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Empty Triangles in Generalized Twisted Drawings of K_n

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Abstract. Simple drawings are drawings of graphs in the plane such that vertices are distinct points, edges are Jordan arcs connecting their endpoints, and edges intersect at most once (either in a proper crossing or in a shared endpoint). Simple drawings are generalized twisted if there is a point O such that every ray emanating from O crosses every edge of the drawing at most once, and there is a ray emanating from O which crosses every edge exactly once. We show that all generalized twisted drawings of K_n contain exactly $2n-4$ empty triangles, by this making a substantial step towards proving the conjecture that any simple drawing of K_n contains at least $2n-4$ empty triangles.

1 Introduction

Simple drawings are drawings of graphs in the plane such that vertices are distinct points, edges are Jordan arcs connecting their endpoints and not passing through any point corresponding to another vertex, and edges intersect at most once, either in a proper (that is, transversal) crossing or in a shared endpoint. Moreover, we assume for convenience that no three edges cross at a common point, however, all statements in this work also hold without this additional assumption. The edges

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and vertices of a drawing partition the plane minus those objects into open connected regions, which are called the *cells* of the drawing. When the edges are straight-line segments, the drawings are usually called straight-line drawings.

Two simple drawings D and D' are *strongly isomorphic* if there is a homeomorphism of the sphere such that D is mapped to D' (strictly speaking, a projection of the plane to the sphere followed by a homeomorphism and a projection of the sphere to the plane). Two simple drawings D and D' are *weakly isomorphic* if there is a bijection between the vertices of D and D' such that a pair of edges in D crosses exactly when the corresponding pair of edges in D' crosses.

A *triangle* in a simple drawing D is a subdrawing of D which is a drawing of K_3, the complete graph on three vertices. By the definition of simple drawings, any triangle is crossing free and thus splits the plane (or the sphere) into two connected regions. We call those open regions the *sides* of the triangle. If one side of a triangle does not contain any vertices of D, that side is called an *empty side* of the triangle, and the triangle is called *empty triangle*. Note that empty (sides of) triangles might be intersected by edges. We observe that simple drawings of K_3 consist of exactly one triangle, which has two empty sides. Triangles in simple drawings of graphs with $n \geq 4$ vertices have at most one empty side.

In this work, we consider the number of empty triangles in simple drawings of graphs. We focus on drawings of K_n, the complete graph on n vertices, since in general, a drawing of a graph might not contain triangles, even if the graph is dense, like for example the complete bipartite graph. In particular, we study the number of empty triangles in a special family of such drawings, which we call the family of generalized twisted drawings.

The number of empty triangles in any simple drawing of K_n is at most $\binom{n}{3}$ because every such drawing contains exactly $\binom{n}{3}$ triangles (not all of which have to be empty). This upper bound is tight, as there are simple drawings of K_n where every triangle is empty, for example, straight-line drawings with vertices in convex position.

Much more interesting and challenging is to find a lower bound on the number of empty triangles in a simple drawing of K_n. This has been studied quite intensively before and is also the subject of this paper.

For straight-line drawings, it is easy to see that every such drawing of K_n contains $\Omega(n^2)$ empty triangles (since every edge is on the boundary of at least one empty triangle and every triangle has exactly three edges). Further, motivated by questions of Erdős on the existence and number of (empty) convex polygons in point sets, there has been a large amount of research on the number of empty triangles in straight-line drawings of K_n. The currently best known bounds are that every such drawing contains at least $n^2 + \Omega(n \log^{2/3} n)$ empty triangles [3] and there exist straight-line drawings with at most $1.6196n^2 + o(n^2)$ empty triangles [8].

A drawing of a graph is *pseudolinear* if the edges can be extended to an arrangement of pseudolines, where a pseudoline is a homeomorphic image of the real line in the plane so that its complement is disconnected, and an arrangement of pseudolines is a set of pseudolines in which every two cross exactly once. Any pseudolinear drawing of K_n has at least $n^2 + O(n \log n)$ empty triangles, as shown by Arroyo, McQuillan, Richter, and Salazar [6]. In the same work, the authors also considered simple drawings of K_n with the following property: Every triangle has a side S such that for any two vertices in the closure $\bar{S}$ of S, the edge between them is also contained in $\bar{S}$. They showed that any drawing of K_n that fulfills this property[1] has at least $\frac{n^2}{3} + O(n)$ empty triangles.

While in the previous cases there is always at least a quadratic number of empty triangles, the situation changes drastically for general simple drawings of K_n. Harborth [11] showed in 1989 that there are simple drawings of K_n that contain only $2n - 4$ empty triangles (see Figure 1 for an

[1] In [6], these drawings are called convex drawings and pseudolinear drawings are also called f-convex drawings.

example). Note that this especially implies that, in contrast to straight-line drawings, most edges in these drawings are not incident to any empty triangles. Moreover, it can be easily checked that all empty triangles in Harborth's drawings are incident to one of two edges.

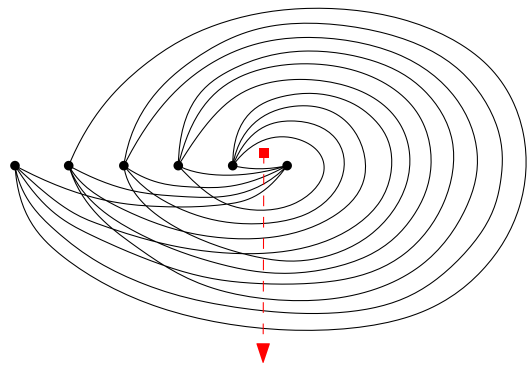


Figure 1: A twisted drawing of K_6. The red dashed arrow indicates a ray crossing every edge exactly once. Further, any ray starting at the red square crosses every edge at most once.

On the other hand, Harborth observed that every vertex in those drawings is incident to at least two empty triangles, a property which he conjectured to be true for any simple drawing of K_n. In 2013, Fulek and Ruiz-Vargas [10, 15][2] proved this conjecture to be true, by this showing that every simple drawing of K_n contains at least $\frac{2n}{3}$ empty triangles.

The currently best lower bound on the number of empty triangles in simple drawings of K_n is n [5]. The authors of [5] conjectured that Harborth's upper bound of $2n-4$ should actually be the true lower bound for $n \geq 4$. This conjecture has been confirmed via computations for all simple drawings of K_n of small cardinality, namely, for all such drawings with $4 \leq n \leq 9$ [2].

Conjecture 1 ([5]) *For any $n \geq 4$, every simple drawing of K_n contains at least $2n-4$ empty triangles.*

The drawings that Harborth used for his upper bound were later called *twisted drawings* by Pach, Solymosi, and Tóth [14], who defined them as follows. A simple drawing is *twisted* if there is a labeling $v_1, v_2, ..., v_n$ of the vertices such that v_iv_j $(i < j)$ crosses v_kv_l $(k < l)$ if and only if $i < k < l < j$ or $k < i < j < l$. Figure 1 shows a classic way of representing a twisted drawing: The vertices are all placed on a line (and labeled from left to right). Edges v_iv_n are drawn below that line; edges v_iv_j with $i < j < n$ start at v_i below the line, surround all the vertices to the right, and enter v_j from above the line.

Pach, Solymosi, and Tóth [14] showed that every simple drawing of K_n contains an induced subdrawing on $c \log^{\frac{1}{8}}(n)$ vertices (for some constant c) which is either a twisted drawing or a *convex* drawing. A simple drawing is called *convex* in this context[3] if there is a labeling $v_1, v_2, ..., v_n$ of the vertices such that v_iv_j $(i < j)$ crosses v_kv_l $(k < l)$ if and only if $i < k < j < l$ or $k < i < l < j$. Note that, up to weak isomorphism, there is exactly one convex and one twisted drawing of K_n, as all crossings are determined by the definitions. The result shown in [14] has recently been improved

[2]The results appeared 2013 as a joint publication of Fulek and Ruiz-Vargas in [10] combined with work on disjoint edges; and in 2015 as a publication by Ruiz-Vargas in [15] with a focus on empty triangles.

[3]Note that definition of convex drawing from [14] is different from the one given in [6]. The class of drawings implied by the latter definition includes the ones by the former definition.

by Suk and Zeng [16], who showed that every simple drawing of K_n contains a subdrawing which is either a twisted or a convex drawing of $K_{n'}$, where $n' = (\log n)^{\frac{1}{4}-o(1)}$. Further, it has been generalized to the setting of (abstract) rotation systems by Arroyo, Richter, Salazar, and Sullivan [7]. We remark that there are several more results in the context of twisted drawings, see [1, 9, 12, 13] and references therein.

A generalization of twisted drawings was introduced in [4] as a special type of c-monotone drawings. A simple drawing D in the plane is *c-monotone* if there is a point O of the plane such that any ray emanating from O intersects any edge of D at most once. A c-monotone drawing D is *generalized twisted* if there exists a ray r emanating from O that intersects every edge of D (see Figure 1 and Figure 2(c) for examples). In [4], generalized twisted drawings are used to improve the lower bound on the number of disjoint edges and the lower bound on the length of the longest plain path in simple drawings of K_n.

The name "generalized twisted" comes from the fact that each twisted drawing admits a generalized twisted representation. For instance, in the twisted drawing shown in Figure 1 the red square in the middle of the drawing satisfies the properties required for the point O in the definition of generalized twisted. Moreover, while for any n, there is only one twisted drawing of K_n up to weak isomorphism, there are many generalized twisted drawings of K_n for $n \geq 6$. Up to weak isomorphism, Figure 2(a) shows the only generalized twisted drawing of K_5 (which is also twisted) and Figures 2(b)-(d) show all generalized twisted drawings of K_6 (Figure 2(b) is the twisted drawing among them).

As twisted drawings and the upper bound obtained by them are crucial in the study of empty triangles, it is natural to ask about the number of empty triangles in their generalization. The initial goal of this work was to prove Conjecture 1 for generalized twisted drawings of K_n. One might expect that two generalized twisted drawings of K_n have a different number of empty triangles. However, we show that, surprisingly, the conjectured bound is tight for all generalized twisted drawings of K_n.

Theorem 1 *For any $n \geq 4$, every generalized twisted drawing of K_n contains exactly $2n-4$ empty triangles.*

In Section 2, we introduce some properties of generalized twisted drawings and empty triangles in simple drawings. Then, in Section 3, we show several results about empty triangles in generalized twisted drawings, which we finally put together to obtain a proof of Theorem 1. We conclude the paper with some open problems in Section 4.

2 Preliminaries

It is well known that all weakly isomorphic drawings of K_n have the same empty triangles. One way to see this is the following: Consider any triangle Δ in a simple drawing of K_n and any pair of vertices a and b that are not vertices of Δ. The vertices a and b lie on the same side of Δ if and only if the edge (a, b) crosses the boundary of Δ an even number of times. Thus, Δ is empty if and only if every edge between two vertices that are not vertices of Δ crosses Δ an even number of times. Therefore, drawings with the same pairs of crossing edges also have the same empty triangles. As a consequence, the number of empty triangles of a generalized twisted drawing D of K_n is the same as the number of empty triangles of any simple drawing of K_n that is weakly isomorphic to D. To prioritize readability, several of our figures show drawings that are weakly isomorphic to generalized twisted (sub-)drawings rather than a generalized twisted drawing.

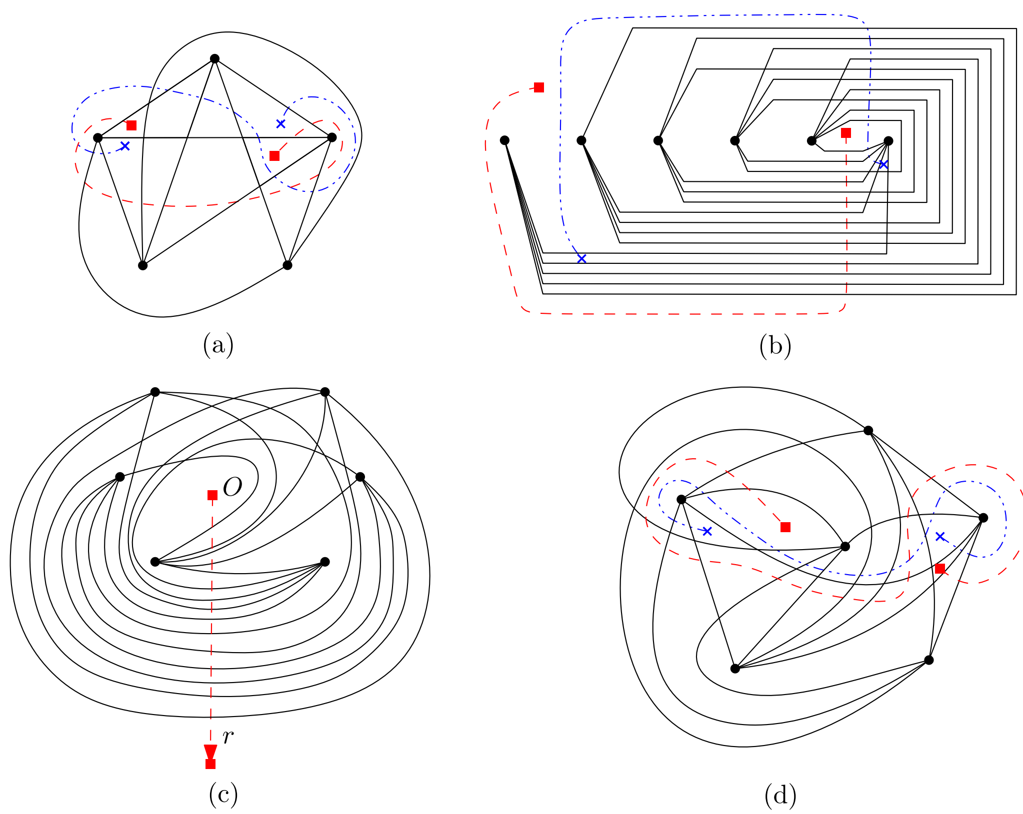


Figure 2: All (up to weak isomorphism) generalized twisted drawings of K_5 and K_6 [4]. O and Z have to lie in cells marked with red squares or in cells with blue crosses. One possible curve OZ for each cell pair that could contain O and Z is drawn dashed or dash-dotted (and as ray in c).

We start by recalling some of the definitions and results from [4], which we will need later on. Given a simple drawing D of K_n, two cells of D are called *antipodal* if for each triangle of D, they lie on different sides. A cell with at least one vertex on its boundary is called a *vertex-incident-cell* or, for short, a *vi-cell*.

For a generalized twisted drawing D of K_n, we can put a point Z into the unbounded cell of D, on the ray r that crosses every edge. Thus, in a generalized twisted drawing of K_n there is always a curve OZ crossing every edge once. One can prove [4] that the cells containing O and Z are antipodal vi-cells (see Figure 2(c)). We denote the vi-cells containing O and Z by C_O and C_Z, respectively.

Theorem 2 **([4])** *Let D be a generalized twisted drawing of K_n and let Z be a point on the ray that crosses every edge such that the curve OZ crosses every edge once. Then C_O and C_Z are antipodal vi-cells.*

Theorem 2 implies that, for every drawing D that is weakly isomorphic to a generalized twisted drawing, there exists a simple curve OZ crossing every edge once and two antipodal vi-cells C_O and C_Z. The converse is also true, as the following theorem proven in [4] implies.

Theorem 3 **([4])** *Let D be a simple drawing of K_n. Then the following properties are equivalent.*

1. *D is weakly isomorphic to a generalized twisted drawing.*
2. *D contains two antipodal vi-cells.*
3. *D can be extended by a simple curve c such that c crosses every edge of D exactly once and does not pass through any vertex.*

We remark that, given a simple drawing D of K_n, there might be several antipodal vi-cell pairs that could be used as C_O and C_Z such that D is weakly isomorphic to a generalized twisted drawing. For instance, Figure 2 shows all generalized twisted drawings of K_5 and K_6 up to weak isomorphism [4], together with all possible antipodal vi-cell pairs C_O and C_Z, and some curve OZ for each pair.

The following lemma summarizes some properties of generalized twisted drawings that we will use later. Parts of them have also implicitly been shown in [4]. For the sake of completeness, we include short proofs.

Lemma 1 *Let D be a simple drawing in the plane that is weakly isomorphic to a generalized twisted drawing of K_n. Let C_O and C_Z be a pair of antipodal vi-cells of D. Then the following statements hold.*

1. *D does not contain three interior-disjoint triangles.*
2. *Every subdrawing of D induced by four vertices of D contains exactly one crossing. If p is this crossing in such a subdrawing D', then the cells C_O and C_Z lie in two cells of D' that are both incident to p and opposite to each other (that is, they are interior disjoint and the intersection of their boundaries is exactly p).*

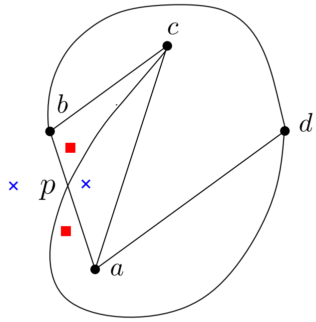
Proof: To prove Lemma 1(1), suppose that Δ_1, Δ_2 and Δ_3 are three interior-disjoint triangles. Since C_O and C_Z are antipodal, C_O must be inside one of the triangles, say Δ_1, and hence C_Z is outside that triangle. Since C_O is inside Δ_1, it is outside Δ_2 and thus C_Z must be inside Δ_2. But then C_O and C_Z lie on the same side of Δ_3, a contradiction. Therefore, D cannot contain three interior-disjoint triangles.

For the first part of Lemma 1(2), recall that there are only two different drawings of K_4 (up to strong isomorphism), one without crossings and one with exactly one crossing. The drawing without crossings contains three interior-disjoint triangles. Thus, since D cannot contain three interior-disjoint triangles by Lemma 1(1), any subdrawing of D induced by four vertices contains a crossing.

To prove the second part of Lemma 1(2), consider the subdrawing induced by the four vertices. There is (up to strong isomorphism) only one way to draw this (see Figure 3(a) for a drawing). It contains two pairs of triangles whose empty sides are disjoint. With the labeling of Figure 3(a), these are the pair of triangles *abc* and *abd* and the pair of triangles *acd* and *bcd*. Since C_O and C_Z are antipodal, C_O and C_Z have to be on different sides of each of those triangles.

Assume without loss of generality that C_O lies on the empty side of *abc* (otherwise, invert the roles of C_O and C_Z). Since the empty sides of *abc* and *abd* are disjoint, it follows that C_Z needs to lie on the empty side of *abd*. If C_O lies on the empty side of *acd*, then C_Z lies on the empty side of *bcd*, which leads to C_O and C_Z being the cells marked by blue crosses in Figure 3(a). Otherwise, C_Z lies on the empty side of *acd*, and hence C_O lies on the empty side of *bcd*. This leads to C_O and C_Z being the cells marked by red squares in Figure 3(a). □

We will also need the following technical lemma for simple drawings.



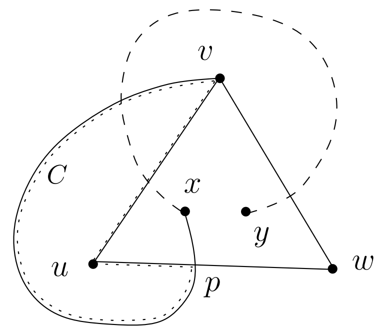


Figure 3: (a) Proof of Lemma 1(2): C_O and C_Z have to be either the cells marked with red squares or the cells with blue crosses. (b) Proof of Lemma 2: The edge (x, y) cannot cross both of the edges (v, u) and (v, w).

Lemma 2 *Let D be a simple drawing of K_n and let Δ be a triangle of D with vertices u, v, w. Let x and y be two vertices on the same side of Δ. If the edge (x, v) crosses (u, w), then the edge (x, y) can cross at most one of (v, u) and (v, w).*

Proof: Assume for a contradiction that (x, y) crosses both (v, u) and (v, w). Since x and y are on the same side of Δ, the edge (x, y) must cross the boundary of Δ an even number of times. Thus, if (x, y) crosses (v, u) and (v, w), it cannot cross (u, w). Let p be the crossing point between (x, v) and (u, w) (see Figure 3(b) for an illustration). Suppose that (x, y), when walking from x to y, crosses (v, u) before it crosses (v, w). Consider the closed curve C consisting of (v, u), the part of the edge (u, w) from u to p and the part of the edge (x, v) from p to v. The vertex y and the part of (x, y) directly after the crossing with (v, u) are in different regions defined by C; see Figure 3(b). Thus, after crossing (v, u), the edge (x, y) must cross C to reach y. However, as (x, y) cannot cross (u, w) as argued before and, by simplicity, it can neither cross (v, x) nor cross (u, v) a second time. Thus, (x, y) cannot cross (v, w) if it crossed (v, u) before. An analogous analysis can be done if (x, y) crosses (v, w) before it crosses (v, u), leading to a contradiction also for that case. Therefore, (x, y) crosses at most one of the edges (v, w) and (v, u). $\square$

In addition to the properties of generalized twisted drawings, we will use the concept of star triangles as introduced in [5]. A triangle Δ with vertices x, y, z is a *star triangle at x* if the edge (y, z) is not crossed by any edge incident to x. In [5], star triangles were used to prove that any simple drawing of K_n contains at least n empty triangles. For our purposes, we will need the following properties of star triangles in simple drawings of K_n, the first two of which have been shown in [5].

Lemma 3 *Let D be a simple drawing of K_n in the plane and x be a vertex of D. Then the following statements hold.*

1. *A star triangle xyz at a vertex x is an empty triangle if and only if the vertices y and z are consecutive in the rotation around x [5, Proposition 1].*

2. *There are at least two empty star triangles at x [5, Corollary 1].*

3. *For any two different empty star triangles at x, xyz and $xy'z'$, the empty sides of xyz and $xy'z'$ are disjoint.*

Proof: To prove Lemma 3(3), consider the triangles xyz and $xy'z'$ and their boundary edges. Since both triangles are different, at most one of $\{y, z\}$ can coincide with one of $\{y', z'\}$.

Suppose first that x is the only common vertex to both triangles. As xyz and $xy'z'$ are star triangles of x, no edge incident to x can cross (y, z) or (y', z'). Edges incident to x cannot cross each other because the drawing is simple. Thus, of the boundary edges of xyz and $xy'z'$, the only pair that could cross is (y, z) and (y', z'). However, y' and z' lie on the same side of the triangle xyz, so (y', z') has to cross the boundary of xyz an even number of times. This is not possible if exactly (y, z) and (y', z') cross. Hence, no edges on the boundary of the star triangles cross, and therefore their empty sides are disjoint.

Suppose now that one of $\{y, z\}$ coincides with one of $\{y', z'\}$, say $z = y'$, so xyz and xzz' share the edge (x, z). As before, no edge incident to x can cross (y, z) or (z, z'). Moreover, (y, z) and (z, z') cannot cross each other as they share a common endpoint. Thus, again the edges on the boundary of the star triangles cannot cross, so their empty sides are disjoint. □

3 Proof of Theorem 1

In this section, we derive several lemmata about empty triangles in generalized twisted drawings. These lemmata put together will yield the proof of Theorem 1. Throughout the section, we always assume that given a generalized twisted drawing of K_n, the antipodal vi-cells C_O and C_Z are also given.

The first of these lemmata shows that in a generalized twisted drawing, every vertex is incident to exactly two empty star triangles.

Lemma 4 *Let D be a generalized twisted drawing of K_n in the plane with $n \geq 4$ and v be a vertex of D. Then v is incident to exactly two empty star triangles at v, where one has C_O on the empty side and the other has C_Z on the empty side. Further, these star triangles have disjoint empty sides.*

Proof: By Lemma 3(2 and 3), for every vertex v there are at least two empty star triangles at v and the empty sides of any two star triangles at a vertex are disjoint. Any triangle of a generalized twisted drawing has C_O on one side and C_Z on the other side, and D cannot contain three interior-disjoint triangles by Lemma 1(1). Thus, for any vertex v in a generalized twisted drawing, the following three properties hold. (i) one empty star triangle at v has C_O on the empty side, (ii) another empty star triangle at v has C_Z on the empty side, and (iii) there cannot be a third empty star triangle at v. □

The following lemma proves that in any generalized twisted drawing, all empty triangles that contain C_O on the empty side are star triangles and have a common incident vertex.

Lemma 5 *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Let v be a vertex on the boundary of C_O. Let Δ be an empty triangle in D that has C_O on the empty side. Then the following statements hold.*

1. *The vertex v is a vertex of Δ, that is, $\Delta = xyv$ for some x, y.*

2. *The triangle $\Delta = xyv$ is an empty star triangle at x or y or both.*

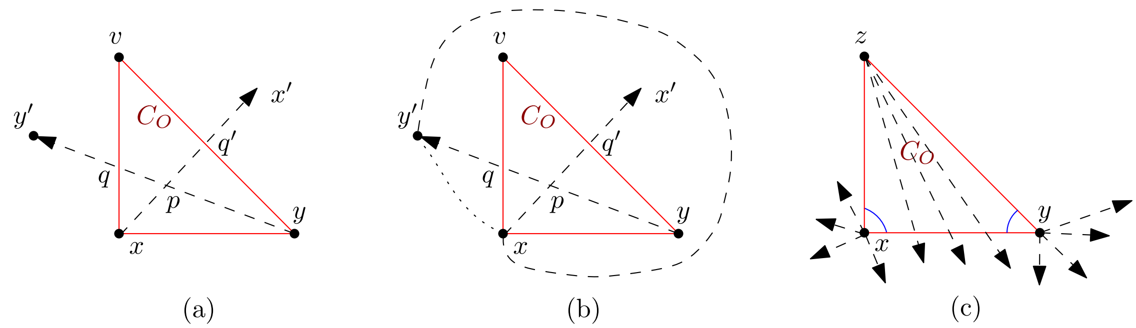


Figure 4: (a) and (b) Illustrations for the proof of Lemma 5(2). (c) Proof of Lemma 5(3): Any empty triangle of D that has C_O on the empty side cannot be a star triangle at three vertices. In all of (a), (b), and (c), the label C_O indicates that the cell C_O lies in the interior of the triangle. For (a) and (b), C_O is in addition incident to the vertex v.

3. *If $\Delta = xyz$ is a star triangle at two vertices, say x and y, then all edges from the third vertex z, except (z, x) and (z, y), emanate from z to the empty side of Δ and cross (x, y).*

4. *Δ is a star triangle for at most two of its vertices.*

Proof: Since Δ contains C_O on its empty side, and since v is on the boundary of C_O, Δ has to contain v on its empty side or on its boundary. As Δ is empty, v must be on the boundary of Δ and hence one of the vertices of Δ, implying that Lemma 5(1) is satisfied.

To prove Lemma 5(2), assume to the contrary that neither at x nor at y, Δ is an empty star triangle. Then from each of x and y, at least one edge must emanate into the empty side of Δ. Let these edges be (x, x') and (y, y'), respectively. Since there are no vertices on the empty side of Δ, the edge (x, x') must cross (v, y) at a point q' and the edge (y, y') must cross (v, x) at a point q. Further, these two edges must cross at a point p on the empty side of Δ. See Figure 4(a) for an illustration.

Now consider the subdrawing D' induced by x, x', y and y'. Observe that since (x, y') must not cross either of (x, v) and (y, y') by the simplicity of D, the cell of D' defined by x, p and y' cannot contain C_O, regardless of how (x, y') is drawn. (In Figure 4(b), the two possibilities for (x, y') are indicated with a dotted and dashed curve, respectively. In the former case, the corresponding cell is bounded, while in the latter it is unbounded.) Thus, by Lemma 1(2) applied to D', C_O and C_Z must be contained in the cells of D' defined by x', p and y', and x, p and y, respectively. Since v is on the boundary of C_O and in the cell of D' defined by x', p and y', the cell C_O of D must be contained in the cell of D' defined by x', p and y'. Consequently, C_Z is contained in the cell of D' defined by x, p and y. This contradicts that C_O and C_Z lie on different sides of Δ. As a consequence, Δ is a star triangle at x or y or both.

To prove Lemma 5(3), we may assume without loss of generality that $\Delta = xyz$ is a star triangle at x and y (and v is an arbitrary vertex $\in \{x, y, z\}$). Let w be any vertex of D that is not a vertex of Δ. By Lemma 1(2), the subdrawing induced by x, y, z, and w has a crossing. As Δ is a star triangle at x and y, any edge incident to x or y emanates from x or y on the non-empty side of Δ, so neither (x, w) nor (y, w) can cross Δ. Hence (z, w) must cross (x, y) and emanate from z to the empty side of Δ (see Figure 4(c)). As we did not make any additional assumption about w, this completes the proof of Lemma 5(3).

Finally, Lemma 5(4) is an immediate consequence of Lemma 5(3): If Δ is a star triangle at two vertices, then by Lemma 5(3), it cannot be a star triangle at the third vertex. Therefore, Δ is a star triangle for at most two of its vertices. □

In the previous lemma we proved that any empty triangle containing C_O is a star triangle at one or two vertices. The following lemma shows that exactly two such triangles are star triangles at two vertices.

Lemma 6 *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Then D contains exactly two empty triangles with C_O on the empty side that are star triangles at two vertices.*

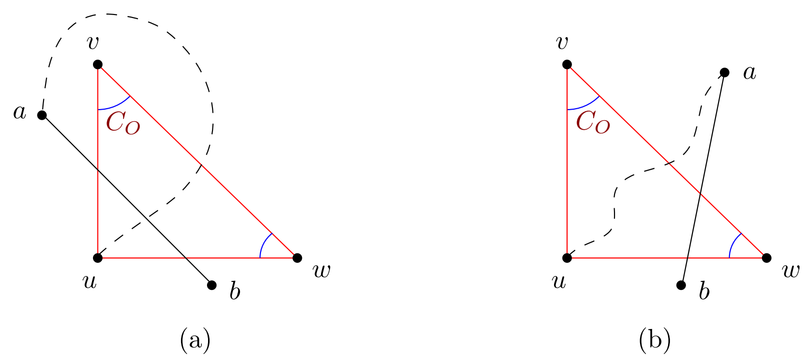


Figure 5: Proof of Claim 1: Forbidden cases for an edge (a, b) to define part of the boundary of C_O. The label C_O indicates that the cell C_O lies in the interior of Δ and is incident to v. The blue angular markings at vertices indicate that no further edge emanates from the vertex in this area.

Proof: Let v be a vertex on the boundary of C_O. Note that v might not be unique; see for example the drawing in Figure 2(a). By Lemma 4, there is an empty star triangle $\Delta = vuw$ at v that has C_O on the empty side. By Lemma 5(2 and 4), Δ is a star triangle at exactly one of u or w, say w. Thus, Δ is an empty star triangle at two vertices (v and w) with C_O on the empty side. Moreover, by Lemma 5(3), all edges emanating from u (except (u, v) and (u, w)) cross (v, w).

We next show the following claim, which guarantees a second empty triangle that is a star triangle at two vertices.

Claim 1 *Let (a, b) be an edge different from (v, u) and (v, w) that defines part of the boundary of C_O. Then the triangle vab is a star triangle at a and b, and the side F of vab that contains C_O is empty.*

Proof of Claim 1: We distinguish two cases depending on whether a and b are different from u and w or not.

Case 1: Both a and b are different from u and w. Since Δ is empty and C_O is on its empty side, necessarily (a, b) has to cross two edges of Δ. As all edges incident to u cross (v, w), also (u, a) crosses (v, w). Thus, if (a, b) crosses (v, u) and (u, w) (see Figure 5(a)), then (a, b) would also cross (u, a), contradicting the simplicity of the drawing. Further, if (a, b) crosses (v, w) and (u, w) (see Figure 5(b)), then (u, a) would separate (a, b) from the cell C_O, contradicting that (a, b) defines part of the boundary of C_O. Therefore, (a, b) has to cross (v, w) and (v, u).

We now show that F is empty. Note that in the plane, F can be bounded (as in Figure 6(a)) or unbounded (as in Figure 6(b)). However, the two drawings shown in Figure 6 are strongly isomorphic. Likewise, any drawing with a side of a closed curve being unbounded is strongly isomorphic to a drawing where that side is bounded. Hence, for the rest of the proof we assume without loss of generality that F is bounded.

Let p be the crossing of (a, b) with (v, w) and let q be the crossing of (a, b) with (v, u), as indicated in Figure 6. F is partitioned into three triangular regions vaq, vqp, and vpb, where C_O lies inside vqp. Note that the sides of vaq, vqp, and vpb that are contained in F are the insides of vaq, vqp, and vpb because F is bounded. We first show that vpb is empty.

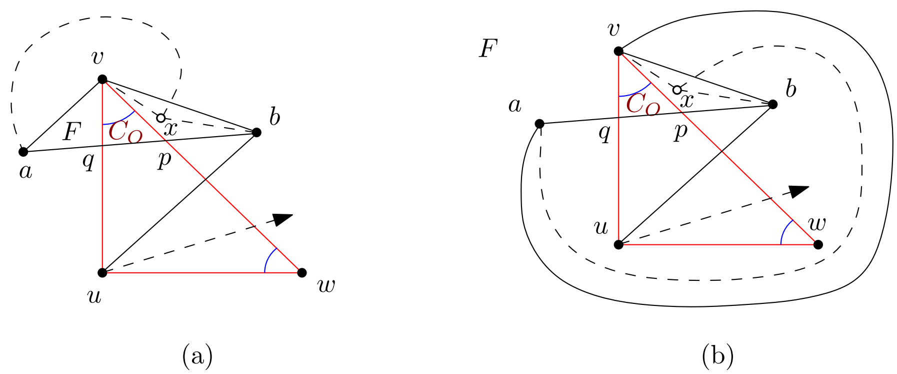


Figure 6: Proof of Claim 1: The case where a and b are different from u and w. The label C_O indicates that the cell C_O lies in the interior of Δ and is incident to v. The blue angular markings at vertices indicate that no further edge emanates from the vertex in this area.

Assume for a contradiction that a vertex x lies inside vpb. We show that (a, x) cannot intersect vqp and has to cross (v, b). As a and x both lie outside vpb, the edge (a, x) has to cross the boundary of vpb an even number of times. Since the drawing is simple, (a, x) cannot cross (a, b), and hence (a, x) cannot cross the arc (q, p). Thus, (a, x) either crosses neither of the arcs (v, p) and (v, q), or it crosses both of them. If (a, x) crosses both (v, p) and (v, q), the edge (a, b) cannot lie on the boundary of C_O, by this contradicting the definition of (a, b). Thus, (a, x) does not intersect vqp. Further, since (a, x) cannot cross (a, b) by simplicity, it must cross (v, b) to connect a and x (see again Figure 6).

As the subdrawing induced by a, b, v and x is a simple drawing of K_4 and contains the crossing between (a, x) and (v, b), the edges (v, x) and (b, x) cannot cross any of the edges (a, b), (v, b), and (v, a). Hence, the edges (v, x) and (b, x) are completely contained in F. Moreover, the edges (v, x) and (b, x) lie completely inside vpb for the following reasons. The edge (v, x) cannot cross (v, w) because the drawing is simple and thus (v, x) cannot cross the arc (v, p) and lies completely inside vbp. The edge (b, x) can cross neither (b, u) nor (a, b) because the drawing is simple. Hence, if (b, x) crosses the arc (v, p), in order to connect x with b, the edge (b, x) has to either lie partly outside F, contradicting the previous conclusions, or cross the arc (v, p) and thus the edge (v, w) twice, contradicting the simplicity of the drawing. Hence, (x, v) and (x, b) lie inside vbp and one side F' of the triangle vxb is contained inside vbp. As C_O is not in F' and C_O and C_Z are antipodal, it follows that C_Z has to be in F'. This implies that both C_O and C_Z are in F, a contradiction. Therefore, vbp is empty.

Using a similar argument, one can prove that vqa is also empty. Since by assumption on Δ, vpq is empty, then F is empty. Moreover, any edge incident to a or b must emanate from a or b

outside F because F is empty and (a, b) defines part of the boundary of C_O. As a consequence, the side F of the triangle vab that contains C_O is empty and is a star triangle at a and b.

Case 2: At least one of a and b is identical to u or w. As Δ is a star triangle at w, any edge incident to it cannot enter into Δ. As (a, b) is different from (v, u) and (v, w), this implies that w cannot coincide with a or b. So suppose that some of a and b, say a, is u and b is a vertex v' on the non-empty side of Δ. In this case, the reasoning to show that F is empty and is a star triangle at a and b is analogous to the previous one where neither a nor b are vertices of Δ. The only difference is that F is now partitioned into only two triangular regions vup and vpb, where p is the crossing point between (u, b) and (v, w).

As in each of the two cases, F is the empty side of a star triangle at both a and b, this completes the proof of Claim 1. □

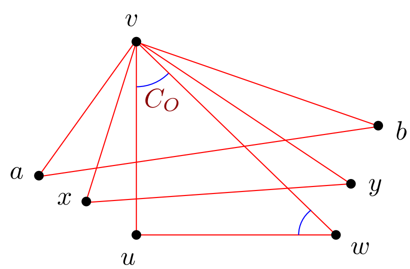


Figure 7: Proof of Lemma 6: The triangle vxy cannot be a star triangle at v. The label C_O indicates that the cell C_O lies in the interior of Δ and is incident to v. The blue angular markings at vertices indicate that no further edge emanates from the vertex in this area.

The triangle vab of Claim 1 is different from the triangle Δ. Both triangles, vab and Δ, are empty star triangles at two vertices that contain C_O on their empty side. What remains to show is that there is no third empty triangle with C_O on the empty side that is a star triangle at two vertices. Assume for a contradiction that such a triangle xyz exists. By Lemma 5(1), one of x, y and z must be v, say $z = v$. By Lemma 4, v, a, and b are each incident to exactly one empty star triangle with C_O on the empty side. Hence, x and y must be different from a and b, and vxy cannot be a star triangle at v. Consequently, vxy is a star triangle at x and y.

Consider now the triangle vab. By Lemma 5(3), the edges (v, x) and (v, y) must cross (a, b) (see Figure 7). In addition, by Lemma 2, since x and y are on the same side of vab and since (v, x) crosses (a, b), the edge (x, y) cannot cross both (v, a) and (v, b). But then applying Lemma 5(3) on the triangle vxy, all edges emanating from v whould have to cross (x, y), a contradiction. Therefore, xyz cannot exist. As a consequence, D contains exactly two empty triangles with C_O on the empty side that are star triangles at two vertices. □

We remark that Lemmata 4–6 and their proofs hold for every choice of C_O and any vertex v on the boundary of C_O. However, whether a triangle is empty, and at how many vertices it is a star triangle, does not depend on the choice of C_O or v (it does not even change between weakly isomorphic drawings). As a consequence, the empty star triangles obtained in these lemmata and proofs must be the same, regardless of the choice of C_O and the vertex v on the boundary of C_O.

We also remark that the reasoning in the proofs of Lemmata 5 and 6 for empty triangles having C_O on the empty side works analogously for empty triangles having C_Z on the empty side. This implies that these triangles are as well star triangles at one or two vertices, and that exactly two of them are star triangles at two vertices. By Lemma 6 and these remarks, we obtain the following result.

Lemma 7 *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Then D contains exactly four empty triangles that are star triangles at two vertices.*

Now we have all ingredients to prove our main theorem, Theorem 1, whose statement we repeat here for convenience.

Theorem 1 *For any $n \geq 4$, every generalized twisted drawing of K_n contains exactly $2n-4$ empty triangles.*

Proof: When summing up the number of empty star triangles over all vertices, we obtain $2n$ empty star triangles by Lemma 4 (n triangles with C_O on the empty side and n with C_Z on the empty side). By Lemma 5 (and the previous remarks), all empty triangles of D have been counted this way, but the triangles that are empty star triangles at two vertices have been counted twice. By Lemma 7, there are exactly four triangles that are empty star triangles at two vertices. Thus, there are exactly four triangles that have been counted twice, implying that the precise number of empty triangles in D is $2n-4$. □

Corollary 1 *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Then C_O and C_Z are triangular cells.*

Proof: Let v be a vertex on the boundary of C_O. At the beginning of the proof of Lemma 6, we have seen that there is an empty triangle vuw that contains C_O on its empty side and is a star triangle at v and w. Moreover, by Claim 1, any edge (a,b) defining part of the boundary of C_O defines a triangle vab that is an empty star triangle at a and b containing C_O on its empty side. Since by Lemma 6, there are exactly two triangles that are empty star triangles at two vertices and contain C_O on its empty side, there is only one edge (a,b) defining part of the boundary of C_O, in addition to (v,u) and (v,w). Therefore, the cell C_O is a triangular region. The same reasoning applies to show that the cell C_Z is triangular as well. □

4 Conclusion and Open Problems

We have shown that the number of empty triangles in any generalized twisted drawing of K_n is exactly $2n-4$. Twisted drawings, which are included in the class of generalized twisted drawings, were the first drawings for which it was shown that they do not contain more than $2n-4$ empty triangles. We believe that the fact that also the wider class of generalized twisted drawings has exactly $2n-4$ empty triangles can be a step towards obtaining results for general simple drawings.

While it was known before that also other simple drawings of K_n also have only $2n-4$ empty triangles, to the best of our knowledge, generalized twisted drawings of K_n are the only defined class with more than one different weak isomorphism class (for fixed $n \geq 6$) such that each drawing has only $2n-4$ empty triangles. For n up to 8, computational results on how many different weak isomorphism classes of drawings of K_n there are, and how many of them have exactly $2n-4$ empty triangles, are listed in [5].

One example of a simple drawing of K_8 that has exactly $2n-4$ empty triangles has been presented in [5] and is also depicted in Figure 8. As it contains seven triangles with pairwise disjoint empty sides (the triangles are marked in bold, red edges in Figure 8), by Lemma 1(1), it is not generalized twisted. This raises the question of which drawings of K_n, other than generalized twisted drawings, have $2n-4$ empty triangles.

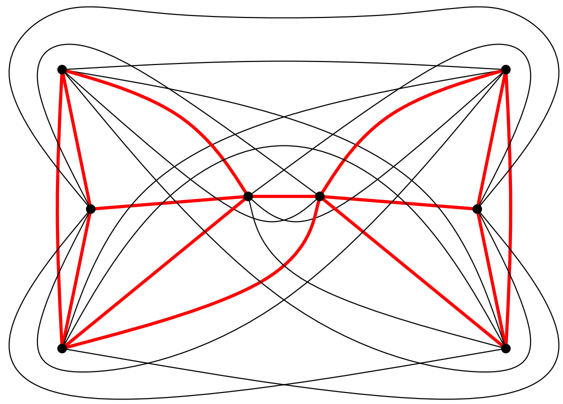


Figure 8: A drawing that is not generalized twisted and has only $2n-4$ empty triangles [5]. The bold, red edges are edges of seven triangles with pairwise disjoint empty sides.

Question 1 *What are characterizations of simple drawings of K_n that contain exactly $2n-4$ empty triangles?*

A complete characterization might help to better understand empty triangles in simple drawings of K_n and to progress towards proving Conjecture 1, or to at least get more insight to approach the big open question behind the conjecture.

Question 2 *How many empty triangles does every simple drawing of K_n contain at least?*

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