/2401.11610, 2024. arXiv:2401.11610, doi:10.48550/ARXIV.2401.11610.

[26] S. Hong and T. Tokuyama, editors. *Beyond Planar Graphs*. Springer, 2020. doi:10.1007/978-981-15-6533-5.

[27] M. Kaufmann and T. Ueckerdt. The density of fan-planar graphs. *Electron. J. Comb.*, 29(1), 2022. doi:10.37236/10521.

[28] S. G. Kobourov, G. Liotta, and F. Montecchiani. An annotated bibliography on 1-planarity. *Comput. Sci. Rev.*, 25:49–67, 2017. doi:10.1016/j.cosrev.2017.06.002.

[29] J. Pach, R. Radoicic, G. Tardos, and G. Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete & Computational Geometry*, 36(4):527–552, 2006. doi:10.1007/s00454-006-1264-9.

[30] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. In S. C. North, editor, *Graph Drawing, GD '96*, volume 1190 of *LNCS*, pages 345–354. Springer, 1996. doi:10.1007/3-540-62495-3_59.

[31] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997. doi:10.1007/BF01215922.

[32] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. Math. Sem. Univ. Hamb.*, 29:107–117, 1965.

[33] D. R. Wood and J. A. Telle. Planar decompositions and the crossing number of graphs with an excluded minor. In M. Kaufmann and D. Wagner, editors, *Graph Drawing, GD 2006*, volume 4372 of *LNCS*, pages 150–161. Springer, 2006. doi:10.1007/978-3-540-70904-6_16.

[34] D. R. Wood and J. A. Telle. Planar decompositions and the crossing number of graphs with an excluded minor. *New York Journal of Mathematics*, 13:117–146, 2007. URL: https://nyjm.albany.edu/j/2007/13-8.pdf.