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Edge k-q-Colorability of Graphs

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Abstract

Given positive integers k, q, we say that a graph is edge k-q-colorable if its edges can be colored in such a way that the number of colors incident to each vertex is at most q and that the size of a largest color class is at most k. The problem of minimizing k for a given q was considered in [T. Larjomaa and A. Popa, *The min-max Edge q-coloring Problem*, Journal of Graph Algorithms and Applications, vol 19(1) pp. 505-528 (2015)]. In this paper, we first fix $k = 2$ and give an $O(\min\{m^2\sqrt{n/\log m}, nm^{1.5}\})$-time algorithm which given an arbitrary graph G with n vertices and m edges, and a positive integer q decides whether G is 2-q-colorable and outputs a 2-q-coloring if such a coloring exists. Then, we fix $q = 2$ and we focus on cubic graphs. In particular, we prove that every cubic graph admits a 4-2-coloring such that the corresponding edge decomposition uses only paths. We give an $O(n \log^2 n)$-time algorithm constructing such a decomposition.

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1 Introduction

We consider undirected graphs with n vertices and m edges. Given a graph G and a positive integer q, an edge q-coloring of G is an assignment of colors to the edges of G such that each vertex is incident to at most q colors. A *maximum edge q-coloring* of G is a q-coloring of G that uses the maximum number of colors. The *maximum edge q-coloring problem* is the problem of finding a maximum edge q-coloring for a given graph. This problem is motivated by the design of *multi-channel wireless mesh networks* (WMN) architectures (see [7, 8] for references). In such networks, each node can be equipped with multiple network interface cards (NIC), each interface card providing a different channel frequency. Allowing each node to use multiple frequencies tends to reduce the interference phenomenen in the network. There is certainly a limitation on the number q of interface cards attached to a given node. Each link (edge) has to be assigned a frequency (a color) such that at each node (vertex), the number of different channels does not exceed q. At first thought, maximizing the number of used frequencies (colors) tends to reduce the interference. However, as observed by the authors in [7, 8], in optimal maximum q-colorings, a same color is assigned to many edges while each other color occurs once. This led the authors in [7] to define the *min-max edge q-coloring problem* where the goal is to balance the size of the groups of edges of different colors, rather than to maximize the number of colors. The min-max edge q-coloring problem can be stated as follows. Given a graph G, and a positive integer q, find an edge q-coloring of G such that the size of a largest color class is minimized. In [8], the authors proved the NP-hardness of the min-max edge q-coloring problem for every $q \geq 2$, and gave an exact polynomial time algorithm for the min-max 2-coloring problem in trees. They also provided exact formulas or tight bounds for some classes of graphs, and an approximation algorithm for planar graphs.

The following example shows the difference between the maximum edge q-coloring and the min-max edge q-coloring problems. Consider the biconnected cubic graph G of order $n = 4p$ with $p \geq 2$ and size $m = 6p$, formed by a cycle $C_{4p} = x_1 x_2 \cdots x_{4p} x_1$ and the $2p$ diametral chords $x_i x_{i+2p}$. Coloring each of p pairwise disjoint cycles C_4 with a different color, and then assigning a different color to each remaining edge gives a 2-coloring of G. It uses $3p$ colors. Hence the optimum of G for the maximum 2-coloring problem is at least $3p$. We will see in Section 3 that G has no 2-coloring such that all color classes have size less than 3 but admits 2-colorings using only paths of length 3. Hence the optimum of G for the min-max 2-coloring problem is 3 and it uses $m/3 = 2p$ balanced classes.

In this paper, we consider a property of graphs that we call *k-q-colorability* which is related to the min-max q-coloring problem. A graph is said to be (edge) k-q-colorable if it admits an edge q-coloring such that the size of a largest color class is at most k. We note that if this problem has a solution then there exists a solution where each color class induces a connected subgraph (give different colors to the edges of the different components of a non connected class if any).

In section 2, we fix $k = 2$, and we study the following problem:

Instance: A graph G, a positive integer q.
Solution: A partition of $E(G)$ into connected subgraphs of size at most 2, no more than q of them share a vertex.

We provide an algorithm solving this problem and running in time $O(\min\ \{m^2\sqrt{n/\log m}\ ,\ nm^{1.5}\})$ for arbitrary graphs. We also prove that d-regular graphs are 2-q-colorable if and only if $q \geq \lceil 3d/4 \rceil$.

In section 3, we fix $q = 2$ and we consider cubic graphs. We prove the 3-2-colorablity of some classes of cubic graphs, and that there are classes of cubic graphs that cannot be 3-2-colorable. However, they all admit a 4-2-coloring using only paths. We give an $O(nlog^2n)$-time algorithm constructing such a decomposition. Lovász [9] showed that every simple odd graph G (a graph such that every vertex has odd degree) admits a path decomposition where every vertex of G is an extremity of exactly one path of the corresponding partition of $E(G)$. To our knowledge, there is no result concerning the length of a longest path in such a decomposition. For cubic graphs, a Lovász path decomposition is clearly a 2-coloring such that all color classes are paths. Our result shows that a Lovász path decomposition of cubic graphs can be done with paths of length at most 4.

In Section 4, we generalize the previous results to subcubic graphs and multigraphs. *Faithful path decompositions* were defined by Cai [2] as a generalization of the Lovász path decompositions to all graphs and multigraphs. We prove that subcubic multigraphs have faithful path decompositions using paths of length at most 4.

A graph of maximum degree Δ is k-q-colorable for every k and every $q \geq \Delta$ by letting one different color per edge, and if G is k-q-colorable, then $\Delta \leq kq$. Moreover, a graph is k-q-colorable if and only if each of its connected components is k-q-colorable. We suppose henceforth that G is connected and that $\lceil \Delta/k \rceil \leq q < \Delta$.

Clearly, every subgraph of a k-q-colorable graph is k-q-colorable. We deduce the following general property.

Theorem 1 *1. If every d-regular graph is k-q-colorable for some k and $q < d$, then every graph with maximum degree $\Delta \leq d$ is k-q-colorable.*
2. If every bipartite d-regular graph is k-q-colorable for some k and $q < d$, then every bipartite graph with maximum degree $\Delta \leq d$ is k-q-colorable.
3. Moreover, if every d-regular (bipartite d-regular) graph admits a k-q-coloring where each color class is a path, then so does every graph (bipartite graph) with maximum degree d.

Proof: 1. Let $G = (V, E)$ be a graph with maximum degree $\Delta \leq d$ and minimum degree δ. The proof is by induction on $\Delta - \delta$. If $\Delta = \delta$, we are done by the hypothesis. Otherwise, let $W = \{x \in V \mid d_G(x) < \Delta\}$, and let

$G' = (V', E')$ be a graph isomorphic to G. We denote by $\phi : V \to V'$ the corresponding bijection. Let H be obtained from the disjoint union of G and G' and by adding the matching $M = \{x\phi(x) \mid x \in W\}$. Then $\Delta(H) = \Delta(G)$, $\delta(H) = \delta(G) + 1$, and H is k-q-colorable by the induction hypothesis. Its subgraph G is k-q-colorable too.
2. If G is bipartite then the previous graph H is also bipartite since the addition of the edges of the matching M cannot create any odd cycle.
3. At each step of the induction, each color class induces a path which is a subpath of a color class in the path decomposition of the d-regular (bipartite d-regular) graph. □

2 2-q-colorability

We begin with two definitions relative to a graph $G = (V, E)$ of maximum degree Δ and a positive integer $q < \Delta$. We let $V_q = \{v \in V \mid d_G(v) \geq q + 1\}$. We say that an orientation $D = (V, A)$ of G has Property (P) if $d^-_D(v) \geq 2(d_G(v) - q)$ for each $v \in V_q$.

Theorem 2 *Let $G = (V, E)$ be a connected graph of maximum degree Δ, and let q be a positive integer with $\Delta/2 \leq q < \Delta$. There exists an orientation $D = (V, A)$ of G with Property (P) if and only if G is 2-q-colorable.*

Proof: Let D be an orientation of G satisfying (P). At each vertex v with $d_G(v) > q$, we pair $2(d_G(v) - q)$ edges entering into v, and we give each pair a new color. After treating all vertices v of V_q in this way, if there are still edges in G that have not yet received a color, assign each of them a new color. We obtain a coloring of the edges of G, where each color class has size at most two. The number of colors present at each vertex v with $d_G(v) \leq q$ is $d_G(v)$. At each vertex v of V_q, the number of present colors is at most $d_G(v) - \lfloor d^-_D(v)/2 \rfloor \leq d_G(v) - (d_G(v) - q) = q$. Hence the resulting coloring is a 2-q-coloring of G.

Conversely, assume that G is 2-q-colorable and consider a 2-q-coloring of G, where each color class induces a connected subgraph. Hence, two edges with the same color are incident. Orient them to their common extremity. Let v be a vertex with $d_G(v) > q$. At least $(d_G(v) - q)$ pairs of edges incident to v have the same color and are oriented to v. After treating all vertices v with $d_G(v) > q$ in this way, if there are still edges in G that have not yet got an orientation, orient them in an arbitrary way. We obtain an orientation $D = (V, A)$ of G satisfying (P). □

Theorem 3 *For any connected graph G and any positive integer q, the 2-q-colorability of G can be checked in time $O(\min\{m^2\sqrt{n/\log m}\ ,\ nm^{1.5}\})$.*

Proof: Let $V = \{v_1, \cdots, v_{n'}, \cdots, v_n\}$ where $V_q = \{v_1, \cdots, v_{n'}\}$. For each i, $1 \leq i \leq n'$, let $l(v_i) = d_G(v_i) - q$ and $X_i = \{x_{i,1}, \cdots, x_{i,2l(v_i)}\}$. Consider the bipartite graph $B = (X, Y, F)$ with $X = \cup_{i=1}^{i=n'} X_i$ and $Y = \{y_1, \ldots, y_m\}$. We put an edge in B between $x_{i,j}$ and y_k if and only if the edge e_k is incident to v_i in the graph G. The number of vertices of B is $|X|+|Y| = 2\sum_{i=1}^{n'}(d_G(v_i) - q) + m < 2\sum_{i=1}^{n} d_G(v_i) + m = 5m$. If $e_k = v_k^1 v_k^2$, the degree in B of the vertex y_k is $d_B(y_k) = \max\{0, 2(d(v_k^1) - q)\} + \max\{0, 2(d(v_k^2) - q)\} < 2(d(v_k^1) + d(v_k^2))$. Thus the number $|E(B)| = \sum_{k=1}^{m} d_B(y_k)$ of edges of B is at most equal to $2\sum_{v_i \in V(G)} d^2(v_i)$.

De Caen [3] proved that $\sum_{v_i \in V(G)} d^2(v_i) \leq m(n - 2 + \frac{2m}{n-1})$ in every graph G. Therefore $|V(B)| = O(m)$ and $|E(B)| = O(mn)$.

Claim. B has a matching that covers all the vertices of the side X if and only if G has an orientation satisfying Property (P).

Proof. Assume that such a matching covering X exists in B. For each i, $1 \leq i \leq n'$, we consider the matching B_i covering the vertices $x_{i,j}$, $1 \leq j \leq 2l(v_i)$. It defines a subset Y_i of Y that is in bijection with a subset E_i of edges of G, each of them incident with v_i. Orient the edges of E_i such that they are entering into v_i. This operation can be done for each i without any conflict because the sets Y_i are disjoint. Then arbitrarily orient the remaining edges. The resulting orientation satisfies Property (P). Conversely, assume G has an orientation satisfying Property (P). For each i, $1 \leq i \leq n'$, consider the subset E_i of edges oriented into v_i. It defines a subset Y_i of Y in bijection with E_i and satisfying $y \in N_B(x_{i,j})$ for each $y \in Y_i$ and each $j, 1 \leq j \leq 2l(v_i)$, where $N_B(x_{i,j})$ denotes the neighborhood of the vertex $x_{i,j}$ in B. The subgraph $B[X_i, Y_i]$ is complete bipartite, with $|X_i| = 2l(v_i) \leq |Y_i|$. Hence $B[X_i, Y_i]$ has a matching F_i that covers all the vertices of X_i. Since $E_i \cap E_j = \emptyset$, for every i, j, $1 \leq i < j \leq n'$, it follows that $\cup_{i=1}^{i=n'} F_i$ is a matching of B covering all the vertices of the side X. This ends the proof of the claim.

From the previous claim and Theorem 2, we can construct a 2-q-coloring of G if and only if the matching number of B is equal to $|X| = \sum_{i=1}^{i=n'} 2(d_G(v_i) - q)$. The matching number can be found in time $O(|V|^{1.5}\sqrt{|E|/\log|V|})$ for bipartite graphs ([1]) or $O(\sqrt{|V|} \cdot |E|)$ for arbitrary graphs ([10, 13]). Depending on the density of G, we apply one or the other algorithm. □

Corollary 1 *1. A d-regular graph is 2-q-colorable if and only if $q \geq \lceil 3d/4 \rceil$.*
2. Every graph with maximum degree Δ is 2-$\lceil 3\Delta/4 \rceil$-colorable.

Proof: 1. If G is d-regular then $V_q = V$, and the bipartite graph B in the proof of Theorem 3 satisfies $|X_i| = 2(d-q)$ for $1 \leq i \leq n$, $|X| = \sum_{i=1}^{n} |X_i| = 2n(d-q)$ and $|Y| = m = nd/2$. Moreover, B is biregular since $\forall x \in X$, $d_Y(x) = d$ and $\forall y \in Y$, $d_X(y) = 4(d-q)$. If B admits a matching covering X then $|X| \leq |Y|$, which is equivalent to $q \geq 3d/4$. Conversely, if $q \geq 3d/4$ then $d_X(y) \leq d_Y(x)$. Let X' be any subset of X, and let $N(X') \subseteq Y$ be the neighborhood of X'. Let $e(X', N(X'))$ be the number of edges between X' and $N(X')$. The inequality $d_Y(x)|X'| = e(X', N(X')) \leq d_X(y)|N(X')|$ implies that $|N(X')| \geq |X'|$. By the König-Hall's theorem, B admits a matching covering X. The result follows from Theorem 2 and the claim in the proof of Theorem 3.
2. This is a consequence of Theorem 1. □

This proves that cubic graphs are not 2-2-colorable. This will be seen again in 1. of Theorem 4.

3 k-2-colorability of cubic graphs

When $\Delta = 3$, the relation $\lceil \Delta/k \rceil \leq q < \Delta$ implies $q = 2$. In this case, the property of k-2-colorability is related to that of $*$-decomposition defined below. A biconnected component of a graph G is a maximal biconnected subgraph of G with a least 2 vertices. We denote by Q_i the path with i edges.

Definition 1 Let $\mathcal{S}$ be a set of connected graphs.
1. A graph G is $\mathcal{S}$-decomposable if $E(G)$ can be partitioned into subgraphs isomorphic to graphs from $\mathcal{S}$.
2. A graph G is $\mathcal{S}^*$-decomposable or $*$-decomposable into subgraphs from $\mathcal{S}$ if it is $\mathcal{S}$-decomposable and each vertex of G belongs to at most two subgraphs of the partition.

Remark 1 1. If a graph is k-2-colorable, then it is $\mathcal{S}^*$-decomposable where $\mathcal{S}$ is the set of all the connected graphs with at most k edges.
2. If a graph is $\mathcal{G}^*$-decomposable where $\mathcal{G}$ is a set of connected graphs with at most k edges, then it is k-2-colorable.
3. If $\mathcal{S}$ is an hereditary family of connected graphs, then every subgraph of a $\mathcal{S}^*$-decomposable graph is $\mathcal{S}^*$-decomposable.
4. A cubic graph is 3-2-colorable if and only if it is $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposable since $K_{1,3}$ is $*$-decomposable into one Q_1 and one Q_2.

We give a property relating the numbers ν_i of paths Q_i and t of triangles in a $*$-decomposition of a cubic graph into paths and triangles. We limit ourselves to paths of length at most 4 since as we will see, every cubic graph is $\{Q_2, Q_3, Q_4\}^*$-decomposable.

Theorem 4 *If a cubic graph admits a $*$-decomposition into ν_i paths Q_i and t triangles, then $3t = \Sigma_i(3-i)\nu_i$. In particular:*
1. No cubic graph is $\{Q_1, Q_2\}^*$-decomposable.
2. Any $\{Q_1, Q_2, Q_3\}^*$-decomposition of a cubic graph is a $\{Q_3\}^*$-decomposition.
3. Any $\{Q_1, Q_2, Q_3, Q_4\}^*$-decomposition of a cubic graph satisfies $\nu_4 = 2\nu_1 + \nu_2$.

Proof: In a $*$-decomposition of a cubic graph into paths and triangles, each vertex of G is an inner vertex of exactly one path or triangle. Hence $n = \Sigma_i(i-1)\nu_i + 3t$. On the other hand, the number of edges of G satisfies $m = \Sigma_i i\nu_i + 3t$. Since G is cubic, $2m = 3n$. Therefore $3t = \Sigma_i(3-i)\nu_i$.
1. If G admits a $\{Q_1, Q_2\}^*$-decomposition, then $2(\nu_1 + 2\nu_2) = 3\nu_2$. Hence $2\nu_1 + \nu_2 = 0$ and thus $\nu_1 = \nu_2 = 0$. Such a decomposition does not exist.
2. If G admits a $\{Q_1, Q_2, Q_3\}^*$-decomposition, then $2(\nu_1 + 2\nu_2 + 3\nu_3) = 3(\nu_2 + 2\nu_3)$. Hence $2\nu_1 + \nu_2 = 0$ and thus $\nu_1 = \nu_2 = 0$.
3. For any $\{Q_1, Q_2, Q_3, Q_4\}^*$-decomposition of G, $2(\nu_1 + 2\nu_2 + 3\nu_3 + 4\nu_4) = 3(\nu_2 + 2\nu_3 + 3\nu_4)$. Hence $\nu_4 = 2\nu_1 + \nu_2$. □

Corollary 2 *Let G be a cubic graph. The following properties are equivalent.*
1. G has a perfect matching.
2. G has a $\{Q_3\}^*$-decomposition.
3. G has a $\{Q_1, Q_2, Q_3\}^*$-decomposition.

Proof: $1 \Leftrightarrow 2$. Kotzig [5] proved that a cubic graph has a decomposition into paths Q_3 if and only if it has a perfect matching M. The edges of $E(G) - M$ form a 2-factor, each cycle of which can be cyclically oriented. Then each edge of M is extended to a path Q_3 by adding the outgoing edge at both extremities. The edges of M are the middle edges of the paths and the $\{Q_3\}$-decomposition is a $\{Q_3\}^*$-decomposition.
$2 \Rightarrow 3$. This is obvious from the definition.
$3 \Rightarrow 2$. This is a consequence of 2. of Theorem 4. □

Corollary 3
1. Cubic bipartite graphs are $\{Q_3\}^*$-decomposable and thus they are 3-2-colorable.
2. Biconnected cubic graphs are $\{Q_3\}^*$-decomposable and thus they are 3-2-colorable.
3. Triangle-free cubic graphs without a perfect matching are not 3-2-colorable.

Proof:
1. It is known from König-Hall's theorem that every regular bipartite graph has a perfect matching.
2. Petersen showed that every biconnected cubic graph has a perfect matching (this is even true if the graph is not biconnected but its bridges are all contained in one path). Hence biconnected cubic graphs are $\{Q_3\}^*$-decomposable by Corollary 2.
3. If G has no perfect matching, then it is not $\{Q_1, Q_2, Q_3\}^*$-decomposable. Since it is triangle-free, then it is not $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposable and thus it is not 3-2-colorable by Remark 1. □

Theorem 5 *Let G be a cubic graph such that every cutvertex of G belonging to a biconnected component is contained in a triangle. Then G is $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposable and thus it is 3-2-colorable.*

Proof: Let C be a biconnected component of G different from a triangle and let $a_1, \cdots, a_p$, be the cutvertices of G that belong to C. If C is a diamond, we decompose it into a triangle and a path Q_2. Otherwise, let $a_ib_ic_i$, $1 \leq i \leq p$, be the triangle of C containing a_i. All these triangles are pairwise disjoint. Let C' be the graph obtained from C by replacing for each i, the edge $[b_i, c_i]$ by a path $[b_i, a'_i, c_i]$ and adding the edge $[a_i, a'_i]$. The graph C' is cubic and biconnected. Hence, it admits a $\{Q_3\}^*$-decomposition by Corollary 3. By Remark 1, the subgraph induced by $V(C') - \cup_{1 \leq i \leq p}\{a_i, a'_i\}$ admits a $\{Q_1, Q_2, Q_3\}^*$-decomposition. The addition of the p triangles $a_ib_ic_i$ gives a $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposition of C. The forest separating the biconnected components of G consists of trees with vertices of degree 3 or 1. Each of these trees is $\{Q_1, Q_2\}^*$-decomposable with exactly one Q_1, as can be seen by rooting the tree at some vertex of degree 1. Removing the edge incident to the root leaves a binary tree which is Q_2^*-decomposable. In the recomposition of G, each leaf of the forest belongs to a triangle of the $*$-decomposition of a biconnected component, or to the inner vertex of a path Q_2 in the case of a diamond. Hence we get a global $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposition of G. □

As a consequence of Theorem 5, claw-free cubic graphs are 3-2-colorable. We note that it was already known that connected claw-free graphs of even order have a perfect matching [12].

Theorem 5 shows a class of cubic graphs, some of them have not a perfect matching but are nevertheless 3-2-colorable thanks to the presence of many triangles. We describe now an arbitrarily large family of 3-2-colorable cubic graphs without a perfect matching and with a unique triangle. For $1 \leq j \leq 3$, let F^j be the graph on seven vertices a_i^j, $1 \leq i \leq 7$, and edge set $\{a_i^j a_{(i+1) \bmod 7}^j, a_2^j a_5^j, a_3^j a_6^j, a_4^j a_7^j\}$. We still denote by F^1 the graph obtained from F^1 by inserting k vertices b_i, $1 \leq i \leq k$, on the edge $a_1^1 a_2^1$, k vertices c_i, $1 \leq i \leq k$, on the edge $a_1^1 a_7^1$, and the k edges $b_i c_i$. Let G be the cubic graph obtained by connecting a new vertex x to the three vertices a_1^j, $1 \leq j \leq 3$. We can find a $\{Q_1, Q_2, Q_3, C_3\}^*$-decomposition of G using the unique triangle, one path Q_1 in F^1, one path Q_2 in $G - F^1$ and $k+9$ paths Q_3.

Since by Corollary 3, a cubic graph without a perfect matching cannot be 3-2-colorable if it is triangle-free, we study below its 4-2-colorability.

Theorem 6 *Every connected cubic graph G is $\{Q_2, Q_3, Q_4\}^*$-decomposable and thus it is 4-2-colorable.*

Proof: To make a proof by induction, we first define two cubic graph transformations which allow us to deduce a $\{Q_2, Q_3, Q_4\}^*$-decomposition of G from a $\{Q_2, Q_3, Q_4\}^*$-decomposition of a smaller graph. In the proof, we precise the notation which will be used in the writing of the algorithm.

A triangle $tt't''$ is said to be isolated if it shares no edge with another triangle. If $tt't''$ is an isolated triangle in G, and w, w', w'' are the three respective neighbors of vertices t, t', t'', we write $G' = f(G; tt't'', ww'w'')$ to denote the graph obtained from G by deleting the vertices t', t'', and connecting by an edge vertices t and w', and connecting by an edge vertices t and w'' (equivalently, the triangle $tt't''$ is replaced by a single vertex adjacent to w, w', w''). Assume that G' admits a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}'$. By exchanging the names of the involved vertices, we can assume w.l.o.g. that the edges $[w', t]$ and $[w'', t]$ are on the same path P' from $\mathcal{D}'$. Let P_1', P_2' be the two paths of the subgraph induced by the subset of edges $E(P') - \{[w', t], [w'', t]\}$. Assume that $l(P_1') \geq l(P_2')$, and that $w' \in V(P_1')$. We have $0 \leq l(P_1') \leq 2$, and $0 \leq l(P_2') \leq 1$. We denote by P_1 the path of G obtained by extending P_1' with the path $[w', t', t'']$, and by P_2 the path of G obtained by extending P_2' with the path $[w'', t'', t, t']$. We have $2 \leq l(P_1) \leq 4$, and $3 \leq l(P_2) \leq 4$. We get a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}$ of G where $\mathcal{D} = \mathcal{D}' \cup \{P_1, P_2\} - \{P'\}$. We write $\mathcal{D} = \triangle(tt't'', ww'w'')\mathcal{D}'$.

If $vv'v''w$ is a diamond of G (that is two triangles $vv'v''$, and $v'v''w$) such that the third neighbors w' of w, and u of v are distinct and non-adjacent, we write $G' = g(G; vv'v''w, uw')$ to denote the graph obtained from G by deleting the vertices v, v', v'', w, and connecting by an edge vertices u and w'. Assume that G' admits a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}'$. Let P' be the path from $\mathcal{D}'$ containing the edge $[u, w']$. Let P'_1, P'_2 be the two paths of the subgraph induced by the subset of edges $E(P') - \{[u, w']\}$. Assume that $l(P'_1) \geq l(P'_2)$, and that $w' \in V(P'_1)$. We have $1 \leq l(P'_1) \leq 3$, and $0 \leq l(P'_2) \leq 1$. We denote by P_1 the path of G obtained by extending P'_1 with the edge $[w', w]$, and P_2 is the path of G obtained by extending P'_2 with the path $[u, v, v', v'']$. We have $2 \leq l(P_1) \leq 4$, and $3 \leq l(P_2) \leq 4$. Let P_3 be the path $[v, v'', w, v']$ of G. We get a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}$ of G where $\mathcal{D} = \mathcal{D}' \cup \{P_1, P_2, P_3\} - \{P'\}$. We write $\mathcal{D} = \diamondsuit(vv'v''w, uw')\mathcal{D}'$.

We prove the theorem by induction on $b + n$ where b is the number of biconnected components of G. The smallest value, 5, of $b + n$ corresponds to $G = K_4$. Suppose $b + n > 5$. If $b = 1$, then G is a biconnected cubic graph which admits a $\{Q_3\}^*$-decomposition by Corollary 3. Hence we assume that G is a connected cubic graph with $b \geq 2$ biconnected components. We consider an extremal biconnected component B of G, that is a biconnected component containing exactly one cutvertex u of G. Then u has two neighbors u', u'' in B and is the extremity of a bridge $[u, v]$ of G. Let v', v'' be the neighbors of v other than u. Assume that $uu'u''$ form a triangle (such a triangle is not part of a diamond since otherwise B would contain two cutvertices of G). By induction hypothesis, $G' = f(G; uu'u'', vw'w'')$, where w', w'' are the respective third neighbors of u', u'', admits a $\{Q_2, Q_3, Q_4\}^*$-decomposition. We get a $\{Q_2, Q_3, Q_4\}^*$-decomposition of G as previously described. Similarly, if v is involved in a diamond $vv'v''w$, or in an isolated triangle $vv'v''$, then by induction hypothesis, $G' = g(G; vv'v''w, uw')$, where w' is the third neighbor of w, or $G' = f(G; vv'v''w, uw'w'')$, where w', w'' are the respective third neighbors of v', and v'', has a $\{Q_2, Q_3, Q_4\}^*$-decomposition. We get a $\{Q_2, Q_3, Q_4\}^*$-decomposition of G as described above. Assume that both u and v are not involved in a triangle. We let $G' = h(G; u, v, u', u'', v', v'')$ be the graph with two connected components obtained by deleting the vertices u and v, and connecting by an edge the vertices u', u'', on one hand, and the vertices v', v'', on the other hand. The component G'_1 of G' containing u' and u'' is a biconnected cubic graph. By Corollary 3, G'_1 has a $\{Q_3\}^*$-decomposition $\mathcal{D}'_1$. We let P be the path from $\mathcal{D}'_1$ containing the edge $[u', u'']$. Let G_1 be the graph obtained from G'_1 by inserting vertex u in the edge $[u', u'']$. We turn $\mathcal{D}'_1$ into a $\{Q_3, Q_4\}^*$-decomposition $\mathcal{D}_1$ of G_1 by inserting u in P between u' and u''. This lengthens by 1 the path P without changing its endvertices. We write $\mathcal{D}_1 = insert(\mathcal{D}'_1; uu'u'')$. The other connected component G'_2 of G' is cubic and has a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}_2$ by the induction hypothesis. Let P' be the path from $\mathcal{D}_2$ containing the edge $[v', v'']$. Let P'_1, P'_2 be the two paths of the subgraph induced by $E(P') - \{[v', v'']\}$. Assume that $l(P'_1) \geq l(P'_2)$, and that $v' \in V(P'_1)$. We have $1 \leq l(P'_1) \leq 3$, and $0 \leq l(P'_2) \leq 1$. We denote by P_1 the path of G obtained by extending P'_1 with the edge $[v', v]$, and P_2 is the path of G obtained by extending P'_2 with the path $[v'', v, u]$. We have $2 \leq l(P_1) \leq 4$, and $2 \leq l(P_2) \leq 3$. We get a $\{Q_2, Q_3, Q_4\}^*$-decomposition $\mathcal{D}$ of G, where $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{P_1, P_2\} - \{P'\}$. We write $\mathcal{D} = \mathcal{D}_1 \oplus_{u,vv'v''} \mathcal{D}_2$. □

Theorem 7
We can 4-2-*color simple cubic graphs in time* $O(n \log^2 n)$.

Proof: First we note that a maximum matching M in a biconnected cubic graph can be computed in time $O(nlog^2n)$ ([4]), and that the extension of the edges of M to paths Q_3 is linear time. Hence a Q_3^*-decomposition $\mathcal{D}$ of a given biconnected cubic graph can be obtained in time $O(nlog^2n)$. We will write $\mathcal{D} = Kotzig(M)$.

Given a simple cubic graph G, we compute its vertex decomposition $\mathcal{B}$ into the biconnected components. It is well known that the biconnected components of an arbitrary graph can be computed in time $O(n+m)$ (see [11] for example). Hence, this computation is done in time $O(n)$ for cubic graphs. An endblock is a biconnected component with at most one cutvertex of G. If v is a vertex, $B(v)$ denotes the biconnected component containing v. The variable *list* maintains a list of operations on path decompositions. The notation $< op_1 \centerdot op_2 > \mathcal{D}$ means that we will perfom op_2 on $\mathcal{D}$, and then op_1 on the resulting path decomposition. We denote by $<>$ the empty list of operations. We start by performing $\mathcal{A}(G; \mathcal{B}; <>)$.

$\mathcal{A}(G; \mathcal{B}; list)$
begin
Pick an endblock B from $\mathcal{B}$;
$\mathcal{B} := \mathcal{B} - \{B\}$;
if $\mathcal{B} = \emptyset$
  then Compute a perfect matching M of G ;
    $\mathcal{D} := kotzig(M)$;
    return $\mathcal{D}$;
  else u is the cutvertex lying in B.
    $e = [u, v]$ is the bridge attaching B.
    u', u'' are the neighbors of u in B.
    v', v'' are the neighbors of v other than u.
    $lst := <>$;
    while u is on a triangle $uu'u''$ **do**
      $\{w'$ (resp. w'') is the third neighbor of u' (resp. u'')$\}$
      $G := f(G; uu'u'', vw'w'')$;
      $lst := lst \centerdot \triangle(uu'u''; vw'w'')$;
      $u' := w'$; $u'' := w''$;
    while v is on a triangle $vv'v''$ **do**
      if v is on a diamond $vv'v''w$
        then $G := g(G; vv'v''w, uw')$; $\{w'$ is the third neighbor of $w\}$
          $lst := lst \centerdot \diamondsuit(vv'v''w, uw')$;
          $\mathcal{B} := \mathcal{B} - \{B(v)\}$;
          $v := w'$;
        else $\{v$ is on an isolated triangle $vv'v''\}$
          $\{w'$ (resp. w'') is the third neighbor of v' (resp. v'')$\}$

$G := f(G; vv'v'', uw'w'')$;
$lst := lst \centerdot \triangle(vv'v'', uw'w'')$;
$v' := w'$; $v'' := w''$;
$G' = h(G; u, v, u', u'', v', v'')$ is the subgraph induced by $V(G) - \{u, v\}$ in which the vertices u', u'' are connected by an edge as well as the vertices v', v''.
G'_1 is the connected component of G' containing u', u''.
G'_2 is the other connected component.
Compute a perfect matching M in G'_1 ;
$\mathcal{D}_1 := kotzig(M)$;
Exchange the names of u' and u'' if necessary so that, in the path P from $\mathcal{D}_1$, the length of the subpath ending at u'' and not containing u', is no more than the length of the other subpath of the subgraph induced by $E(P) - \{[u', u'']\}$;
$\mathcal{D}_1 := insert(\mathcal{D}_1; uu'u'')$;
$list := lst$;
$\mathcal{D}_2 := \mathcal{A}(G'_2; \mathcal{B}; list)$;
Exchange the names of v' and v'' if necessary so that, in the path P from $\mathcal{D}_2$, the length of the subpath ending at v'' and not containing v', is no more than the length of the other subpath of the subgraph induced by $E(P) - \{[v', v'']\}$;
$\mathcal{D} := \mathcal{D}_1 \oplus_{u,vv'v''} \mathcal{D}_2$;
$\mathcal{D} := list\ \mathcal{D}$;
return $\mathcal{D}$;
end

There are at most $b(G)$ calls to the procedure $\mathcal{A}$, where $b(G)$ is the number of biconnected components. The overall time needed for reduction operations f, g, and restore operations $\triangle$ and $\diamondsuit$ is $O(n)$. So, the running time is due to the computation of perfect matchings in biconnected cubic graphs. We can prove by induction on $b(G)$ that the algorithm runs in time $O(n \log^2 n)$. This is true for biconnected cubic graphs since we compute once a perfect matching. Assume that cubic graphs with b biconnected components consume time $O(n \log^2 n)$, and consider a cubic graph G with $b+1$ biconnected components. After picking an endblock and performing a sequence of reductions if necessary, the algorithm ends up by computing a perfect matching in a graph having $O(n)$ vertices, and recalling the procedure $\mathcal{A}$ on a graph having no more than b biconnected components and $O(n)$ vertices. By the induction hypothesis, the running time in the latter graph is $O(n \log^2 n)$. □

4 k-2-colorability of subcubic graphs and multigraphs

Theorem 8 *1. Every subcubic graph is $\{Q_1, Q_2, Q_3, Q_4\}^*$-decomposable and thus it is 4-2-colorable.*
2. Every subcubic bipartite graph is $\{Q_1, Q_2, Q_3\}^*$-decomposable and thus it is 3-2-colorable.

Proof: 1. This is a consequence of Theorems 6 and 1.
2. This is a consequence of Corollary 3 and Theorem 1. □

Corollary 4
Every subcubic multigraph has a $\{Q_1, Q_2, Q_3, Q_4, C_2\}^*$-decomposition and thus it is 4-2-colorable.

Proof: Let G be a subcubic multigraph of order n. If there exists a triple edge then $n = 2$, and G is $\{Q_1, C_2\}^*$-decomposable. Suppose that $n \geq 3$ and let G' be the graph obtained from G by deleting all the double edges. Each component of G' is a simple subcubic graph admitting a $\{Q_1, Q_2, Q_3, Q_4\}^*$-decomposition by Theorem 8. The addition of the deleted C_2's gives a $\{Q_1, Q_2, Q_3, Q_4, C_2\}^*$-decomposition of G. □

In his generalization of Lovász path decompositions, Cai [2] introduced a function $\phi : V(G) \to \mathbb{Z}^+$ that depends on the parity of the degree of the vertex x and on the maximum multiplicity of the edges of G incident to x. For a subcubic multigraph, $\phi(x) = 1$ if x is the extremity of one or three simple edges, $\phi(x) = 2$ if $d(x) = 2$, and $\phi(x) = 3$ if $d(x) = 3$ and x is the extremity of a double or a triple edge. Cai defined a path decomposition as being faithful if each vertex is the end of exactly $\phi(x)$ paths from the decomposition. He proved that every multigraph admits a faithful path decomposition. We show that if $\Delta \leq 3$, this can be done by paths of length at most 4.

Corollary 5 *Every subcubic multigraph admits a faithful decomposition by paths of length at most 4.*

Proof: For simple cubic graphs, this is a consequence of Theorem 6 since every $*$-decomposition by paths is faithful. For subcubic graphs or multigraphs, we start from a $\{Q_1, Q_2, Q_3, Q_4, C_2\}^*$-decomposition and we replace each C_2 of the decomposition by two paths Q_1. Then for each vertex x of degree two which is an inner vertex of a path Q_i, $i \geq 2$, we cut the path at x and replace it by two paths Q_j and Q_k with $j + k = i$. In the resulting path decomposition, each vertex x is the end of exactly $\phi(x)$ paths. □

For even regular graphs, there exist some results on the length of the paths in a faithful path. In [6], Kouider and Lonc proved that every $2p$-regular graph with girth (the length of a shortest cycle) at least $(p + 3)/2$ admits a decomposition into paths of length p such that every vertex is the extremity of exactly two paths and conjectured that the result remains true without any restriction on the girth. Such a decomposition is faithful since in Cai's definition, $\phi(x) = 2$ for each vertex of even degree of a simple graph. This shows that every $2p$-regular graph with girth $g \geq (p + 3)/2$ has a p-$(p + 1)$-coloring by paths Q_p.

We note that every $2p$-regular graph of any girth admits a p-$(p+1)$-coloring by considering an eulerian cycle and giving color i to all the edges going into vertex v_i. In this case, the color classes have a large maximum degree while in the former p-$(p + 1)$-coloring, all color classes have maximum degree 2.

5 Concluding remarks

Our results show that, for cubic graphs, the smallest k such that G is k-2-colorable is 3 or 4. The complexity of deciding whether it is 3 or 4 remains an open problem.

The case $q = 2$ is the most interesting in the design of WMN architectures as it requires the minimum on the number of NICs per node. However, it is worthwhile to investigate other values of q. We recall that the goal in such architectures is to reduce the interference caused by close nodes. In this work, we used an approach that consists in balancing the sizes of groups of links set up with different frequency channels. Another approach would be to seek a tradeoff between the sizes of the color classes and the maximum degree of the subgraphs induced by the q-coloring. For example, consider the case $q = 3$, and a node x with five incident links. In an assignment of frequencies resulting in a group of four links around x with the same frequency, the signal along each link in this group may be distorted by the interference of three other incident links. If the frequencies are assigned to the links in such a way that each group of links with the same frequency channel is a path, then each link is potentially altered by the noise of at most two other incident links. This makes worthwhile the study of q-colorings resulting in color classes of low maximum degrees as done in this work by considering path decompositions.

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