

## On a Tree and a Path with no Geometric Simultaneous Embedding

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### Abstract

Two graphs  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  admit a geometric simultaneous embedding if there exist a set of points  $P$  and a bijection  $M : V \rightarrow P$  that induce planar straight-line embeddings both for  $G_1$  and for  $G_2$ . The most prominent problem in this area is the question of whether a tree and a path can always be simultaneously embedded. We answer this question in the negative by providing a counterexample. Additionally, since the counterexample uses disjoint edge sets for the two graphs, we also negatively answer another open question, that is, whether it is possible to simultaneously embed two edge-disjoint trees. Finally, we study the same problem when some constraints on the tree are imposed. Namely, we show that a tree of height 2 and a path always admit a geometric simultaneous embedding. In fact, such a strong constraint is not so far from closing the gap with the instances not admitting any solution, as the tree used in our counterexample has height 4.

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## 1 Introduction

Embedding planar graphs is a well-established field in graph theory and algorithms with a great variety of applications. keystones in this field are the works of Thomassen [17], of Tutte [18], and of Pach and Wenger [16], dealing with planar and convex representations of graphs in the plane.

Recently, motivated by the need of concurrently representing several different relationships among the same elements, a major focus in the research lies on *simultaneous graph embedding*. In this setting, given a set of graphs with the same vertex-set, the goal is to find a set of points in the plane and a mapping between these points and the vertices of the graphs that yield a planar embedding for each of the graphs, when displayed separately. Problems of this kind frequently arise when dealing with the visualization of evolving networks and with the visualization of huge and complex relationships, such as the graph of the Web.

Among the many variants of this problem, the most important and natural one is the *geometric simultaneous embedding* problem. Given two graphs  $G_1 = (V, E')$  and  $G_2 = (V, E'')$ , the task is to find a set of points  $P$  and a bijection  $M : V \rightarrow P$  that induce planar *straight-line* embeddings for both  $G_1$  and  $G_2$ .

In the seminal paper on this topic [2], Brass *et al.* proved that geometric simultaneous embeddings of pairs of paths, pairs of cycles, and pairs of caterpillars always exist. A *caterpillar* is a tree such that deleting all its leaves yields a path. On the other hand, many negative results have been shown. Brass *et al.* [2] presented a pair of outerplanar graphs not admitting any geometric simultaneous embedding and provided negative results for three paths as well. Erten and Kobourov [5] proved negative results for a planar graph and a path, while Geyer *et al.* [13] proved the same for two edge-disjoint trees. Finally, Cabello *et al.* [3] showed a planar graph and a matching that do not admit any geometric simultaneous embedding and presented algorithms to obtain a geometric simultaneous embedding of a matching and a wheel, an outerpath, or a tree. An *outerpath* is an outerplanar graph whose weak dual is a path. The most important open problem in this area is the question of whether a tree and a path always admit a geometric simultaneous embedding or not, which is the subject of this paper.

Many variants of the problem, where some constraints are relaxed, have been studied. In the *simultaneous embedding* setting, where the edges do not need to be straight-line segments, any number of planar graphs admit a simultaneous embedding, since any planar graph can be planarly embedded on any given set of points in the plane [15, 16]. However, the same result does not hold if the edges that are shared by the two graphs have to be represented by the same Jordan curve. In this setting the problem is called *simultaneous embedding with fixed edges* [10, 12, 7]. Finally, the research on this problem opened a new exciting field of problems and techniques, like ULP trees and graphs [6, 8, 9], colored simultaneous embedding [1], near-simultaneous embedding [11], and matched drawings [4], deeply related to the general fundamental question of point-set embeddability.

In this paper we study the geometric simultaneous embedding problem of a tree and a path. We answer the question in the negative by providing a counterexample, that is, a tree and a path that do not admit any geometric simultaneous embedding. Moreover, since the tree and the path used in our counterexample do not share any edge, we also negatively answer the question on two edge-disjoint trees.

The main idea behind our counterexample is to use the path to enforce a part of the tree to be in a certain configuration which cannot be drawn planar. Namely, we make use of level nonplanar trees [6, 9], that is, trees not admitting any planar embedding if their vertices have to be placed inside certain regions according to a particular leveling. The tree of the counterexample contains many copies of such trees, while the path is used to create the regions.

To prove that at least one copy has to be in the particular leveling that determines a crossing, we need quite a huge number of vertices. However, such a number is often needed just to ensure the existence of particular structures playing a role in our proof. A much smaller counterexample could likely be constructed with the same techniques, but as the end result would be the same, we opted not to minimize the size.

The paper is organized as follows. In Section 2 we give preliminary definitions and we introduce the concept of level nonplanar trees. In Section 3 we describe the tree  $\mathcal{T}$  and the path  $\mathcal{P}$  used in the counterexample. In Section 4 we give a proof that  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding, leaving some of the more complex proofs for Section 5. In Section 6 we give an algorithm to construct a geometric simultaneous embedding of a tree of height 2 and a ()path and in Section 7 we make some final remarks.

## 2 Preliminaries

An (undirected)  $k$ -level tree  $T = (V, E, \phi)$  is a tree  $T' = (V, E)$ , called the *underlying tree* of  $T$ , together with a leveling  $\phi : V \mapsto \{1, \dots, k\}$  of its vertices such that for every edge  $(u, v) \in E$ , it holds  $\phi(u) \neq \phi(v)$  (See [6, 9]). A drawing of  $T$  is a *level drawing* if each vertex  $v \in V$  is placed on a horizontal line  $l_{\phi(v)} = \{(x, \phi(v)) \mid x \in \mathbb{R}\}$ . A level drawing of  $T$  is *planar* if no two edges intersect except, possibly, at common end-points. A  $k$ -level tree  $T = (V, E, \phi)$  is *level nonplanar* if it does not admit any planar level drawing.

We extend this concept to the one of a *region-level drawing* by enforcing the vertices of each level to lie inside a certain region rather than on a horizontal line. Let  $l_1, \dots, l_k$  be  $k$  non-crossing straight-line segments and let  $r_1, \dots, r_{k+1}$  be the regions of the plane such that any straight-line segment connecting a point in  $r_i$  and a point in  $r_h$ , with  $1 \leq i < h \leq k + 1$ , cuts all and only the segments  $l_i, l_{i+1}, \dots, l_{h-1}$ , in this order. We say that regions  $r_1, \dots, r_{k+1}$  are *linearly-separated*. A drawing of a  $k$ -level tree is a *region-level drawing* if each vertex  $v \in V$  is placed inside region  $r_{\phi(v)}$ . A  $k$ -level tree is *region-level nonplanar* if it does not admit any planar region-level drawing for any set of linearly-separated regions. The 4-level tree  $T$  whose underlying tree is shown in Fig. 1(a) is level

nonplanar [9] (see Fig. 1(b)). We show that  $T$  is also region-level nonplanar.

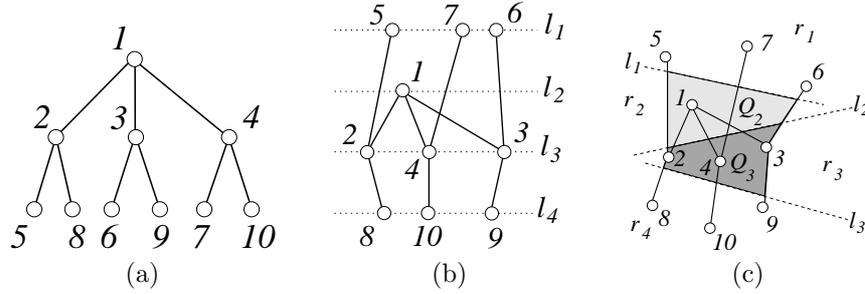


Figure 1: (a) A tree  $T_u$ . (b) A level nonplanar tree  $T$  whose underlying tree is  $T_u$ . (c) A region-level nonplanar tree  $T$  whose underlying tree is  $T_u$ .

**Lemma 1** *The 4-level tree  $T$  whose underlying tree is shown in Fig. 1(a) is region-level nonplanar.*

**Proof:** Refer to Fig. 1(c). First observe that, in any possible region-level planar drawing of  $T$ , there exists a polygon  $Q_2$  inside region  $r_2$  delimited by paths  $p_1 = \{5, 2, 8\}$  and  $p_2 = \{6, 3, 9\}$ , and by segments  $l_1$  and  $l_2$ , and a polygon  $Q_3$  inside region  $r_3$  delimited by paths  $p_1$  and  $p_2$ , and by segments  $l_2$  and  $l_3$ . We have that vertex 1 is inside  $Q_2$ , as otherwise one of edges  $(1, 2)$  or  $(1, 3)$  would cross one of  $p_1$  or  $p_2$ . Hence, vertex 4 is inside  $Q_3$ , as otherwise edge  $(1, 4)$  would cross one of  $p_1$  or  $p_2$ . However, in this case, there is no placement for vertex 7 that avoids a crossing between edge  $(4, 7)$  and one of the other edges.  $\square$

Lemma 1 will be vital for proving that there exist a tree  $\mathcal{T}$  and a path  $\mathcal{P}$  not admitting any geometric simultaneous embedding. In fact,  $\mathcal{T}$  contains many copies of the underlying tree of  $T$ , while  $\mathcal{P}$  connects vertices of  $\mathcal{T}$  in such a way as to create the regions satisfying the above conditions and to enforce at least one of these copies to lie inside these regions according to the leveling that makes it nonplanar.

### 3 The Counterexample

In this section we describe a tree  $\mathcal{T}$  and a path  $\mathcal{P}$  not admitting any geometric simultaneous embedding.

#### 3.1 Tree $\mathcal{T}$

The tree  $\mathcal{T}$  contains a root  $r$  and  $q$  vertices  $j_1, \dots, j_q$  at distance 1 from  $r$ , called *joints*. Each joint  $j_h$ , with  $h = 1, \dots, q$ , is connected to  $(s - 1)^4 \cdot 3^2 \cdot b$  vertices of degree 1, called *stabilizers*, and to  $b$  subtrees  $B_1, \dots, B_b$ , called *branches*, each one consisting of a root  $r_i$ ,  $(s - 1) \cdot 3$  vertices of degree  $(s - 1)$  adjacent

to  $r_i$ , and  $(s - 2) \cdot (s - 1) \cdot 3$  leaves at distance 2 from  $r_i$ . See Fig. 2(a) for a schematization of  $\mathcal{T}$  and Fig. 2(b) for a schematization of a branch. Vertices belonging to branches are called *B-vertices* and denoted by 1-, 2-, or 3-vertices, according to their distance from their joint.

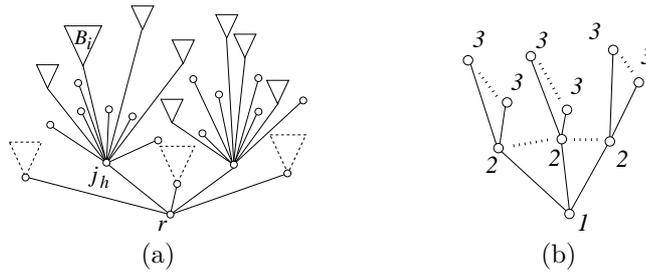


Figure 2: (a) A schematization of  $\mathcal{T}$ . Joints and stabilizers are small circles. A solid triangle represents a branch, while a dashed triangle represents the subtree connected to a joint. (b) A schematization of a branch  $B_i$ . Vertices are labeled with their distance from the joint to which the branch is connected.

Because of the huge number of vertices, in the rest of the paper, for the sake of readability, we use variables  $q$ ,  $s$ , and  $b$  as parameters describing the size of certain structures. Such parameters will be given a value when the technical details are described. At this stage we just claim that a total number  $n \geq \binom{2^7 \cdot 3 \cdot b + 2}{3}$  of vertices (see Lemmata 4 and 5) suffices for the counterexample.

As a first observation we note that, despite the oversized number of vertices, tree  $\mathcal{T}$  has limited *height*, that is, every vertex is at distance from the root at most 4. This leads to the following property:

**Property 1** *Any simple path of tree-edges starting at the root has at most 3 bends.*

### 3.2 Path $\mathcal{P}$

Path  $\mathcal{P}$  is given by describing some basic and recurring subpaths on the vertices of  $\mathcal{T}$  and how such subpaths are connected to each other. The idea is to partition the set of branches adjacent to each joint  $j_h$  into subsets of  $s$  branches each and to connect the vertices of each set with path-edges, according to some features of the tree structure, so defining the first building block, called a *cell*. Then, cells belonging to the same joint are connected to each other to create structures, called *formations*, for which we can ensure certain properties regarding the intersection between tree- and path-edges. Further, different formations are connected to each other by path-edges in such a way as to create bigger structures, called *extended formations*, which in turn are connected to create *sequences of extended formations*.

All of these structures are constructed in such a way that there exists a set of cells, connected to the same joint and being part of the same formation or extended formation, such that any four of these cells contain a copy of a region-level nonplanar tree, where the level of a vertex is determined by the cell it belongs to. Hence, proving that four of such cells lie in different regions satisfying the properties of separation described above is equivalent to proving the existence of a crossing in  $\mathcal{T}$ . This allows us to consider only bigger structures instead of dealing with single copies of the region-level nonplanar tree.

In the following we define such structures more formally and state their properties.

**Cell:** The most basic structure is defined by determining how  $\mathcal{P}$  connects the vertices of a set of  $s$  branches connected to the same joint of  $\mathcal{T}$ .

For each joint  $j_h$ ,  $h = 1, \dots, q$ , we partition the set of branches connected to  $j_h$  into sets of  $s + 3 \cdot s \cdot (s - 1)^2$  branches each. Then, for each such set, we construct a set of  $s$  cells as follows.

Each cell  $c_i(h)$ ,  $i = 1, \dots, s$ , is composed of its *head*, its *tail*, and a number of stabilizers to be determined later.

Let  $B_i$ ,  $i = 1, \dots, s$ , be  $s$  branches of the considered set of  $s + 3 \cdot s \cdot (s - 1)^2$  branches. The *head* of  $c_i(h)$  consists of the unique 1-vertex of  $B_i$ , the first three 2-vertices of each branch  $B_k$ , with  $1 \leq k \leq s$  and  $k \neq i$ , not belonging to any other cell and, for each 2-vertex in  $c_i(h)$  that belongs to branch  $B_m$ , the first 3-vertex of each branch  $B_k$ , with  $1 \leq k \leq s$  and  $k \neq i, m$ , not connected to a 2-vertex in  $c_i(h)$  and not belonging to any other cell. The *tail* of  $c_i(h)$  is created by considering the remaining  $3 \cdot s \cdot (s - 1)^2$  branches of the set, and by distributing their vertices to the cells in the same way as for the vertices of the head.

Path  $\mathcal{P}$  visits the vertices of  $c_i(h)$  in the following order: It starts at the unique 1-vertex of the head, then it reaches all the 2-vertices of the head, then all the 3-vertices of the head, then all the 2-vertices of the tail, and finally all the 3-vertices of the tail, visiting each set in arbitrary order. After each occurrence of a 2- or 3-vertex of the head,  $\mathcal{P}$  visits a 1-vertex of the tail, and after each occurrence of a 2- or a 3-vertex of the tail, it visits a stabilizer of joint  $j_h$  (see Fig. 3(a)).

This implies that each cell contains one 1-vertex,  $3 \cdot (s - 1)$  2-vertices, and  $3 \cdot (s - 2) \cdot (s - 1)$  3-vertices of the head, an additional  $3 \cdot (s - 1)^2$  1-vertices,  $3^2 \cdot (s - 1)^3$  2-vertices, and  $3^2 \cdot (s - 2) \cdot (s - 1)^3$  3-vertices of the tail, plus  $3^2 \cdot (s - 1)^4$  stabilizers.

Note that each set of  $s$  cells constructed as above is such that each subset of size 4 contains a region-level nonplanar tree, where the levels correspond to the membership of the vertices to a cell. Namely, consider four cells  $c_1, \dots, c_4$  belonging to the same set, leveled in this order. A region-level nonplanar tree as in Fig. 1(c) is illustrated in Fig. 3(b) and consists of the unique 1-vertex  $v$  of the head of  $c_2$ , the three 2-vertices of  $c_3$  connected to  $v$  and, for each of them, the 3-vertex of  $c_1$  and the 3-vertex of  $c_4$  connected to it.

**Formation:** In the definition of cells we described how the path traverses one set of  $s + 3 \cdot s \cdot (s - 1)^2$  branches connected to the same joint. Now we

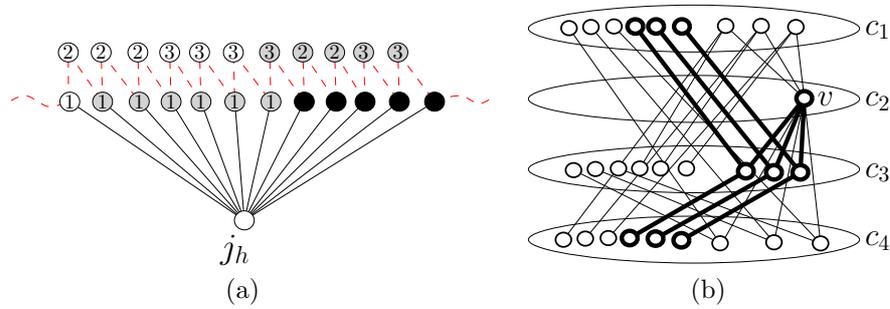


Figure 3: (a) An illustration of how path  $\mathcal{P}$  traverses the vertices of a cell. Vertices of the head are white, vertices of the tail are grey, and stabilizers are black. Throughout the paper, tree-edges are (black) solid segments and path-edges are (red) dashed segments. (b) The region-level nonplanar tree (represented by solid fat lines) as in Fig. 1(c) contained in a set of four cells  $c_1, \dots, c_4$  belonging to the same set.

describe how cells from four different sets are connected to each other.

A formation  $F(H)$ , where  $H = (h_1, h_2, h_3, h_4)$  is a 4-tuple of indices of joints, consists of 592 cells. Namely, for each joint  $j_{h_i}$ ,  $1 \leq i \leq 4$ ,  $F(H)$  contains 148 cells belonging to the same set of cells connected to  $j_{h_i}$ . Path  $\mathcal{P}$  connects these cells in the order  $((h_1 h_2 h_3)^{37} h_4^{37})^4$ , that is,  $\mathcal{P}$  repeats four times the following sequence: It connects  $c_1(h_1)$  to  $c_1(h_2)$ , then to  $c_1(h_3)$ , then to  $c_2(h_1)$ , and so on until  $c_{37}(h_3)$ , from which it then connects to  $c_1(h_4)$ , to  $c_2(h_4)$ , and so on until  $c_{37}(h_4)$  (see Fig. 4(a)). A connection between two consecutive cells  $c(h_a)$  and  $c(h_b)$  is done with an edge between the end vertices of the subpaths of  $\mathcal{P}$  induced by the vertices of  $c(h_a)$  and  $c(h_b)$ , respectively.

Since, by construction, the cells of  $F(H)$  that are connected to the same joint belong to the same set of cells, and since, by construction, any four cells belonging to the same set contain a region-level nonplanar tree, the following property holds:

**Property 2** For any formation  $F(H)$  and any joint  $j_h$ , with  $h \in H$ , if four cells  $c(h) \in F(H)$  lie in a set of linearly-separated regions, then there exists a crossing in  $\mathcal{T}$ .

**Extended Formation:** Formations are connected by the path in a special sequence, called an *extended formation* and denoted by  $EF(H)$ , where  $H = (H_1 = (h_1, \dots, h_4), H_2 = (h_5, \dots, h_8), \dots, H_x = (h_{4x-3}, \dots, h_{4x}))$  is an  $x$ -tuple of 4-tuples of disjoint indices of joints. For each 4-tuple  $H_i$ ,  $EF(H)$  contains  $y - \frac{y}{x}$  formations  $F_1(H_i), \dots, F_{y-\frac{y}{x}}(H_i)$  not belonging to any other extended formation and composed of cells of the same set of  $s$  cells connected to the same joint (see Fig. 4(b)). Formations inside  $EF(H)$  are connected in  $\mathcal{P}$  in the order  $(H_1, H_2, \dots, H_x)^y$ , that is,  $\mathcal{P}$  connects  $F_1(H_1)$  to  $F_1(H_2)$ , then to  $F_1(H_3)$ , and so on until  $F_1(H_x)$ , then to  $F_2(H_1)$ , to  $F_2(H_2)$ , and so on until  $F_{y-\frac{y}{x}}(H_x)$ . However, in each of these  $y$  repetitions one  $H_i$  is missing. Namely, in the  $k$ -th

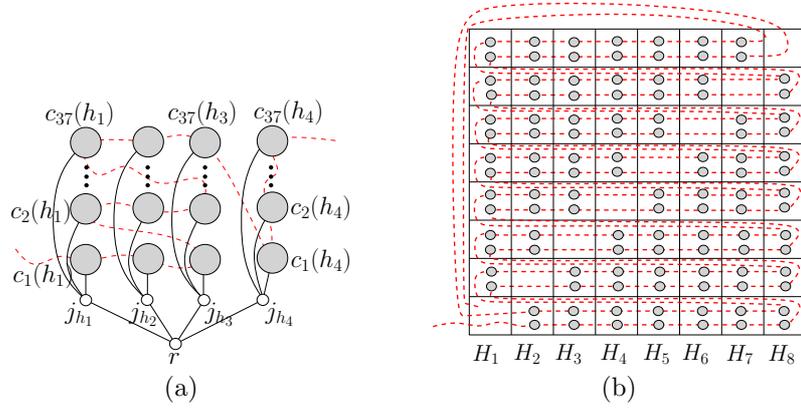


Figure 4: (a) A subsequence  $(h_1 h_2 h_3)^{37} h_4^{37}$  of a formation. (b) A subsequence  $(H_1, \dots, H_x)^{2^x}$  of an extended formation, with  $x = 8$ . Formations are placed inside a table in such a way that formations belonging to the same 4-tuple are in the same column and repetitions  $(H_1, \dots, H_x)$  in which the same 4-tuple is missing because of a defect are in the same row.

repetition the path does not reach any formation at  $H_m$ , with  $m \equiv k \pmod{x}$ . We say that the  $k$ -th repetition has a *defect* at  $m$ . Observe that in a subsequence  $(H_1, H_2, \dots, H_x)^x$ , that we call *full repetition*, there is one defect at each tuple. Thus, after  $y$  repetitions, there are  $y(x-1)/x$  formations used per tuple.

Note that the size of  $s$  can now be fixed as the number of formations creating repetitions inside one extended formation times the number of cells inside each of these formations, that is,  $s := (y - \frac{y}{x}) \cdot 37 \cdot 4$ . We claim that  $x = 7 \cdot 3^2 \cdot 2^{23}$  and  $y = 7^2 \cdot 3^3 \cdot 2^{26}$  is sufficient throughout the proofs. However, for readability reasons, we will keep on using variables  $x$  and  $y$  in the remainder of the paper.

**Sequence of Extended Formations:** Extended formations are connected by the path in a special sequence, called a *sequence of extended formations* and denoted by  $SEF(H)$ , where  $H = (H_1^*, \dots, H_{12}^*)$  is a 12-tuple of  $x$ -tuples of 4-tuples of disjoint indices of joints. For each  $x$ -tuple  $H_i^*$ , with  $i = 1, \dots, 12$ , consider 105 extended formations  $EF_j(H_i^*)$ , with  $j = 1, \dots, 105$ , not already belonging to any other sequence of extended formations. These extended formations are connected by  $\mathcal{P}$  in the order  $(H_1^*, \dots, H_{12}^*)^{120}$ , that is,  $\mathcal{P}$  connects  $EF_1(H_1^*)$  to  $EF_1(H_2^*)$ , then to  $EF_1(H_3^*)$ , and so on until  $EF_1(H_{12}^*)$ , then to  $EF_2(H_1^*)$ , to  $EF_2(H_2^*)$ , and so on until  $EF_{105}(H_{12}^*)$ .

There exist two types of sequences of extended formations, that are alternated in  $SEF$ . In the first type, in each repetition  $(H_1^*, \dots, H_{12}^*)$  one extended formation  $EF(H_m)$  is missing, as in the case of extended formations. In this case, we say that the repetition has a *defect* at  $m$ . In the second type, in each repetition  $(H_1^*, \dots, H_{12}^*)$  two consecutive extended formations are missing. Namely, in the  $k$ -th repetition the path skips the extended formations  $EF(H_m^*)$  and  $EF(H_{m+1}^*)$ , with  $m \equiv k \pmod{12}$  and  $m+1 = 1$  when  $m = 12$ . In this case,

we say that the repetition has a *double defect* at  $m$ . Thus, after 24 repetitions there are 21 formations used per tuple, which implies that after 120 repetitions each tuple has 105 formations.

Since we need a 12–tuple of  $x$ -tuples of 4–tuples of disjoint indices of joints, we can fix the number  $q$  of joints of  $\mathcal{T}$  as  $q = 48x$ .

## 4 $\mathcal{T}$ and $\mathcal{P}$ do not Admit any Geometric Simultaneous Embedding

In this section we present the main arguments leading to the final conclusion that the tree  $\mathcal{T}$  and the path  $\mathcal{P}$  described in Section 3 do not admit any geometric simultaneous embedding. For the sake of readability, we decided to give the outline of the proof in this section and to defer some of the longest proofs to Section 5.

The main idea in this proof scheme is to use the structures given by the path to fix a part of the tree in a specific shape creating restrictions for the placement of the further substructures of  $\mathcal{T}$  and of  $\mathcal{P}$  attached to it. Then, we show that such restrictions lead to a crossing in any possible drawing of  $\mathcal{P}$  and  $\mathcal{T}$ . In the following, we will perform an analysis of the geometrical properties of all possible embeddings in order to show that none of them is feasible. Hence, throughout the proof, we will assume that an embedding of the graph has been fixed and show that such an embedding determines a crossing.

We first give some further definitions and basic topological properties on the interaction among cells that are enforced by the preliminary arguments about region-level planar drawings and by the order in which the subtrees are connected inside one formation.

A *tree-route* is a path composed of edges of  $\mathcal{T}$ , while a *path-route* is a subpath of  $\mathcal{P}$ . We say that two cells  $c$  and  $c'$  are *separated* by a polyline  $l$  if  $l$  crosses all the tree-edges connecting vertices of  $c$  to vertices of  $c'$ .

**Passage:** Consider two cells  $c_1(h)$  and  $c_2(h)$  connected to a joint  $j_h$  that cannot be separated by a straight line. Further, consider a cell  $c'(h')$  connected to a joint  $j_{h'}$ , with  $h' \neq h$ . We say that  $c_1$  and  $c_2$  create a *passage*  $P$  with  $c'$  if the polyline given by the path-route connecting vertices of  $c'$  separates  $c_1$  and  $c_2$  (see Fig. 5). Alternatively, we also say that joints  $j_h$  and  $j_{h'}$  create a *passage*. Observe that, since  $c_1(h)$  and  $c_2(h)$  cannot be separated by a straight line, there exists at least one vertex of  $c'$  inside the convex hull of the vertices of  $c_1 \cup c_2$ , and there exist at least two path-edges  $e_1, e_2$  of  $c'$  that are intersected by tree-edges connecting vertices of  $c_1$  to vertices of  $c_2$ .

Let  $c_1(h_1)$  and  $c_2(h_1)$  be two cells creating a passage  $P_1$  with a cell  $c'(h'_1)$ , and let  $c_3(h_2)$  and  $c_4(h_2)$  be two cells creating a passage  $P_2$  with a cell  $c'(h'_2)$ , with  $h_1, h'_1 \neq h_2, h'_2$ . We distinguish three different configurations. Consider any linear order of the joints around the root, and restrict such an order to  $h_1, h'_1, h_2$ , and  $h'_2$ :

- If  $h_1$  and  $h'_1$  are the first and the second elements in such an order, then

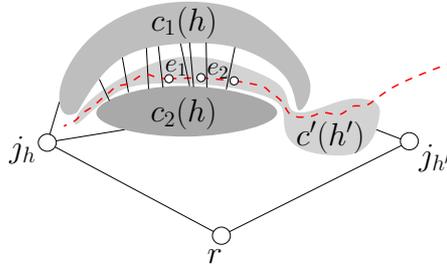


Figure 5: Two cells  $c_1$  and  $c_2$  creating a passage with a cell  $c'$ .

$P_1$  and  $P_2$  are *independent*;

- if  $h_1$  and  $h'_1$  are the first and the last elements in such an order, then  $P_2$  is *nested* into  $P_1$ ; and
- if  $h_1$  and  $h'_1$  are the first and the third (or the second and the fourth) elements in such an order, then  $P_1$  and  $P_2$  are *interconnected* (examples of interconnected passages are in Fig. 6).

In the following, in order to determine whether two passages are independent, nested, or interconnected, we will either explicitly describe the linear order of the joints around the root or, when presenting argumentations about some structures (formations, extended formations, and sequences of extended formations), we will implicitly assume the linear order given by such structures.

**Doors:** Let  $c_1(h)$  and  $c_2(h)$  be two cells creating a passage with a cell  $c'(h')$ . Consider any triangle given by a vertex  $v'$  of  $c'$  inside the convex hull of  $c_1 \cup c_2$  and by any two vertices of  $c_1 \cup c_2$ . This triangle is a *door* if it encloses neither any other vertex of  $c_1, c_2$  nor any vertex of  $c'$  belonging to the tree-route between  $v'$  and  $j_{h'}$ . A door is *open* if no tree-edge incident to  $v'$  crosses the opposite side of the triangle, that is, the side between the vertices of  $c_1$  and  $c_2$  (see Fig. 6(a)), otherwise it is *closed* (see Fig. 6(b)).

Observe that, if two passages  $P_1$  and  $P_2$  are interconnected, then either all the doors of  $P_1$  are traversed by a tree-route composed of edges of  $P_2$  or all the doors of  $P_2$  are traversed by a tree-route composed of edges of  $P_1$ . Suppose the former (see Figs. 6(a) and (b)). Then, as the polyline determined by the tree-route of  $P_2$  traversing all the doors of  $P_1$  can not cross tree-edges, it must traverse each door by crossing both the sides adjacent to  $v'$ . As shown in Fig. 6(b), if a door is closed then such a polyline has to bend after crossing one side adjacent to  $v'$  and before crossing the other one.

In the rest of the argument we will show that a closed door is present in each passage, which implies that the tree-route of  $P_1$  traversing all the doors of  $P_2$  creates at least one bend. Then, we will use further properties to show that a large part of  $\mathcal{T}$  has to create more than one bend. In view of this, we state the following lemmata relating doors, passages, and formations.

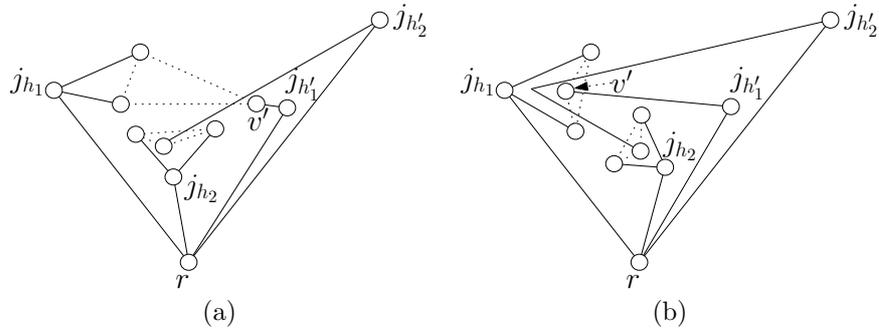


Figure 6: Two interconnected passages  $P_1$ , between  $j_{h_1}$  and  $j'_{h_1}$ , and  $P_2$ , between  $j_{h_2}$  and  $j'_{h_2}$ . Doors are represented as dotted lines. (a) The door of  $P_1$  is open. (b) The door of  $P_1$  is closed and a bend is needed in the poly-line determined by the tree-route of  $P_2$  traversing all the doors of  $P_1$ .

**Lemma 2** For each formation  $F(H)$ , with  $H = (h_1, \dots, h_4)$ , there exist two cells  $c_1(h_a), c_2(h_a) \in F(H)$  creating a passage with a cell  $c'(h_b) \in F(H)$ , with  $1 \leq a, b \leq 4$ .

**Lemma 3** Each passage contains at least one closed door.

**Proof:** Refer to Fig. 7. Let  $c_1(h)$  and  $c_2(h)$  be two cells creating a passage  $P_1$  with a cell  $c'(h')$ . Consider any vertex  $v$  of  $c'$  inside the convex hull of  $C := c_1 \cup c_2$ . Further, consider all the triangles  $\Delta(v, v_1, v_2)$  created by  $v$  with any two vertices  $v_1, v_2 \in C$  such that  $\Delta(v, v_1, v_2)$  does not enclose any other vertex of  $C$ . The tree-route connecting  $v$  to  $j_{h'}$  enters one of the triangles. Then, either it leaves the triangle on the opposite side, thereby creating a closed door, or it encounters a vertex  $v'$  of  $c'$ . Since at least one vertex of  $c'$  lies outside the convex hull of  $C$ , otherwise  $c_1(h)$  and  $c_2(h)$  would not be separated by  $c'(h')$ , it is possible to repeat the argument on triangle  $\Delta(v', v_1, v_2)$  until a closed door is found.  $\square$

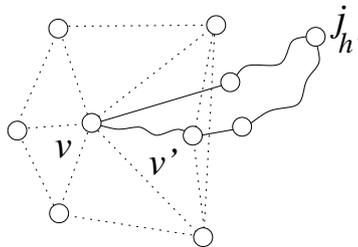


Figure 7: There exists a closed door in each passage.

Hence, each formation contains at least one closed door. In the following we prove that the effects of closed doors belonging to different formations can be combined to obtain more restrictions on the shape of the tree. First, we exploit a combinatorial argument based on the Ramsey Theorem [14] to state that there exists a set of joints such that any two joints in this set contain cells creating a passage.

**Lemma 4** *Given a set of joints  $J = \{j_1, \dots, j_q\}$ , with  $|J| = \binom{2^7 \cdot 3 \cdot b + 2}{3}$ , there exists a subset  $J' = \{j'_1, \dots, j'_k\}$ , with  $|J'| \geq 2^7 \cdot 3 \cdot b$ , such that for each pair of joints  $j'_i, j'_h \in J'$  there exist two cells  $c_1(i), c_2(i)$  creating a passage with a cell  $c(h)$ .*

**Proof:** By construction of the tree, for each 4-tuple of indices of joints, there exist formations that visit only cells of these joints. By Lemma 2, there exists a passage inside each of these formations, which implies that for each set of four joints there exists a subset of two joints creating a passage.

The number of joints needed to ensure the existence of a subset of joints  $J'$  of size  $k$  such that passages exist between each pair of joints is given by the Ramsey Number  $R(k, 4)$ . This number is defined as the minimal number of vertices of a graph  $G$  such that  $G$  either has a complete subgraph of size  $k$  or an independent set of size 4. Since in our case we can never have an independent set of size 4, we conclude that a subset of size  $k$  exists with the claimed property. The Ramsey number  $R(k, 4)$  is not exactly known, but we can use the upper bound directly extracted from the proof of the Ramsey theorem [14] to obtain the stated bound.  $\square$

Then, we give further definitions concerning the possible shapes of the tree.

**Enclosing bendpoints:** Consider two tree-routes  $p_1 = \{u_1, v_1, w_1\}$  and  $p_2 = \{u_2, v_2, w_2\}$ . The bendpoint  $v_1$  of  $p_1$  *encloses* the bendpoint  $v_2$  of  $p_2$  if  $v_2$  is internal to triangle  $\triangle(u_1, v_1, w_1)$ . See Fig. 8(a).

**Channels:** Consider a set of joints  $J = \{j_1, \dots, j_k\}$  in clockwise order around the root. The *channel*  $ch_i$  of a joint  $j_i$ , with  $i = 2, \dots, k - 1$ , is the region defined by a pair of tree-routes starting at  $r$ , one containing  $j_{i-1}$  and one containing  $j_{i+1}$ , with the maximum number of enclosing bendpoints with each other. We say that  $ch_i$  is an *m-channel* if the number of enclosing bendpoints is at least  $m$ . Observe that, by Prop. 1,  $m \leq 3$ . A 3-channel is depicted in Fig. 8(b). Note that, given an *m-channel*  $ch_i$  of  $j_i$ , all the vertices of the subtree rooted at  $j_i$  that are at distance at most  $m$  from the root lie inside  $ch_i$ .

**Channel segments:** An *m-channel*  $ch_i$  is composed of  $m + 1$  *channel segments*. The first channel segment  $cs_1$  is the part of  $ch_i$  that is visible from the root. The  $h$ -th channel segment  $cs_h$  is the part of  $ch_i$  disjoint from  $cs_{h-1}$  that is bounded by the elongations of the paths of  $j_{i-1}$  and  $j_{i+1}$  after the  $h$ -th bend. The *bending area*  $b(a, a + 1)$  of  $ch_i$  is the region that is visible from all the points of channel segments  $cs_a$  and  $cs_{a+1}$ .

Observe that, as the channels are delimited by tree-routes, any tree-edge connecting vertices inside the channel has to be drawn inside the channel, while

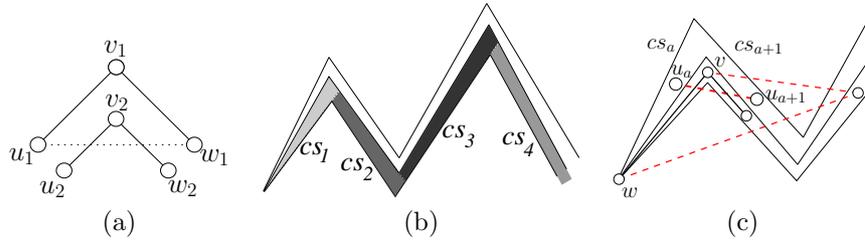


Figure 8: (a) An enclosing bendpoint. (b) A 3-channel and its channel segments. (c) A blocking cut  $(u_a, u_{a+1})$  where  $a = 1$ . As in Property 4, a vertex in a different channel segment is needed.

path-edges can cross the boundaries of the channel, hence possibly crossing other channels. We study the relationships between path-edges and channels.

The following property descends from the fact that, by construction, every second vertex reached by  $\mathcal{P}$  in a cell is either a 1-vertex or a stabilizer.

**Property 3** For any path-edge  $e = (a, b)$ , at least one of  $a$  and  $b$  lies inside either  $cs_1$  or  $cs_2$ .

**Blocking cuts:** A blocking cut is a path-edge connecting two consecutive channel segments by cutting some of the other channels twice. See Fig. 8(c).

**Property 4** Let  $ch$  be a channel that is cut twice by a blocking cut. If  $ch$  has vertices in both the channel segments cut by the blocking cut, then it has some vertices in a different channel segment.

**Proof:** Consider the vertices  $u_a$  and  $u_{a+1}$  lying in the two consecutive channel segments  $cs_a$  and  $cs_{a+1}$  of  $ch$  cut by the blocking cut. Observe that, at least one vertex  $v$  of the tree-route connecting  $u_a$  and  $u_{a+1}$  lies inside bending area  $b(a, a + 1)$ . Also, at least one vertex  $w$  of the tree-route connecting  $v$  to the root  $r$  of  $\mathcal{T}$  has to lie in the bending area  $b(a - 1, a)$  (note that, if  $a = 1$ ,  $w$  can be the root itself). Hence,  $v$  and  $w$  are separated by the blocking cut in  $cs_a$ . Since the path-route between  $v$  and  $w$  cannot cross the blocking cut, it has to pass through at least a vertex lying in a different channel segment.  $\square$

Now we are ready to prove that the subtrees connected to most of the joints create the same shape.

From now on, we identify a joint with the channel it belongs to. Then, when dealing with a passage between two joints  $j_h$  and  $j_{h'}$ , we might also say that there is a passage between the channels of  $j_h$  and  $j_{h'}$ .

First, based on Prop. 4, we show that any set of joints as in Lemma 4 contains a particular subset, composed of joints creating interconnected passages, such that each pair of tree-routes starting at  $r$  and containing such joints has at least two common enclosing bendpoints, which implies that most of them create 2-channels.

**Lemma 5** Consider a set of joints  $J = \{j_1, \dots, j_k\}$  such that there exists a passage between every two joints  $j_i, j_h$ , with  $1 \leq i, h \leq k$ . Let  $\mathcal{P}_1 = \{P \mid P \text{ is a passage between } j_i \text{ and } j_{\frac{3k}{4}+1-i}, \text{ for } i = 1, \dots, \frac{k}{4}\}$  and  $\mathcal{P}_2 = \{P \mid P \text{ is a passage between } j_{\frac{k}{4}+i} \text{ and } j_{k+1-i}, \text{ for } i = 1, \dots, \frac{k}{4}\}$  be two sets of passages between pairs of joints in  $J$  (see Fig. 9). Then, for at least  $\frac{k}{4}$  of the joints of one set of passages, say  $\mathcal{P}_1$ , there exist tree-routes with at least 2 and at most 3 bends, starting at the root and containing these joints, which traverse all the doors of  $\mathcal{P}_2$ . Also, at least  $\frac{k}{8}$  joints create a 2-channel.

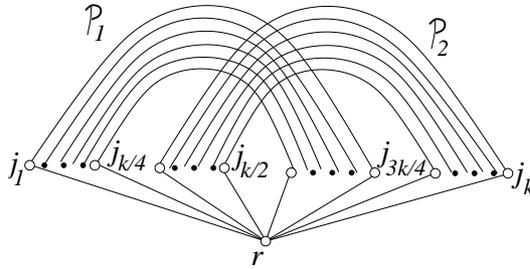


Figure 9: Two sets of passages  $\mathcal{P}_1$  and  $\mathcal{P}_2$  as described in Lemma 5.

By Lemma 5, any formation attached to a certain subset of joints creates channels with at least three channel segments. In the remainder of the argument we focus on this subset of joints and give some properties holding for it, in terms of interaction between different formations with respect to channels.

Since we need a full sequence of extended formations attached to these joints,  $k$  has to be at least eight times the number of channels inside a sequence of extended formations, that is,  $k \geq 8 \cdot 48b = 2^7 \cdot 3b$ .

**Nested formations** A formation  $F$  is *nested* in a formation  $F'$  if there exist four path-edges  $e_1, e_2 \in F$  and  $e'_1, e'_2 \in F'$  cutting a boundary  $cb$  of a channel  $ch$  such that all the vertices of the path-route in  $F$  between  $e_1$  and  $e_2$  lie inside the region delimited by  $cb$  and by the path-route in  $F'$  between  $e'_1$  and  $e'_2$  (see Fig. 10(a)). Since  $F'$  can also be nested in  $F$ , we say that two formations  $F_1$  and  $F_2$  are *nested* if  $F_1$  is nested in  $F_2$  or  $F_2$  is nested in  $F_1$  (or both hold).

A set of pairwise nested formations  $F_1, \dots, F_k$  have a nesting of *depth*  $d$  if there exist  $d$  formations  $F_{q_1}, \dots, F_{q_d}$ , with  $1 \leq q_1, \dots, q_d \leq k$ , such that the 4-tuples of  $F_{q_1}, \dots, F_{q_d}$  have at least one common joint  $j$ , and such that for each pair  $F_{q_p}, F_{q_{p+1}}$ , with  $1 \leq p < d$ , there exists at least one formation  $F_z$ , with  $1 \leq z \leq k$ , such that the 4-tuple of  $F_z$  does not contain  $j$ ,  $F_{q_p}$  is nested in  $F_z$  and  $F_z$  is nested in  $F_{q_{p+1}}$ . A set of formations with a nesting of depth 4 is depicted in Fig. 10(b).

**Independent sets of formations** Let  $S_1, \dots, S_k$  be sets of formations of one extended formation  $EF(H)$  such that each set  $S_i$ , for  $i = 1, \dots, k$ , contains formations  $F_i(H_1), \dots, F_i(H_q)$ , with  $(H_1, \dots, H_q) \subset H$ . Let  $F_a(H_c)$  and  $F_b(H_d)$  be not nested, for each  $1 \leq a, b \leq k$ ,  $a \neq b$ , and  $1 \leq c, d \leq q$ . Let  $cs_y$  and  $cs_{y+1}$

be two consecutive channel segments. If for every two sets  $S_a, S_b$  there exists a line  $l_y$  separating the vertices of  $S_a$  from the vertices of  $S_b$  inside  $cs_y$  and a line  $l_{y+1}$  separating the vertices of  $S_a$  from the vertices of  $S_b$  inside  $cs_{y+1}$ , then sets  $S_1, \dots, S_k$  are *independent* (see Fig. 10(c)).

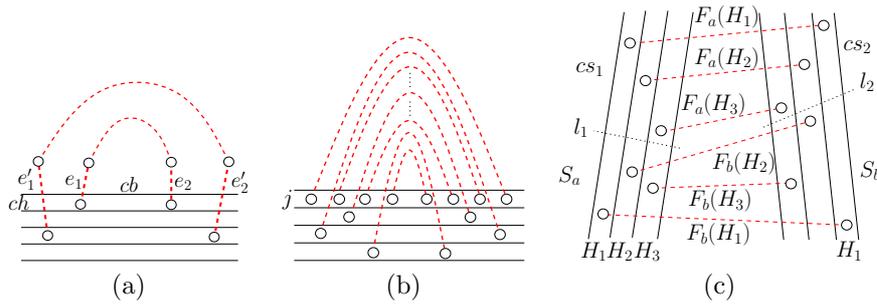


Figure 10: (a) A formation  $F$  nested in a formation  $F'$ . (b) A set of formations having a nesting of depth 4. (c) Two independent sets  $S_a$  and  $S_b$ .

In the following lemmata we prove that in any extended formation there exists a nesting of a certain depth (Lemma 8). This important property will be the starting point for the final argument and will be deeply exploited in the rest of the paper. We get to this conclusion by first proving that in an extended formation the number of independent sets of formations is limited (Lemma 6) and then by showing that, although there exist formations that are neither nested nor independent, in any extended formation there exists a certain number of pairs of formations that have to be either independent or nested (Lemma 7).

**Lemma 6** *No extended formation contains  $2^{22} \cdot 14$  independent sets of formations such that each set  $S_i$  contains formations  $F_i(H_1), \dots, F_i(H_q)$ , where  $q \geq 22$ .*

**Lemma 7** *Let  $EF$  be an extended formation and let  $Q_1, \dots, Q_4$  be four subsequences of  $EF$ , each consisting of a whole repetition  $(H_1, H_2, \dots, H_x)$ . Then, either there exists a pair of nested formations or two subsequences  $Q_i$  and  $Q_j$ ,  $i, j \in \{1, \dots, 4\}$ , are independent sets of formations.*

**Lemma 8** *For every extended formation  $EF$  there exists a nesting of depth  $d$ , with  $d \geq 6$ , among the formations of  $EF$ .*

Once the existence of 2-channels (Lemma 5) and of a nesting of a certain depth in each extended formation (Lemma 8) have been shown, we turn our attention to study how such a deep nesting can be performed inside the channels. In our discussion, we will get to the conclusion that, in any possible shape of the tree, either it is not possible to draw the formations creating the nesting without crossings, or that any planar drawing of such formations induces further

geometrical constraints that do not allow for a planar drawing of the rest of the tree.

We give some more formal definitions about the shapes of the channels. Let  $cs_a$  and  $cs_b$ , with  $1 \leq a, b \leq 4$ , be two channel segments of the same channel. If it is possible to connect from  $cs_a$  to  $cs_b$  by cutting either side of  $cs_b$ , then  $cs_a$  has a *2-side connection* to  $cs_b$  (see Fig. 11(b)). Otherwise, if only one side of  $cs_b$  can be used, then  $cs_a$  has a *1-side connection* to  $cs_b$  (see Fig. 11(a)). Note that,  $cs_a$  has a 2-side connection to  $cs_b$  if and only if  $cs_a$  and  $cs_b$  are not consecutive, and the elongation of  $cs_b$  intersects  $cs_a$ .

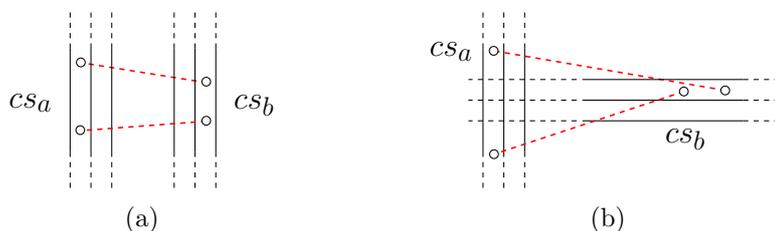


Figure 11: (a) Channel segment  $cs_a$  has a 1-side connection to  $cs_b$ . (b) Channel segment  $cs_a$  has a 2-side connection to  $cs_b$ .

We split our proof into three cases, based on whether only 1-side connections are possible (Proposition 1), at most one 2-side connection is possible (Proposition 2), or two 2-side connections are possible (Proposition 3). In all of such cases, we prove that a crossing is found in either  $\mathcal{T}$  or  $\mathcal{P}$ .

**Proposition 1** *If every two channel segments have a 1-side connection, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

Observe that an example of a shape in which only 1-side connections are possible is provided by the *M*-shape, depicted in Fig. 8(b).

We prove this proposition by showing that, in this configuration, the existence of a deep nesting in a single extended formation, proved in Lemma 8, results in a crossing in either  $\mathcal{T}$  or  $\mathcal{P}$ .

**Lemma 9** *If all the vertices of an extended formation lie inside channel segments that have only 1-side connections with each other, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Proof:** First observe that, by Lemma 8, there exists a nesting of depth  $d$ , with  $d \geq 6$ , in any extended formation  $EF$ . Consider two nested formations  $F, F' \in EF$  belonging to the nesting and the formation  $F'' \in EF$  not sharing any joint with  $F$  and  $F'$  such that  $F$  is nested in  $F''$  and  $F''$  is nested in  $F'$ . Since each pair of channel segments have a 1-side connection,  $F''$  blocks visibility for  $F'$  on the channel segment used by  $F$  for the nesting (see Fig. 12). Hence,  $F'$  has to use a different channel segment to perform its nesting, which increments

the number of used channel segments for each level of nesting. Since the tree supports at most 4 channel segments, the statement follows.  $\square$

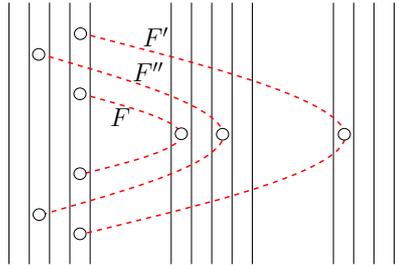


Figure 12: Illustration for the case in which only 1-side connections are possible.

Next, we study the case in which there exist 2-side connections. We distinguish two types of 2-side connections, based on whether the elongation of channel segment  $cs_a$  intersecting channel segment  $cs_b$  starts at the bendpoint that is closer to the root, or not. In the first case we have a *low Intersection*  $I_{(a,b)}^l$  (see Fig. 13(a)), while in the second case we have a *high Intersection*  $I_{(a,b)}^h$  (see Fig. 13(b)). We use notation  $I_{(a,b)}$  to describe both  $I_{(a,b)}^h$  and  $I_{(a,b)}^l$ . We say that two intersections  $I_{(a,b)}$  and  $I_{(c,d)}$  with  $a < c$  are *disjoint* if  $a, d \in \{1, 2\}$  and  $b, c \in \{3, 4\}$ . For example,  $I_{(1,3)}$  and  $I_{(4,2)}$  are disjoint, while  $I_{(1,3)}$  and  $I_{(2,4)}$  are not.

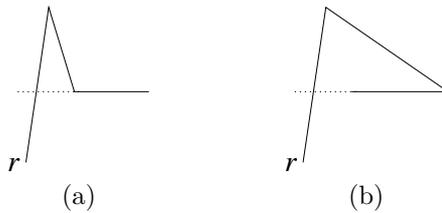


Figure 13: (a) A low Intersection  $I_{(3,1)}^l$ . (b) A high Intersection  $I_{(3,1)}^h$ .

Since consecutive channel segments cannot create 2-side connections, in order to explore all the possible shapes we consider all the combinations of low and high intersections created by channel segments  $cs_1$  and  $cs_2$  with channel segments  $cs_3$  and  $cs_4$ .

With the intent of proving that intersections of different channels have to maintain certain consistencies, we state the following lemma.

**Lemma 10** Consider two channels  $ch_p, ch_q$  with the same intersections. Then, none of channels  $ch_i$ , where  $p < i < q$ , has an intersection that is different from the intersections of  $ch_p$  and of  $ch_q$ .

**Proof:** The statement follows from the fact that the channel boundaries of  $ch_p$  and  $ch_q$  delimit the channel for all the joints between  $p$  and  $q$ . Hence, if any channel  $ch_i$ , with  $p < i < q$ , had an intersection different from the ones of  $ch_p$  and  $ch_q$ , either it would intersect with one of the channel boundaries of  $ch_p$  or  $ch_q$  or it would have to bend around one of the channel boundaries, hence crossing twice a straight line.  $\square$

As with Proposition 1, in order to prove that 2-side connections are not sufficient to obtain a simultaneous embedding of  $\mathcal{T}$  and  $\mathcal{P}$ , we exploit the existence of the deep nesting shown in Lemma 8.

Observe that every extended formation that uses a channel segment to place the nesting has to place vertices inside the adjacent bending area. In the following lemma we prove that not many of the formations involved in the nesting can use the part of the path that creates the nesting to do it, and hence they have to reach the bending area in a different way.

**Lemma 11** *Consider a nesting of formations of depth  $d \geq 6$  inside a sequence of extended formations on an intersection  $I_{(a,b)}$ , with  $a \leq 2$ . Then, one of the nesting formations contains a pair of path-edges  $(u, v)$ ,  $(v, w)$ , with  $v$  lying inside channel segment  $cs_a$ , that separates some of the formations in  $cs_a$  from the bending area  $b(a, a + 1)$  or  $b(a - 1, a)$  (see Fig. 14).*

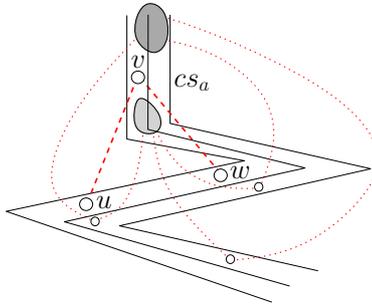


Figure 14: A situation as in Lemma 11. Inner and outer areas are represented by a light grey and a dark grey region, respectively.

Let the *inner area* and *outer area* of  $cs_a$  be the two parts in which  $cs_a$  is split by edges  $(u, v)$ ,  $(v, w)$ , as described in Lemma 11. Since in every extended formation  $EF$  there exists a path connecting the inner and the outer area by going around either vertex  $u$  or vertex  $w$ , we can infer that the extended formations using such paths create a structure that is analogous to the one created by the nested formations. Hence, because of the presence of a defect in every repetition of an extended formation, if only 1-side connections are available to host the vertices of such paths, then a crossing in  $\mathcal{T}$  or  $\mathcal{P}$  is created.

**Lemma 12** *Let  $cs_a$  be a channel segment that is split into its inner area and outer area by two edges in such a way that every extended formation of a sequence of extended formations SEF has vertices in both areas. If the only possibility to connect vertices from the inner to the outer area is with a 1-side connection, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

From Lemma 12 we conclude that having one single 2-side connection is not sufficient to obtain a geometric simultaneous embedding of the tree and the path. In the following we prove that a further 2-side connection is not useful if it is not disjoint from the first one.

**Proposition 2** *If there exists no pair of disjoint 2-side connections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

Observe that, in this setting, it is sufficient to restrict the analysis to cases  $I_{(1,3)}$  (see Figs. 15(a)–(b)) and  $I_{(3,1)}$  (see Figs. 16(a)–(b)), since the cases involving 2 and 4 can be reduced to them.

**Lemma 13** *If a shape contains an intersection  $I_{(1,3)}$  and does not contain any other intersection that is disjoint with  $I_{(1,3)}$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

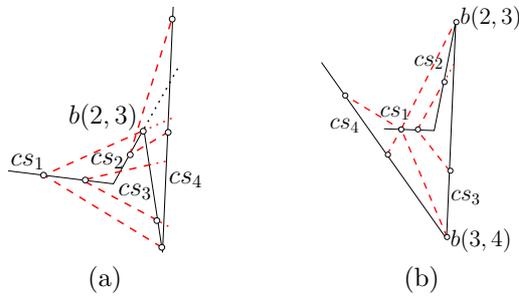


Figure 15: (a) Case  $I_{(1,3)} I_{(2,4)}^h$ . Since a nesting at  $I_{(1,3)}$  has to reach bending area  $b(2,3)$ , it crosses any nesting at  $I_{(2,4)}^h$ . (b) Case  $I_{(1,3)} I_{(2,4)}^l$ . A nesting at  $I_{(1,3)}$  crosses any nesting at  $I_{(2,4)}^l$ . Also, if there exist extended formations nesting at  $I_{(1,4)}$ , then they create a nesting also at  $I_{(1,3)}$ , as they have to reach  $b(2,3)$  and  $b(3,4)$ .

**Lemma 14** *If there exists a sequence of extended formations in any shape containing an intersection  $I_{(3,1)}$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

Observe that, in the latter lemma, we proved a property that is stronger than the one stated in Proposition 2. In fact, we proved that a simultaneous

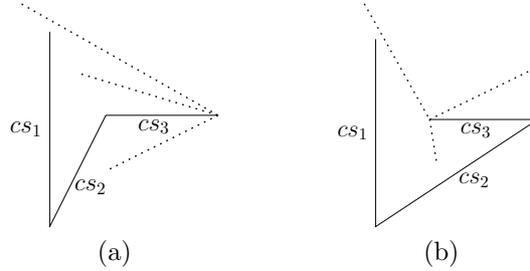


Figure 16: (a) Case  $I_{(3,1)}^l$ . (b) Case  $I_{(3,1)}^h$ . In both cases, if  $cs_4$  is not on the convex hull, then either  $cs_1$  or  $cs_2$  is on the convex hull. The possible placements of  $cs_4$  are represented by dotted lines.

embedding cannot be obtained in any shape containing an intersection  $I_{(3,1)}$ , even if a second intersection that is disjoint with  $I_{(3,1)}$  is present.

Finally, we tackle the general case where two disjoint intersections exist.

**Proposition 3** *If there exist two disjoint intersections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

Since the cases involving intersection  $I_{(3,1)}$  were considered in Lemma 14, we only have to consider the eight different configurations where one intersection is  $I_{(1,3)}$  and the other is one of  $I_{(4,\{1,2\})}^{h,l}$ . In the next three lemmata we cover the cases involving  $I_{(1,3)}^h$  and in Lemma 18 the ones involving  $I_{(1,3)}^l$ .

Consider two consecutive channel segments  $cs_i$  and  $cs_{i+1}$  of a channel  $ch$  and let  $e$  be a path-edge crossing the boundary of one of  $cs_i$  and  $cs_{i+1}$ , say  $cs_i$ . We say that  $e$  creates a *double cut* at  $ch$  if the line through  $e$  cuts  $ch$  in  $cs_{i+1}$ . A double cut is *simple* if the elongation of  $e$  cuts  $cs_{i+1}$  (see Fig. 17(a)) and *non-simple* if  $e$  itself cuts  $cs_{i+1}$  (see Fig. 17(b)). Also, a double cut of an extended formation  $EF$  is *extremal* with respect to a bending area  $b(a, a + 1)$  if there exists no double cut of  $EF$  that is closer than it to  $b(a, a + 1)$ . We can state for double cuts a property that is analogous to the one stated for blocking cuts.

**Property 5** *Any edge  $e$  creating a double cut at a channel  $ch_k$  in channel segment  $cs_i$  blocks visibility to the bending area  $b(i, i + 1)$  for a part of  $cs_i$  in each channel  $ch_h$  with  $h > k$  (with  $h < k$ ).*

In the following lemma we show that a particular ordering of extremal double cuts in two consecutive channel segments leads to a non-planarity in  $\mathcal{T}$  or  $\mathcal{P}$ . Note that an ordering of extremal double cuts corresponds to an ordering of the connections of a subset of extended formations to the bending area. Then, we will show that all shapes  $I_{(1,3)}^h$   $I_{(4,\{1,2\})}^{h,l}$  induce this order (Lemma 17).

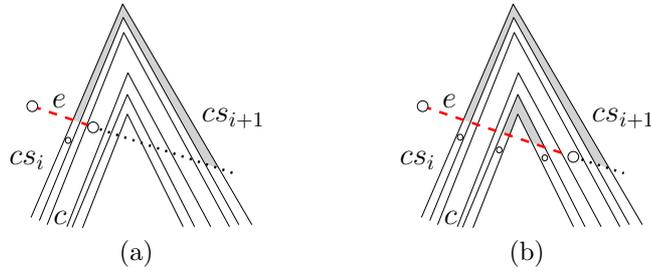


Figure 17: (a) A simple double cut. (b) A non-simple double cut.

**Lemma 15** *Let  $cs_i$  and  $cs_{i+1}$  be two consecutive channel segments. If there exists an ordered set  $S := (1, 2, \dots, 5)^3$  of extremal double cuts cutting  $cs_i$  and  $cs_{i+1}$  in such a way that the order of the intersections of the double cuts with  $cs_i$  (with  $cs_{i+1}$ ) is coherent with the order of  $S$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

First, we state the existence of double cuts in these shapes. While the existence of double cuts in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^l$  can be easily seen (see Fig 18(a)), in order to prove it in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^h$  we state the following lemma.

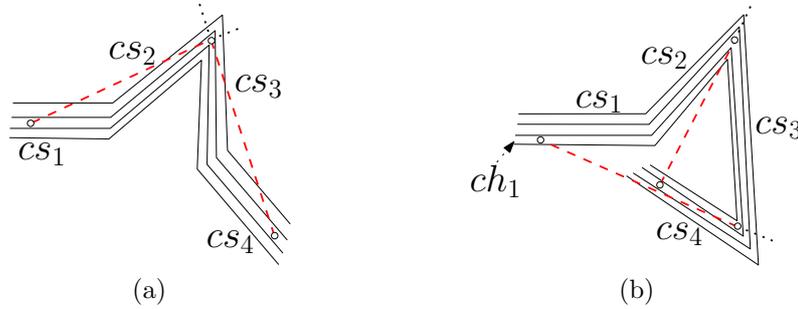


Figure 18: (a) Shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^l$  creates double cuts at  $b(2, 3)$ . (b) Shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^h$  creates double cuts.

**Lemma 16** *Each extended formation in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^h$  creates double cuts in at least one bending area.*

**Proof:** Refer to Fig. 18(b). Assume, without loss of generality, that the first bendpoint of channel  $ch_1$  encloses the first bendpoint of all the other channels. This implies that the second and the third bendpoints of channel  $ch_1$  are enclosed by the second and the third bendpoints of all the other channels, respectively.

Suppose, for a contradiction, that there exists no double cut in  $b(2, 3)$  and in  $b(3, 4)$ . Hence, any edge  $e$  connecting to  $b(2, 3)$  (to  $b(3, 4)$ ) is such that  $e$  and its elongation cut each channel once. Consider an edge connecting to  $b(2, 3)$  in a channel  $ch_i$ . Such an edge creates a triangle together with channel segments  $cs_3$  and  $cs_4$  of channel  $ch_i$  which encloses the bending areas  $b(3, 4)$  of all the channels  $ch_h$  with  $h < i$  by cutting such channels twice. Hence, a connection to such a bending area in one of these channels has to be performed from outside the triangle. However, since in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^h$  both  $b(2, 3)$  and  $b(3, 4)$  are on the convex hull, this is only possible with a double cut, a contradiction.  $\square$

Then, we show that the existence of a double defect in every repetition of an extended formation leads to the existence of the undesired ordering of extremal double cuts in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^{h,l}$ .

**Lemma 17** *Every sequence of extending formations in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^{h,l}$  contains an ordered set  $(1, 2, \dots, 5)^3$  of extremal double cuts with respect to bending area either  $b(2, 3)$  or  $b(3, 4)$ .*

Finally, we consider the configurations where one intersection is  $I_{(1,3)}^l$  and the other is one of  $I_{(4,2)}^{h,l}$ . We solve this cases by exploiting a geometrical property they exhibit, that is, that channel segment  $cs_2$  is on the convex hull of all such configurations.

**Lemma 18** *If channel segment  $cs_2$  is part of the convex hull, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

Based on the above discussion, we state the following theorem.

**Theorem 1** *There exist a tree and a path that do not admit any geometric simultaneous embedding.*

**Proof:** Let  $\mathcal{T}$  and  $\mathcal{P}$  be the tree and the path described in Section 3. Then, by Lemma 5, Lemma 10, and Property 1, a part of  $\mathcal{T}$  has to be drawn inside channels having at most four channel segments. Also, by Lemma 8, there exists a nesting of depth at least 6 inside each extended formation.

By Proposition 1, if there exist only 1-side connections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any simultaneous embedding. By Proposition 2, if there exists no pair of disjoint intersections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any simultaneous embedding. By Proposition 3, even if there exist two disjoint intersections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any simultaneous embedding. Since it is not possible to have more than two disjoint intersections, the statement follows.  $\square$

## 5 Detailed Proofs

In this section we give the details of the proofs of some of the lemmas and properties stated in Section 4.

### 5.1 Proof of Lemma 2

**Lemma 2.** *For each formation  $F(H)$ , with  $H = (h_1, \dots, h_4)$ , there exist two cells  $c_1(h_a), c_2(h_a) \in F(H)$  creating a passage with a cell  $c'(h_b) \in F(H)$ , with  $1 \leq a, b \leq 4$ .*

**Proof:** First observe that, by Property 2, there exists no set of four cells connected to the same joint inside  $F(H)$  that can be separated by straight lines creating linearly-separated regions. Hence, the cells of  $F(H)$  connected to the same joint, say  $j_{h_a}$ , can be grouped into at most 3 different sets  $S_{h_a}^1, S_{h_a}^2$ , and  $S_{h_a}^3$  such that cells from different sets can be separated by straight lines, but cells from the same set cannot. As any two cells  $c_1(h_a), c_2(h_a) \in F(H)$  can only be separated either by a straight-line or by a cell  $c_3(h_a)$  of the same joint  $j_{h_a}$ , every two cells inside one of these sets can only be separated by other cells of the same set.

Consider the connections of the path through  $F(H)$  with regard to these sets of cells. Observe that, for any two joints  $j_{h_q}, j_{h_{q+1}}$ , with  $1 \leq q \leq 4$ , there are nine possible ways to connect between a set  $S_{h_q}^p$ , with  $1 \leq p \leq 3$ , and a set  $S_{h_{q+1}}^{p'}$ , with  $1 \leq p' \leq 3$ . Then, since the part of  $\mathcal{P}$  through  $F(H)$  visits 37 times cells from  $j_{h_1}, j_{h_2}, j_{h_3}$ , in this order, there exist at least two sets  $S_{h_1}^p$  and  $S_{h_2}^{p'}$ , with  $1 \leq p, p' \leq 3$ , that are connected by at least five path-edges  $e_1, \dots, e_5$  (see Fig.19). Observe that edges  $e_1, \dots, e_5$ , together with the cells of  $S_{h_1}^p$  and of  $S_{h_2}^{p'}$ , subdivide the plane into five connected regions. Since the path is continuous in  $F(H)$ , it connects from the end of  $e_1$  (a cell of joint  $j_{h_2}$ ) to the beginning of  $e_2$  (a cell of joint  $j_{h_1}$ ), from the end of  $e_2$  to the beginning of  $e_3$ , and so on. If in the region between two edges  $e_s$  and  $e_{s+1}$ , with  $1 \leq s \leq 4$ , there exists no cell of joint  $j_{h_3}$ , then the path through  $F(H)$  will not traverse such a region in the opposite direction, since  $\mathcal{P}$  contains no edges going from a cell of  $j_{h_2}$  to a cell of  $j_{h_1}$ . Since there exist five edges between  $S_{h_1}^p$  and  $S_{h_2}^{p'}$  but at most 3 sets of cells on joint  $j_{h_3}$ , there exist at least two empty regions, which implies that the part of the path connecting  $e_s$  and  $e_{s+1}$  in a certain repetition of the formation creates a spiral, in the sense that it separates the cells connected to joint  $j_{h_4}$  in the previous repetitions from the analogous cells in the following repetitions.

Note that, having four repetitions would create a separation of four cells in  $j_{h_4}$  inside linearly-separated regions, hence determining, by Property 2, a crossing in  $\mathcal{T}$ . Therefore, at least two of such cells are not separated by a straight line, but are separated by the path. Hence, since the path of the spiral separating them can only consist of a cell belonging to joint  $j_{h_3}$ , a passage inside  $F(H)$  is created. □

### 5.2 Proof of Lemma 5

**Lemma 5.** *Consider a set of joints  $J = \{j_1, \dots, j_k\}$  such that there exists a passage between every two joints  $j_i, j_h$ , with  $1 \leq i, h \leq k$ . Let  $\mathcal{P}_1 = \{P \mid$*

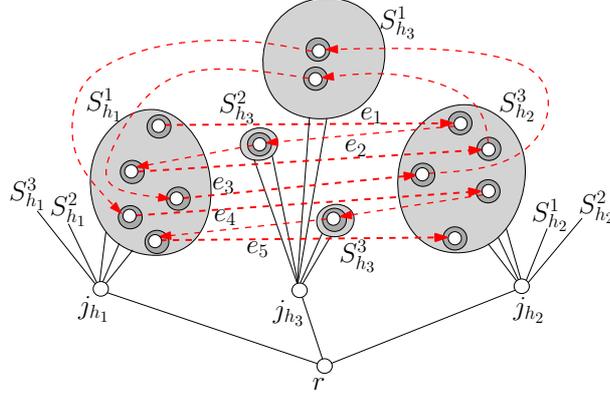


Figure 19: The five path-edges  $e_1, \dots, e_5$  connecting five cells of set  $S_{h_1}^a$  with five cells of set  $S_{h_2}^b$ .

$P$  is a passage between  $j_i$  and  $j_{\frac{3k}{4}+1-i}$ , for  $i = 1, \dots, \frac{k}{4}$  and  $\mathcal{P}_2 = \{P \mid P \text{ is a passage between } j_{\frac{k}{4}+i} \text{ and } j_{k+1-i}, \text{ for } i = 1, \dots, \frac{k}{4}\}$  be two sets of passages between pairs of joints in  $J$  (see Fig. 9). Then, for at least  $\frac{k}{4}$  of the joints of one set of passages, say  $\mathcal{P}_1$ , there exist tree-routes with at least 2 and at most 3 bends, starting at the root and containing these joints, which traverse all the doors of  $\mathcal{P}_2$ . Also, at least  $\frac{k}{8}$  joints create a 2-channel.

**Proof:** First observe that each passage of  $\mathcal{P}_1$  is interconnected with each passage of  $\mathcal{P}_2$  and that all the passages of  $\mathcal{P}_1$  (all the passages of  $\mathcal{P}_2$ ) are nested.

By Lemma 3 and Property 1, for one of  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , say  $\mathcal{P}_1$ , either for every joint of  $\mathcal{P}_1$  between the joints of  $\mathcal{P}_2$  in the order around the root or for every joint of  $\mathcal{P}_1$  not between the joints of  $\mathcal{P}_2$ , there exists a tree-route  $p_i$ , starting at the root and containing these joints, that has to traverse all the doors of  $\mathcal{P}_2$  by making at least 1 and at most 3 bends. Also, tree-routes  $p_1, \dots, p_{\frac{k}{4}}$  can be ordered in such a way that a bendpoint of  $p_i$  encloses a bendpoint of  $p_h$  for each  $h > i$ . It follows that each joint has a 1-channel. Consider now the set of joints  $J' \subset J$  visited by these paths. We assume the joints of  $J' = \{j'_1, \dots, j'_q\}$  to be in this order around the root.

Consider the path  $p_1$  whose bendpoint encloses the bendpoint of each of all the other paths and the path  $p_q$  whose bendpoint encloses the bendpoint of none of the other paths (see Figs. 20(a) and 20(b)). Please note that either  $p_1$  visits  $j'_1$  and  $p_q$  visits  $j'_q$  or vice versa, say  $p_1$  visits  $j'_1$ . By construction, there exists a passage between cells from  $j'_1$  and cells from  $j'_q$ . In this passage, there exist either two path-edges  $e_1, e_2$  of a cell  $c'(1)$  separating two cells  $c_1(q), c_2(q)$ , thereby crossing the channel of  $j'_q$ , or two edges of a cell  $c'(q)$  separating two cells  $c_1(1), c_2(1)$ , thereby crossing the channel of  $j'_1$ . We show that 1-channels are not sufficient to draw these passages.

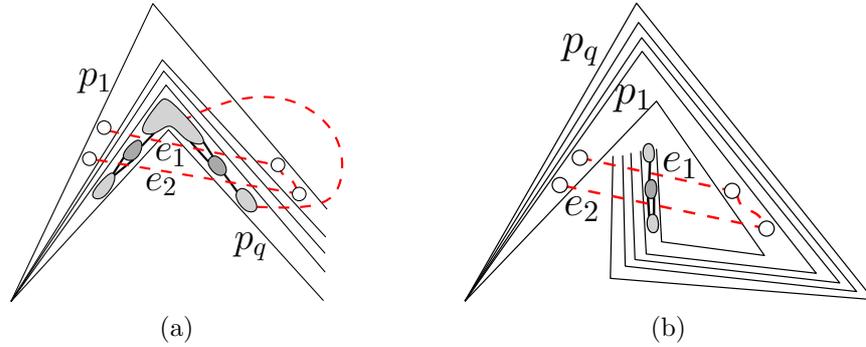


Figure 20: (a) The separating cell  $c'$  is in the outermost channel. (b) The separating cell  $c'$  is in the innermost channel.

In the first case (see Fig. 20(a)), both separating edges  $e_1, e_2$  cross the path  $p_q$  before and after the bend, thereby creating blocking cuts separating vertices of the same cell, say  $c_1$ . By Property 4, an additional bend is needed.

In the second case (see Fig. 20(b)), no edge connecting vertices of  $c'$  ( $j'_q$ ) crosses edges of  $p_1$ , and therefore at least another bend is needed.

Hence, at least one of the channels needs an additional bend. Since there are passages between each pair of joints in  $J'$ , all but one joint  $j^*$  have a path that has to bend an additional time. We note that the additional bendpoint of each path  $p_k$  aside from  $p_1, p_r$ , and  $p^*$  has to enclose all the additional bendpoints either of  $p_1, \dots, p_{k-1}$  or of  $p_{k+1}, \dots, p_q$ . It follows that, for at least half of the  $\frac{k}{4}$  joints, there exist 2-channels.  $\square$

### 5.3 Proofs of Lemmata 6, 7, and 8

**Lemma 6.** *No extended formation contains  $2^{22} \cdot 14$  independent sets of formations such that each set  $S_i$  contains formations  $F_i(H_1), \dots, F_i(H_q)$ , where  $q \geq 22$ .*

**Proof:** Suppose that such independent sets  $S_1, \dots, S_z$  exist. We show that this induces a crossing in either  $\mathcal{T}$  or  $\mathcal{P}$ . By Lemma 2, each formation contains a passage, and thereby an edge cutting the boundary of a channel. By Property 3, every edge has an end-vertex either in channel segment  $cs_1$  or in  $cs_2$ . Hence, for each 4-tuple  $t$ , the formations placed in  $t$  in at least  $z/2$  sets of formations have a common connection, that is, they have connections to vertices in the same channel segment, either  $cs_1$  or  $cs_2$ . Let  $S^1 = \{S_p, \dots, S_q\}$ , with  $1 \leq p < q \leq z$  and  $q - p \geq z/2$ , be the set of set of formations containing such independent sets.

By using the same argument we can find a subset  $S^2 \subset S^1$  of size  $\frac{z}{4}$  such that, for each pair of 4-tuples  $t, t'$ , the sets belonging to  $S^2$  have at least two common connections. By continuing this procedure we arrive at a subset  $S^q$

containing at least  $\frac{z}{2^q}$  sets having at least  $q$  common connections. Since all these common connections have to connect to either  $cs_1$  or  $cs_2$ , we have identified a set  $S = \{S'_1, \dots, S'_{\frac{z}{2^q}}\}$  of size  $\frac{z}{2^q}$  of sets of formations of size at least  $\frac{q}{2}$  that has all its connections to the same channel segment  $cs$ .

We now consider, for each of the formations of  $S$ , the edges cutting the boundary of  $cs$ . Since any of those edges can intersect the channel boundary on two different sides, for every formation  $F_{\frac{q}{4}}$  in each set  $S'_i$ , at least half of such edges cut the same side of the channel, thereby crossing either all the channels  $1, \dots, \frac{q}{4} - 1$  or all the channels  $\frac{q}{4} + 1, \dots, \frac{q}{2}$ , say the former.

Consider now the formations  $F_{\frac{q}{8}}$  in each of the sets. These formations in the sets  $S'_2, S'_4, \dots, S'_{\frac{z}{2^{q+1}}}$  are separated in  $cs$  by the edges of the formations  $F_{\frac{q}{4}}$  of the sets  $S'_3, S'_5, \dots, S'_{\frac{z}{2^q}-1}$ . To avoid a linearly-separated ordering of the separated formations and thereby the existence of a region-level nonplanar tree, formations  $F_{\frac{q}{8}}$  have to place vertices in an adjacent channel segment  $cs'$ . However, in this way they create blocking cuts for either all the channels  $1, \dots, \frac{q}{8} - 1$  or all the channels  $\frac{q}{8} + 1, \dots, \frac{q}{4}$ , say the former.

Consider now the formations  $F_1$  in each of the sets. These formations in the sets  $S'_3, S'_5, \dots, S'_{\frac{z}{2^q}-2}$  are separated in  $CS$  by the edges of the formations  $F_{\frac{q}{8}}$  of the sets  $S'_4, S'_6, \dots, S'_{\frac{z}{2^q}-3}$ . By the same argument as above, also these formations have to place vertices in an adjacent channel segment that is visible from some of the separated areas of  $cs$ . Since the connections of the formations  $F_{\frac{q}{8}}$  block visibility for the connections to  $cs'$ , the formations  $F_1$  have to use the other adjacent channel segment  $cs''$ , thereby blocking all the channels  $ch_1, \dots, ch_{q_2}$ .

Finally, consider the formations  $F_2$  in the sets  $S'_4, S'_6, \dots, S'_{10}$ . These formations are now separated in  $cs$  by the edges connecting formations  $F_{\frac{q}{8}}$  to  $cs'$  and by the edges connecting formations  $F_1$  to  $cs''$ . Therefore, these formations cannot use any channel segment other than  $cs$ , which implies that they are linearly-separated inside such a channel segment. So, by Property 2, there exists a region-level nonplanar tree.  $\square$

**Lemma 7.** *Let  $EF$  be an extended formation and let  $Q_1, \dots, Q_4$  be four subsequences of  $EF$ , each consisting of a whole repetition  $(H_1, H_2, \dots, H_x)$ . Then, either there exists a pair of nested formations or two subsequences  $Q_i$  and  $Q_j$ ,  $i, j \in \{1, \dots, 4\}$ , are independent sets of formations.*

**Proof:** Assume that no pair of nested formations exists. We show that there exists two subsequences that are independent sets.

First, consider how  $Q_1, \dots, Q_4$  use the first two channel segments  $cs_1$  and  $cs_2$  to place their formations. Each of these subsequences uses either only  $cs_1$ , only  $cs_2$ , or both. Observe that, if a subsequence uses only  $cs_1$  and another one uses only  $cs_2$ , then such subsequences are clearly independent sets. So, at least one channel segment, say  $cs_2$ , is used by all of  $Q_1, \dots, Q_4$ .

Then, we show that if there exist three subsequences that use only  $cs_2$ , then at least two of them are separated in  $cs_2$ . In fact, consider two subsequences



Since the number of channels used inside  $EF$  is greater than  $(z - 1) \cdot 3 \cdot 3$ , where  $z \geq 2^{22} \cdot 14$ , we have a nesting consisting of repetitions of formations in  $EF$  with at least 3 different defects.

Let the nesting consist of repetitions  $Q_1^1, \dots, Q_1^q, Q_2^1, \dots, Q_2^q, \dots, Q_k^1, \dots, Q_k^q$ , where  $Q_i^h$  denotes the  $h$ -th occurrence of a repetition of  $EF$  with a defect at the 4-tuple  $H_i$ . Further, let  $\mathcal{P}$  connect such repetitions in the order  $Q_1^1, Q_2^1, \dots, Q_k^1, Q_1^2, \dots, Q_k^2, \dots, Q_k^q$ . We show that there exists a pair of independent formations within this nesting.

Consider now the first two nesting repetitions  $Q_1^1$  and  $Q_2^1$ . Let the nesting consist of a formation  $F(H_k)$  from  $Q_1^1$  nesting in a formation  $F'(H_s)$  from  $Q_2^1$ , where  $1 \leq k, s \leq x$ . Consider the edges  $e_1, e_2 \in F(H_k)$  and  $e'_1, e'_2 \in F'(H_s)$  determining the nesting. Assume, without loss of generality, that the path-route  $p$  between  $e'_2$  and  $e_2$  does not contain  $e'_1$  and  $e_1$ . Consider the two parts  $a$  and  $b$  of the channel boundary that is cut by all such edges, where  $a$  is between  $e_1$  and  $e'_1$  and  $b$  is between  $e_2$  and  $e'_2$ . Consider now the closed region delimited by the path-route through  $F'(H_s)$ , path-route  $p$ , the path-route through  $F(H_k)$ , and  $a$ . Such a region is split into two closed regions  $R_{in}$  and  $R_{nest}$  by  $b$  (see Fig. 22).

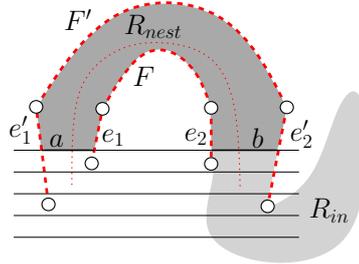


Figure 22: Regions  $R_{in}$  and  $R_{nest}$ .

Observe that, in order to go from  $R_{in}$  to the outer region, any path-route has to cross both  $a$  and  $b$  by using a vertex inside  $R_{nest}$ . We note that the part of  $\mathcal{P}$  starting at  $e'_1$  and not containing  $F(H_k)$  is either completely contained in the outer region or has to cross over between  $R_{in}$  and the outer region by traversing  $R_{nest}$ . Similarly, the part of  $\mathcal{P}$  starting at  $e_1$  and not containing  $F'(H_s)$  either does not reach the outer region or has to cross over between  $R_{in}$  and the outer region by traversing  $R_{nest}$ . Furthermore, any formation  $F''$  using such a path-route either crosses over, thereby cutting both  $a$  and  $b$ , or it does not enter  $R_{in}$  at all. Observe that, in the first case,  $F$  is nested in  $F''$  and  $F''$  is nested in  $F'$ .

Consider now the third nesting repetition  $Q_3^1$  of sequence  $(H_1, H_2, \dots, H_x)$  (see Figs. 23(a) and 23(b)). It is easy to see that, if  $Q_3^1$  is nested between  $Q_1^1$  and  $Q_2^1$ , then there exists a nesting of depth 1, as  $Q_3^1$  contains a defect at a different 4-tuple. Hence, we have only to consider the cases in which the remaining repetitions create the nesting by creating a spiral, that is, by strictly going either outward or inward. By this we mean that the  $i$ -th repetition  $Q_i^1$

has to be placed such that either  $Q_i^1$  is nested inside  $Q_{i-1}^1$  (inward) or vice versa (outward). Without loss of generality, we assume the latter (see Fig. 23(c)).

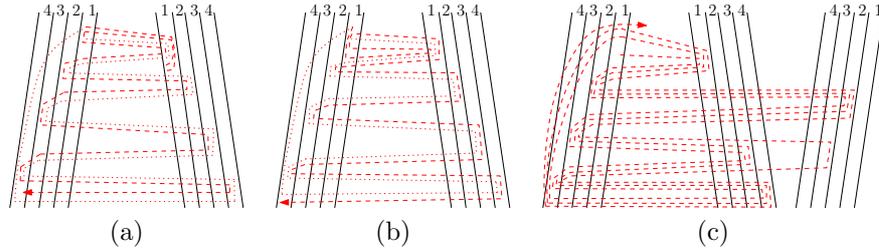


Figure 23: (a) and (b) Possible configurations for  $Q_1^1$  and  $Q_2^1$  when  $Q_3^1$  is considered. Repetition  $Q_1^1$  is represented by dotted (red) lines in order to distinguish it from  $Q_2^1$  (dashed (red) lines). (c) The repetitions follow the outward orientation.

Consider now a defect in a 4-tuple  $H_c$ , with  $1 < c < k$ , at a certain repetition  $Q_i^h$ . Since the path is moving outward, the connection between  $H_{c-1}$  and  $H_{c+1}$  blocks visibility for the following repetitions to the part of the channel segment where vertices of  $H_c$  were placed until that repetition (see Fig. 24(a) for an example with  $c = 3$ ).

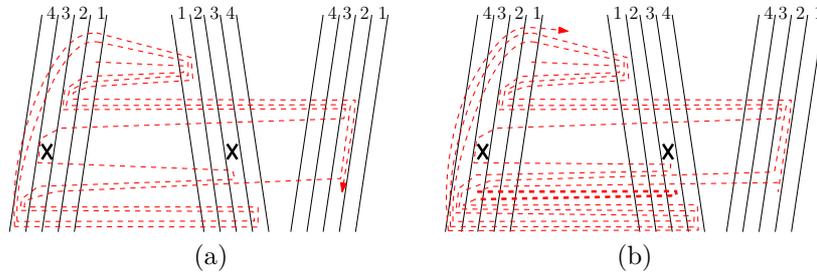


Figure 24: (a) The connection between channels  $ch_2$  and  $ch_4$  blocks visibility for the following repetitions to the part of the channel segment where vertices of channel  $ch_3$  were placed until that repetition. (b) A shift in which the vertices of channel  $ch_3$  (fat lines) are placed in the same part of the channel segment where vertices of channel  $ch_4$  were placed until that repetition.

A possible placement for the vertices of  $H_c$  in the following repetitions that does not increase the depth of the nesting could be in the same part of the channel segment where vertices of a 4-tuple  $H_{c'}$ , with  $c' \neq c$ , were placed until that repetition. We call *shift* such a move. However, in order to place vertices of  $H_c$  and of  $H_{c'}$  in the same zone, all the vertices of  $H_c$  belonging to the current cell have to be placed there (see fat lines in Fig. 24(b), where  $c' = c + 1$ ), which implies that a further defect at  $H_c$  in one of the following repetitions encloses all the vertices of each of the previously drawn cells, hence separating them

with a straight line from the following cells. Hence, also the vertices of  $H_{c'}$  have to perform a shift to a 4-tuple  $H_{c''}$ , with  $c \neq c'' \neq c'$ . Again, if the vertices of  $H_{c'}$  and of  $H_{c''}$  lie in the same zone, we have two cells that are separated by a straight line, and hence also the vertices of  $H_{c''}$  have to perform a shift. By repeating such an argument we conclude that the only possibility for not having vertices of different 4-tuples lying in the same zone is to shift all the 4-tuples  $H_c, \dots, H_x$  and to go back to  $H_1$  for starting the following repetition in a completely different region (see Fig. 25, where the following repetition is performed completely below the previous one). However, this implies that there exist two repetitions in one configuration that have to be separated by a straight line and therefore are independent, in contradiction to our assumption. Hence, after  $3 \cdot x + 1$  repetitions, we arrive at a nesting of depth 1. By repeating this argument, after  $6 \cdot 3 \cdot x + 1$  repetitions we obtain a nesting of depth 6, thus contradicting the initial assumption.  $\square$

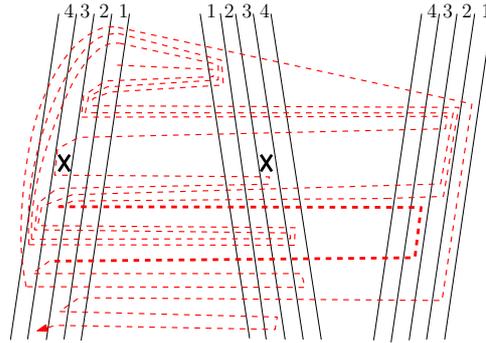


Figure 25: All the channels hosting 4-tuples  $H_c, \dots, H_x$  are shifted and the next repetition starts in a completely different region.

### 5.4 Proof of Lemma 11

**Lemma 11.** *Consider a nesting of formations of depth  $d \geq 6$  inside a sequence of extended formations on an intersection  $I_{(a,b)}$ , with  $a \leq 2$ . Then, one of the nesting formations contains a pair of path-edges  $(u, v), (v, w)$ , with  $v$  lying inside channel segment  $cs_a$ , that separates some of the formations in  $cs_a$  from the bending area  $b(a, a + 1)$  or  $b(a - 1, a)$  (see Fig. 14).*

**Proof:** Consider three extended formations  $EF_1(H_1), EF_2(H_1), EF_3(H_1)$  lying in a 4-tuple of channels  $H_1$  and two extended formations  $EF_1(H_2), EF_2(H_2)$  lying in a 4-tuple of channels  $H_2$  such that all the channels of the sequence of extended formations are between  $H_1$  and  $H_2$  and there is no formation  $F \notin EF(H_1), EF(H_2)$  nesting between  $EF_1(H_1), EF_2(H_1), EF_3(H_1)$  and  $EF_1(H_2), EF_2(H_2)$ . Suppose, without loss of generality, that the bending point of  $H_1$  is

enclosed into the bending point of  $H_2$ .

Refer to Fig. 26(a). Consider a formation  $F_1 \in EF_1(H_1)$  nesting with a formation  $F'_1 \in EF_1(H_2)$ . The connections from  $F'_1$  to channel segment  $cs_a$  and back have to go around the vertex placed by  $F_1$  on channel segment  $cs_a$ . Therefore, at least one of the connections of  $F'_1$  cuts all the channels between  $H_1$  and  $H_2$ , that is, all the channels where the sequence of extended formations is placed. Such a connection separates the vertices of  $F_1$  from the vertices of a formation  $F_2 \in EF_2(H_1)$  in  $cs_a$ . Therefore, at least one of the connections of  $F_2$  to  $cs_a$  cuts either all the channels in  $cs_a$  or all the channels in  $cs_{a+1}$  (or  $cs_{a-1}$ ), hence becoming a blocking cut for such channels. It follows that all the formations nesting inside  $F_2$  on such channels cannot place vertices in the bending area  $b(a, a + 1)$  (or  $b(a - 1, a)$ ) outside  $F_2$ .  $\square$

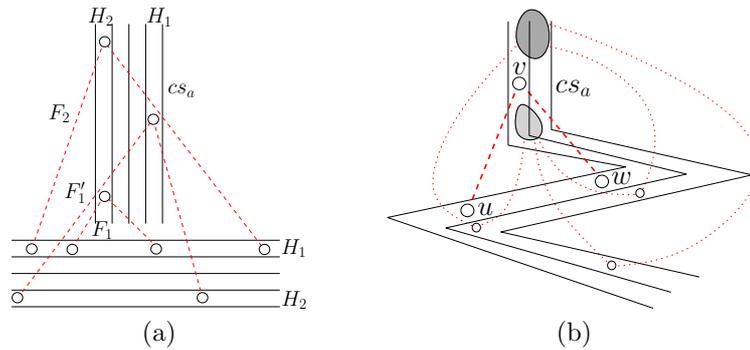


Figure 26: (a) The connections from  $F_1$  to  $cs_a$  enforce  $F'_1$  to cut all the channels and the connections from  $F'_1$  to  $cs_a$  enforce  $F_2$  to cut all the channels. (b) A pair of path-edges  $(u, v)$ ,  $(v, w)$ , with  $v$  lying inside channel segment  $cs_a$ , separates some of the formations in  $cs_a$  from the bending area  $b(a, a + 1)$  or  $b(a - 1, a)$ . The chosen turning vertex is represented by a big black circle and is in configuration  $\beta$ . The inner and the outer areas are represented by a light grey and a dark grey region, respectively.

### 5.5 Proof of Lemma 12

**Lemma 12.** *Let  $cs_a$  be a channel segment that is split into its inner area and outer area by two edges in such a way that every extended formation of a sequence of extended formations SEF has vertices in both areas. If the only possibility to connect vertices from the inner to the outer area is with a 1-side connection, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

In order to prove Lemma 12, we first need to introduce some definitions and to prove an auxiliary lemma.

Refer to Fig. 26(b). Let  $(u, v)$  and  $(v, w)$  be a pair of edges separating a channel segment  $cs_a$  into its inner and outer area. We call such a pair the *open triangle* of the extended formation. Assume that  $(u, v)$  and  $(v, w)$  do not cut any channel segment  $cs_b$  completely, since such a cut would create more restrictions than placing  $u$  or  $w$  inside  $cs_b$ . For each path of an extended formation connecting vertices in the inner area to vertices in the outer area, consider a vertex, called *turning vertex*, which is placed in  $cs_b$  and for which no other path in  $EF$  exists that connects the inner and the outer area by using a channel segment  $cs_c$  such that the subpath to  $cs_c$  intersects either  $cs_c$  or its elongation. If there exist more than one of such vertices, then arbitrarily choose one of them. Observe that the path connecting from the inner area to the outer area through the turning vertex encloses exactly one of  $u$  and  $w$ . If it encloses  $u$ , it is in configuration  $\alpha$ , otherwise it is in configuration  $\beta$ . If there exist both paths in  $\alpha$  and paths in  $\beta$  configuration, then arbitrarily consider one of them. Finally, consider the connections between different extended formations inside a sequence of extended formations. Consider a turning vertex  $v$  in a channel segment  $cs_a$  of a channel  $ch$  such that the edges incident to  $v$  cut a channel  $ch'$ . Then, any connection of an extended formation of  $ch'$  from the inner to the outer area in the same configuration as  $ch$  and with its turning vertex  $v'$  in  $cs_a$  is such that  $v'$  lies inside the convex hull of the open triangle of the extended formation lying in channel  $ch$ .

**Lemma 19** *If the connection between the inner and the outer area can be realized only through a 1-side connection, then not all the extended formations in a sequence of extended formations can place turning vertices in the same channel segment.*

**Proof:** Assume, for a contradiction, that all the turning vertices are in the same channel segment. Consider a sequence of extended formations  $SEF(H)$ , where  $H = (H_1^*, \dots, H_{12}^*)$ , and the extended formations  $EF_j(H_i^*) \in SEF$ , with  $j = 1, \dots, 110$ .

We first show that in  $SEF$  there exist some extended formations  $EF_j(H_i^*) \in SEF$  using connections in  $\alpha$  configuration and some using connections in  $\beta$  configuration. Consider the continuous subsequence of extended formations  $EF_1(H_1^*), \dots, EF_1(H_3^*)$  in  $SEF$ . Assume that all the turning vertices of these extended formations are in  $\alpha$  configuration. Consider a further subsequence  $EF_p(H_1^*), \dots, EF_p(H_3^*)$ , with  $1 \leq p < 110$ , of  $SEF$  with a defect at  $H_2^*$ . Then, the connection between  $EF_p(H_1^*)$  and  $EF_p(H_3^*)$  crosses the channels in  $H_2^*$ , thereby blocking any extended formation  $EF_q(H_2^*)$ , with  $p < q \leq 110$ , from being in  $\alpha$  configuration. Hence, when considering another subsequence of  $SEF$  on the same set of channels which does not contain defects at  $H_1^*, \dots, H_3^*$ , either the extended formation placed in  $H_2^*$  is in  $\beta$  configuration or it uses another channel segment to place the turning vertex.

Consider the open triangle of each extended formation. Note that, if an open triangle of an extended formation  $EF(H_k)$  is inside the inner area of an extended formation  $EF(H_s)$  and the open triangle of  $EF(H_s)$  is inside the inner

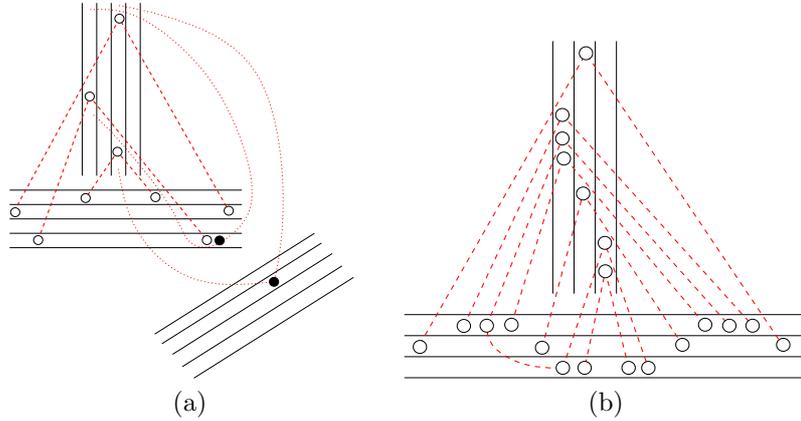


Figure 27: (a) Two triangles from the same channel have to use different channel segments if a triangle of another channel is between them. Turning vertices are represented by black circles. (b) When a defect at  $H_2$  is encountered, the connection between  $EF(H_1)$  and  $EF(H_3)$  does not permit the following  $EF(H_2)$  to respect the ordering of triangles.

area of an extended formation  $EF'(H_k)$ , with  $k < s$ , then  $EF(H_k)$  has to use a different channel segment to place its turning vertex (see Fig. 27(a)). Hence, the open triangles have to be ordered according to the order of the sets of channels used by the extended formations.

We show that it is not possible for the open triangles of all the extended formations to be nested in such an order.

First observe that, if the continuous path connecting two open triangles  $t_1 = (u, v, w)$  and  $t_2 = (u', v', w')$  of two consecutive extended formations  $EF(H_s)$  and  $EF(H_{s+1})$  connects vertex  $u$  to vertex  $w'$  (or  $u'$  to  $w$ ) via the outer area of  $EF(H_{s+1})$ , then an open triangle of an extended formation  $EF(H_1)$  that occurs before  $EF(H_s)$  and an open triangle of an extended formation  $EF'(H_1)$  that occurs after  $EF(H_{s+1})$  are nested with the open triangle given by the connection of  $t_1$  and  $t_2$  in an ordering different from the order of the channels.

Further, consider two  $x$ -tuples  $H_1^*$  and  $H_2^*$  of 4-tuples of indices of channels such that there exists an extended formation  $EF_p(H_1^*)$  in  $\alpha$  configuration and an extended formation  $EF_p(H_2^*)$  in  $\beta$  configuration, and consider two further extended formations  $EF_p(H_3^*)$  and  $EF_p(H_4^*)$  directly following  $EF_p(H_1^*)$  and  $EF_p(H_2^*)$  in  $SEF$ . Also, consider the first extended formation  $EF_{p+1}(H_1^*)$  on the set of channels  $H_1^*$  following  $EF_p(H_4^*)$  in  $SEF$ . Consider now the first repetition  $q$  of  $SEF$  after repetition  $p + 1$  having a defect at  $H_2^*$ . As  $EF_p(H_1^*)$  is in  $\alpha$  configuration and  $EF_p(H_2^*)$  is in  $\beta$  configuration, the connection of  $EF_q(H_1^*)$  to  $EF_q(H_3^*)$  in this repetition blocks access for all the extended formations  $EF_m(H_2^*)$ , with  $m > q$ , to the area where it should place vertices in order to respect the ordering of open triangles (see Fig. 27(b)). Therefore, after 3 full

repetitions of the sequence in  $SEF$ , at least one extended formation has to use a different channel segment to place its turning vertex.  $\square$

Now we are ready to prove the claimed lemma.

**Proof of Lemma 12** Let  $EF_p(H_{12}^*)$  and  $EF_{p+1}(H_1^*)$  be two extended formations that are consecutive in  $SEF$ . First note that the connection between  $EF_p(H_{12}^*)$  and  $EF_{p+1}(H_1^*)$  cuts all the channels between  $H_2^*$  and  $H_{11}^*$  either in  $cs_1$  or in  $cs_2$ , say  $cs_1$ . Since both of these extended formations are also connected to the bending area  $b(3, 4)$  between  $cs_3$  and  $cs_4$ , it is not possible for an extended formation  $EF_m(H_s^*)$ , with  $m > p + 1$  and  $s \in \{2, \dots, 11\}$ , to connect from vertices above the connection between  $EF_p(H_{12}^*)$  and  $EF_{p+1}(H_1^*)$  to vertices below it by using a path that passes through  $b(3, 4)$ . Further, if all such extended formations  $EF_m(H_s^*)$  are in  $cs_2$ , then a connection is needed from  $cs_1$  to  $cs_2$  in  $H_{12}^*$ . However, by Lemma 19, after three defects in the subsequence of  $\{H_2^*, \dots, H_{11}^*\}$  it is no longer possible for any extended formation  $EF_m(H_s^*)$ , with  $s \in \{2, \dots, 11\}$ , to place its turning vertex in the same channel segment. Since the path is continuous and since the connection between two consecutive extended formations  $EF_q(H_{12}^*)$  and  $EF_q(H_1^*)$  is repeated at a certain repetition  $q > p$ , we can follow that the path creates a spiral that is directed either inward or outward. Also, in order to respect the order of the sequence, it is not possible for the path to reverse the direction of the spiral. Hence, once a direction of the spiral has been chosen, either inward or outward, all the following connections use the same. This implies that, if a connection between two consecutive extended formations  $EF_m(H_s^*)$  and  $EF_m(H_{s+1}^*)$  is performed in a different channel segment than the one between  $EF_m(H_{s-1}^*)$  and  $EF_m(H_s^*)$ , then all the connections of this type have to change. However, when a defect at  $H_{s+1}^*$  is encountered, also the connection between  $EF_m(H_s^*)$  and  $EF_m(H_{s+2}^*)$  has to change channel segments, thereby making it impossible for any future connection between  $EF_m(H_s^*)$  and  $EF_m(H_{s+1}^*)$  to change channel segments. We conclude that, after a full repetition of  $SEF$ , which contains defects at each set of channels, all the extended formations should place their turning vertices in the same channel segment, which is not possible, by Lemma 19, hence proving the statement.

## 5.6 Proofs of Proposition 2

**Proposition 2.** *If there exists no pair of disjoint 2-side connections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Lemma 13.** *If a shape contains an intersection  $I_{(1,3)}$  and does not contain any other intersection that is disjoint with  $I_{(1,3)}$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Proof:** First observe that the only intersections that are not disjoint with  $I_{(1,3)}$  and that could occur together with  $I_{(1,3)}$  are  $I_{(2,4)}$  and  $I_{(1,4)}$ . Consider the nesting of depth  $d \geq 6$  that is present in any extended formation (Lemma 8).

Observe that a nesting can only take place at intersections  $I_{(1,3)}$  and either  $I_{(2,4)}$  or  $I_{(1,4)}$ . Recall that, by Property 3, every edge has an end-vertex in either  $cs_1$  or  $cs_2$ . Also, by construction, the stabilizers are placed in  $cs_1$  or  $cs_2$ . Note that the stabilizers also work as 1-vertices in the tails of other cells. This means that if there exist seven sets of tails that can be separated by straight lines, then there exists a region-level nonplanar tree, by Lemma 6. Observe that, by nesting them according to the sequence, the previous condition would be fulfilled.

This means that we have either a sorting or other nestings. We first show that there exists at most one nesting of depth  $d \geq 6$ .

Consider the case  $I_{(2,4)}^h$  (see Fig 15(a)). Observe that intersections  $I_{(1,4)}$  and  $I_{(1,3)}$  are either both high or both low. Also, every connection from  $cs_1$  to  $cs_4$  cuts either  $cs_2$  or  $cs_3$  and, if one of these connections cuts  $cs_2$ , then every nesting cutting  $cs_1$  closer to  $b(1,2)$  than the previous connection has to cut  $cs_2$ , as well. Hence, we can consider all the connections to  $cs_4$  as connections to  $cs_2$  or  $cs_3$ . Also, since any connection cutting a channel segment is more restrictive than a connection placing a vertex inside the same channel segment, the connections to  $cs_2$  or  $cs_3$  can be considered as the same. Finally, since each extended formation in the nesting has to connect to bending area  $b_{(2,3)}$ , it is not possible to have a nesting at  $I_{(2,4)}^h$  together with a nesting at  $I_{(1,3)}$ . Hence, we conclude that only one nesting is possible in this case.

Consider the case  $I_{(2,4)}^l$  (see Fig 15(b)). Observe that 1-vertices can be placed at most in  $cs_2$  and 2-vertices can be placed at most in  $cs_3$ , and that every extended formation belonging to a nesting has to visit these vertices. Therefore, if there exist both a nesting at  $I_{(1,3)}$  and a nesting at  $I_{(1,4)}$ , then the connections to the 1- and 2-vertices in the bending areas  $b(2,3)$  and  $b(3,4)$  are such that every extended formation nesting at  $I_{(1,4)}$  makes a nesting with the extended formations nesting at  $I_{(1,3)}$ . Hence, only one nesting is possible.

So we consider the unique nesting of depth  $d \leq 6$  and we show that any way of sorting the nesting formations in the channels will cause separated cells, hence proving the existence of a nonplanar region-level tree.

Consider four consecutive repetitions of the sequence of formations. These formations visit areas of  $cs_1$  and are separated by previously placed formations from other formations on the same channels. This will result in some cells to become separated in  $cs_1$ . Since, by Property 2, the number of linearly-separated cells in  $cs_1$  cannot be larger than 3, for any set of four such separated formations there exists a pair of formations  $F_1, F_2$  that change their order in  $cs_1$  by using one of the sides of the nesting. If between this pair of formations there is a formation of a different channel, then this formation has to choose the other side to reorder with a formation outside  $F_1, F_2$ . We further note that, if there are two such connections  $F_1, F_4$  and  $F_2, F_3$  on the same side that are connecting formations of one channel, nested in the order  $F_1, F_2, F_3, F_4$ , and another connection on the same side between  $F'_1, F'_2$  such that  $F'_1$  is nested between  $F_1, F_2$  and  $F'_2$  between  $F_3, F_4$ , then this creates a nesting. In the following we show that a nesting of depth at least 6 is reached.

Assume the repetitions of formations in the extended formation to be placed

in the order  $a, b, c, d, e$ . If this order is not coherent with the order in which the channels appear in the sequence of formations inside the extended formation, then we have already some connections that close both sides of the nesting for some formations. So we assume them to be in the order given by the sequence. Then, consider a repetition of formations with a defect at some 4-tuple  $H_i$ . Then, there exists a connection closing off at one side all the previously placed formations of  $H_i$ . However, there are sequences with defects also at 4-tuples  $H_{i+1}$  and  $H_{i-1}$ , which cannot be realized on the same side as the defects at  $H_i$ . We generalize this by saying that all the defects at odd indices are in one side, while the defects at even indices are in the other side. Since the path is continuous and has to reach from the last formation in a repetition to the first one in the following repetition, the continuation of the path can only use either the odd or the even defects. This implies that, when considering three further repetitions of formations, the first and the third having a defect at  $H_i$  and the second having no defect at  $H_i$ , there exists a nesting of depth 2 between them. Since, by Lemma 9, there cannot be a nesting of depth greater than 5 at this place, we conclude that after six repetitions of such a triple of formations there will be at least two formations that are separated from each other. By repeating this argument we arrive, after  $7 \cdot 6 \cdot 2$  repetitions, either at the existence of 7 formations that are separated on  $cs_1$  and  $cs_2$  or at the existence of a nesting of depth 6, both of which will not be drawable without the aid of another intersection that is able to support the second nesting.  $\square$

**Lemma 14.** *If there exists a sequence of extended formations in any shape containing an intersection  $I_{(3,1)}$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Proof:** Consider a sequence of extended formations in a shape containing an intersection  $I_{(3,1)}$ . We show that  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding. Observe that there exist several possibilities for channel segment  $cs_4$  to be placed, which determine one of  $I_{(1,4)}$ ,  $I_{(4,2)}$ ,  $I_{(4,1)}$ , and  $I_{(2,4)}$ .

We note that, if there exists the intersection  $I_{(3,1)}$ , then at least one of  $cs_1$ ,  $cs_2$ , and  $cs_4$  are part of the convex hull (see Figs. 16(a) and (b)).

First, we show that there exists a nesting at  $I_{(3,1)}$  (case  $I_{(4,1)}$  can be considered as the same).

Consider case  $I_{(3,1)}^h$  (see Fig. 16(b)). We have that  $cs_1$  and  $cs_2$  are on the convex hull restricted to the first three channel segments, and  $cs_4$  can force at most one of them out of it. Hence, one of  $cs_1$  and  $cs_2$  is part of the convex hull. We distinguish the two cases.

Suppose that  $cs_2$  is part of the convex hull. Assume that there exists a nesting at  $I_{(2,4)}$ . Since from  $cs_4$  the only possible connection without a 1-side connection is the one to  $cs_2$ , which is on the convex hull, an argument analogous to the one used in Lemma 13 proves that the nesting at  $I_{(2,4)}$  has size smaller than  $7 * 12$ , which implies that the rest of the nesting takes place at  $I_{(3,1)}$ .

Suppose that  $cs_1$  is part of the convex hull. Assume that there exists a nesting at  $I_{(2,4)}$ . Every connection from  $cs_4$  has to be either to  $cs_1$  or to  $cs_2$ ,

by Property 3. However, a nesting is already taking place at  $cs_2$ , and hence we have connections to  $cs_1$ . As  $cs_1$  is on the convex hull, an argument analogous to the one used in Lemma 13 proves that the nesting at  $I_{(2,4)}$  has size smaller than  $7 * 12$ , which implies the rest of the nesting has to take place at  $I_{(3,1)}$ .

Consider case  $I_{(3,1)}^l$  (see Fig. 16(a)). Since  $cs_2$  is not part of the convex hull, either  $cs_1$  or  $cs_4$  are. If  $cs_1$  is on the convex hull, then the same argument as before holds, while if  $cs_4$  is on the convex hull, then no reordering is possible.

Hence, we conclude that a nesting has to take place at  $I_{(3,1)}$ .

Consider a sequence of extended formations  $SEF$  which uses only channels in this particular shape. Then, all the extended formations in  $SEF$  have to place a nesting at  $I_{(\{3,4\},1)}$  with an open triangle whose middle vertex splits either  $cs_3$  or  $cs_4$  into an inner and an outer area. Observe that, by Lemma 11, only a limited part of the nesting can be performed in the bending area. Also, every extended formation  $EF$  having at least one vertex either in  $cs_3$  or in  $cs_4$  has a vertex in the bending area. Hence, every extended formation has to use both of such areas. If  $cs_1$  is on the convex hull, then there exist only 1-sided connections to connect such areas, which implies the statement, by Lemma 12. On the other hand, if  $cs_1$  is not on the convex hull, then there exists intersection  $I_{(1,4)}$ , and  $cs_4$  can be also used to perform connections from the inner to the outer area. However, since  $cs_4$  is on the convex hull, such connections are only 1-side. Hence, by Lemma 12, the statement follows.  $\square$

### 5.7 Proofs of Proposition 3

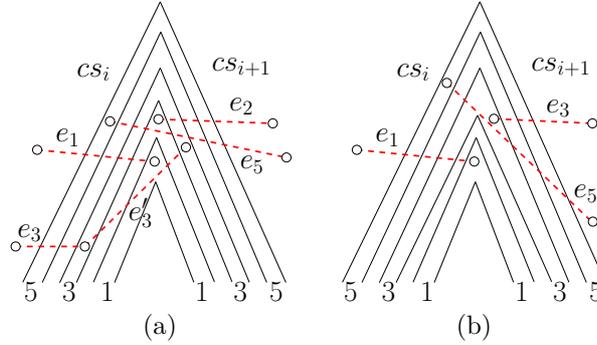
**Proposition 3.** *If there exist two disjoint intersections, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Lemma 15.** *Let  $cs_i$  and  $cs_{i+1}$  be two consecutive channel segments. If there exists an ordered set  $S := (1, 2, \dots, 5)^3$  of extremal double cuts cutting  $cs_i$  and  $cs_{i+1}$  in such a way that the order of the intersections of the double cuts with  $cs_i$  (with  $cs_{i+1}$ ) is coherent with the order of  $S$ , then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Proof:** Assume first that  $cs_i$  and  $cs_{i+1}$  are such that the bendpoint of channel  $ch_5$  encloses the bendpoint of all the other channels. Hence, any edge creating a double cut at a channel  $ch$  has to cut all the channels  $ch'$  with  $ch' > ch$ , either in  $cs_i$  or in  $cs_{i+1}$ . Refer to Fig. 28.

Consider the first repetition  $(1, 2, \dots, 5)$ . Let  $e_1$  be an edge creating a double cut at channel  $ch_1$ . Assume, without loss of generality, that  $e_1$  cuts channel segment  $cs_i$ . Observe that, for channel  $ch_1$ , the visibility constraints determined in channels  $ch_2, \dots, ch_5$  in  $cs_i$  and in  $cs_{i+1}$  by the double cut created by  $e_1$  do not depend on whether it is simple or non-simple. Indeed, by Property 5, edge  $e_1$  blocks visibility to  $b(i, i + 1)$  for the part of  $cs_i$  where edges creating double cuts at channels  $ch_2, \dots, ch_5$  following  $e_1$  in  $S$  have to place their end-vertices.

Then, consider an edge  $e_3$  creating a double cut at channel  $ch_3$  in the first

Figure 28: Proof of Lemma 15. (a)  $e_3$  cuts  $cs_i$ . (b)  $e_3$  cuts  $cs_{i+1}$ .

repetition of  $(1, 2, \dots, 5)$ .

If  $e_3$  cuts  $cs_i$  (see Fig. 28(a)), then it has to create either a non-simple double cut or a simple one. However, in the latter case, an edge  $e'_3$  between  $cs_i$  and  $cs_{i+1}$  in channel  $ch_3$ , which creates a blocking cut in channel  $ch_2$ , is needed. Hence, in both cases, channel  $ch_2$  is cut both in  $cs_i$  and in  $cs_{i+1}$ , either by  $e_3$  or by  $e'_3$ . It follows that an edge  $e_2$  creating a double cut at channel  $ch_2$  in the second repetition of  $(1, 2, \dots, 5)$  has to cut  $cs_{i+1}$ , hence blocking visibility to  $b(i, i + 1)$  for the part of  $cs_{i+1}$  where edges creating double cuts at channels  $ch_3, \dots, ch_5$  following it in  $S$  have to place their end-vertices, by Property 5. Further, consider an edge  $e_5$  creating a double cut at channel  $ch_5$  in the second repetition of  $(1, 2, \dots, 5)$ . Since visibility to  $b(i, i + 1)$  is blocked by  $e_1$  and  $e_3$  in  $cs_i$  and by  $e_2$  in  $cs_{i+1}$ ,  $e_5$  has to create a non-simple double cut (or a simple one plus a blocking cut), hence cutting channel  $ch_4$  both in  $cs_i$  and in  $cs_{i+1}$ . It follows that, by Property 4, an edge  $e_4$  creating a double cut at channel  $ch_4$  in the third repetition of  $(1, 2, \dots, 5)$  can place its end-vertex neither in  $cs_i$  nor in  $cs_{i+1}$ .

If  $e_3$  cuts  $cs_{i+1}$  (see Fig. 28(b)), then it has to create a simple double cut. Again, by Property 5, edge  $e_3$  blocks visibility to  $b(i, i + 1)$  for the part of  $cs_{i+1}$  where edges creating double cuts following  $e_3$  in  $S$  have to place their end-vertices. Hence, an edge  $e_5$  creating a double cut at  $ch_5$  in the first repetition of  $(1, 2, \dots, 5)$  cannot create a simple double cut, since its visibility to  $b(i, i + 1)$  is blocked by  $e_1$  in  $cs_i$  and by  $e_3$  in  $cs_{i+1}$ . This implies that  $e_5$  creates a non-simple double cut (or a simple one plus a blocking cut) at channel  $ch_5$ , cutting either  $cs_i$  or  $cs_{i+1}$ , hence cutting channel  $ch_4$  both in  $cs_i$  and in  $cs_{i+1}$ . It follows that, by Property 4, an edge  $e_4$  creating a double cut at  $ch_4$  in the second repetition of  $(1, 2, \dots, 5)$  can place its end-vertex neither in  $cs_i$  nor in  $cs_{i+1}$ .

The case in which  $cs_i$  and  $cs_{i+1}$  are such that the bendpoint of  $ch_1$  encloses the bendpoint of all the other channels can be proved analogously. Namely, the same argument holds with  $ch_5$  playing the role of  $ch_1$ , with  $ch_1$  playing the role of  $ch_5$ , with  $ch_3$  having the same role as before, with  $ch_4$  playing the role

of  $ch_2$ , and with  $ch_2$  playing the role of  $ch_4$ . Observe that, in order to obtain the needed ordering in this setting, 3 repetitions of  $(1, 2, \dots, 5)$  are needed. In fact, we consider channel  $ch_5$  in the first repetition, channels  $ch_3$  and  $ch_4$  in the second one, and channels  $ch_1$  and  $ch_2$  in the third one.  $\square$

**Lemma 17.** *Every sequence of extending formations in shape  $I_{(1,3)}^h I_{(4,\{1,2\})}^{h,l}$  contains an ordered set  $(1, 2, \dots, 5)^3$  of extremal double cuts with respect to bending area either  $b(2, 3)$  or  $b(3, 4)$ .*

**Proof:** First observe that shapes  $I_{(1,3)}^h I_{(4,2)}^h$  and  $I_{(1,3)}^h I_{(4,2)}^l$  are similar to shapes  $I_{(1,3)}^h I_{(4,1)}^h$  and  $I_{(1,3)}^h I_{(4,1)}^l$ , respectively, with the only difference on the slope of channel segment  $cs_4$ , whose elongation crosses  $cs_2$  and not  $cs_1$ . Hence, it is sufficient to study only such cases. Shapes  $I_{(1,3)}^h I_{(4,2)}^l$  and  $I_{(1,3)}^h I_{(4,1)}^l$  are depicted in Fig. 18.

Further, observe that bending area  $b(2, 3)$  is on the convex hull both in shape  $I_{(1,3)}^h I_{(4,2)}^h$  and in shape  $I_{(1,3)}^h I_{(4,2)}^l$  and that each extended formation has some vertices in  $b(2, 3)$  and in  $b(3, 4)$ , which implies that they have to reach such bending areas with path-edges.

In shape  $I_{(1,3)}^h I_{(4,2)}^h$ , by Lemma 16, there exist double cuts either in  $b(2, 3)$  or in  $b(3, 4)$ , while in shape  $I_{(1,3)}^h I_{(4,2)}^l$  there exist double cuts in  $b(2, 3)$ , since the only possible connections to  $b(2, 3)$  are from channel segments  $cs_1$  and  $cs_4$ , both creating double cuts (see Fig. 18(b)). Hence, we consider the extremal double cuts of each extended formation with respect to one of  $b(2, 3)$  or  $b(3, 4)$ , say  $b(2, 3)$ .

Assume, without loss of generality, that the first bendpoint of channel  $ch_1$  is enclosed by the first bendpoint of all the other channels. This implies that the second bendpoint of  $ch_1$  encloses the second bendpoint of all the other channels. See Fig. 29.

Consider two sets of extended formations creating double cuts in  $b(2, 3)$  at channels  $ch_1, \dots, ch_5$ , respectively. Observe that the extended formations in these two sets could be placed in such a way that the ordering of their extremal double cuts is  $(1, 1, 2, 2, \dots, 5, 5)$ . The same holds for the following occurrences of extended formations creating double cuts in  $b(2, 3)$  at channels  $ch_1, \dots, ch_5$ , respectively. Clearly, in this way an ordering  $(1^n, 2^n, \dots, 5^n)$  could be achieved and hence an ordered set  $(1, 2, \dots, 5)^3$  of double cuts would be never obtained (see Fig. 29(a)).

However, every repetition of extended formations inside a sequence of extended formations contains a double defect at some channel. We show, with an argument similar to the one used in Lemma 8, that the presence of such double defects determines an ordering  $(1, 2, \dots, 5)^3$  of extremal double cuts after a certain number of repetitions of extended formations inside a sequence of extended formations. Namely, consider a double defect at channel  $ch_i$  in a certain repetition. The connection between channels  $ch_{i-1}$  and  $ch_{i+2}$  cannot be performed in the same area as the connection between channels  $ch_{i-1}$  and  $ch_i$  and between channels  $ch_i$  and  $ch_{i+1}$  was performed in the previous repetition.

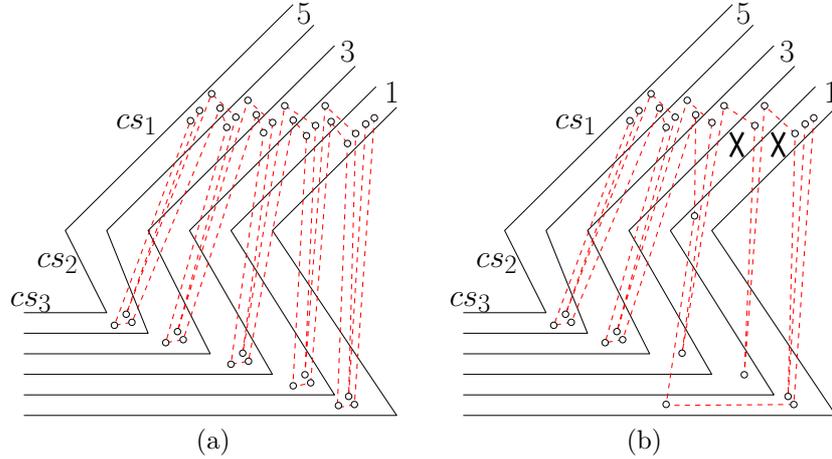


Figure 29: (a) The ordering of the extremal double cuts is  $(1, 1, 2, 2, \dots, 5, 5)$ . (b) When a double defect is encountered, the connection between channels  $ch_{i-1}$  and  $ch_{i+2}$  cannot be performed in the same area as the connections between channels  $ch_{i-1}$  and  $ch_i$  and between channels  $ch_i$  and  $ch_{i+1}$  were performed in the previous repetition. Hence, the connection is performed in the same area as the connection between channels  $ch_{i+1}$  and  $ch_{i+2}$  was performed.

Hence, such a connection has to be performed either in the same area as the connection between channels  $ch_{i+1}$  and  $ch_{i+2}$  was performed (see Fig. 29(b)), or in  $cs_4$  (this is only possible in shape  $I_{(1,3)}^h I_{(4,2)}^l$ , see Fig. 30).

Observe that, if the path goes to  $cs_4$  to make the connection, then to  $cs_1$ , and finally back to  $b(2, 3)$ , hence creating a spiral, then the considered double cut is not extremal (see Fig. 30(b)). Therefore, the only possibility to consider is to connect channels  $ch_{i-1}$  and  $ch_{i+2}$  in  $cs_4$  and then to come back to  $b(2, 3)$  with a double cut (see Fig. 30(a)). Hence, independently on whether  $cs_4$  is used or not, the connection between channels  $ch_{i-1}$  and  $ch_{i+2}$  blocks visibility for the following repetitions to the areas where the connections between some channels were performed in the previous repetition. This implies that the ordering  $(1^n, 2^n, \dots, 5^n)$  of extremal double cuts cannot be respected in the following repetitions. In fact, a partial order  $(i, i+1, i+2)^2$  is obtained in a repetition of formations creating extremal double cuts at channels  $1, \dots, 5$ .

Also, when two different double defects having a channel in common are considered, the effect of such defects is combined. Namely, consider a double defect at channel  $ch_3$  in a certain repetition. The connection between  $ch_2$  and  $ch_5$  blocks visibility to the areas where the connection between  $ch_2$  and  $ch_3$  and between  $ch_3$  and  $ch_4$  were performed at the previous repetitions (see Fig. 31(a)). Then, consider a double defect at  $ch_1$  in a following repetition. We have that the connection between channels  $ch_0$  and  $ch_3$  cannot be performed where the connection between  $ch_2$  and  $ch_3$  was performed in the previous repetitions, since

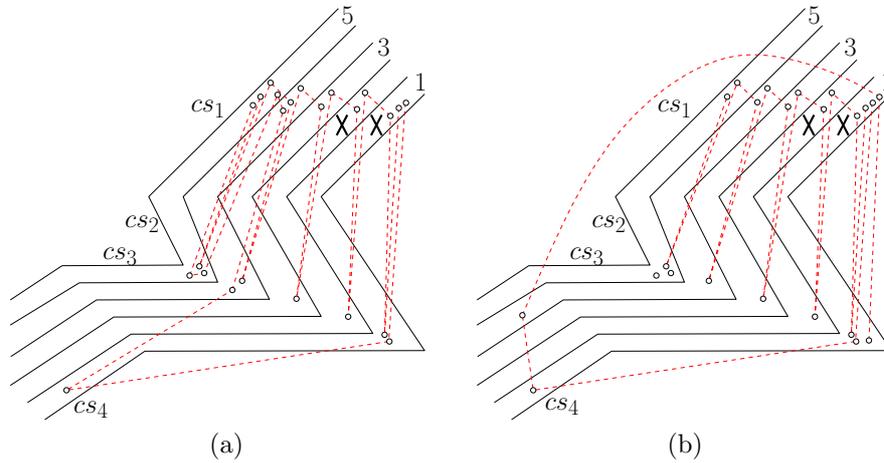


Figure 30: (a) When a double defect is encountered, the connection between channels  $ch_{i-1}$  and  $ch_{i+2}$  is performed in  $cs_4$ . (b) If  $cs_4$  is used to spiral, the considered double cut was not extremal.

such an area is blocked by the presence of the connection between  $ch_2$  and  $ch_5$ . Hence, a double cut at  $ch_3$  has to be placed after the double cut at  $ch_5$  created in the previous repetition (see Fig. 31(b)).

Consider now a further repetition with a defect not involving any of channels  $ch_1, \dots, ch_5$ . The region where the connection from  $ch_1$  to  $ch_2$  was performed in the previous repetitions is blocked by the connection between  $ch_0$  and  $ch_3$ , and hence a double cut at  $ch_1$  has to be placed after the one at  $ch_3$  of the previous repetition, which in turn was created after the one at  $ch_5$  (see Fig. 32(a)). Also, all the double cuts at channels  $ch_2, \dots, ch_5$  have to be placed after the double cut at  $ch_1$ , and hence a shift of the whole sequence  $1, \dots, 5$  after the double cut at  $ch_5$  is performed and an ordered set  $(1, 2, \dots, 5)^2$  is obtained (see Fig. 32(b)). Observe that at most two repetitions of extended formation inside a sequence of extended formations such that each set contains a double defect at each channel are needed to obtain such a shift.

By repeating such an argument we obtain another shift of the whole sequence  $(1, \dots, 5)$ , which results in the desired ordered set  $(1, 2, \dots, 5)^3$ . We have that a set of repetitions of extended formation containing a double defect at each channel is needed to obtain the first sequence  $(1, 2, \dots, 5)$ , then two of such sets are needed to get to  $(1, 2, \dots, 5)^2$ , and two more are needed to get to  $(1, 2, \dots, 5)^3$ , which proves the statement.

Observe that, if it were possible to partition the defects into two sets such that there exists no pair of defects involving a common channel inside the same set, then such sets could be independently drawn inside two different areas and the effects of the defects could not be combined to obtain  $(1, 2, \dots, 5)^3$ . However, since each double defect involves two consecutive channels, at least three sets

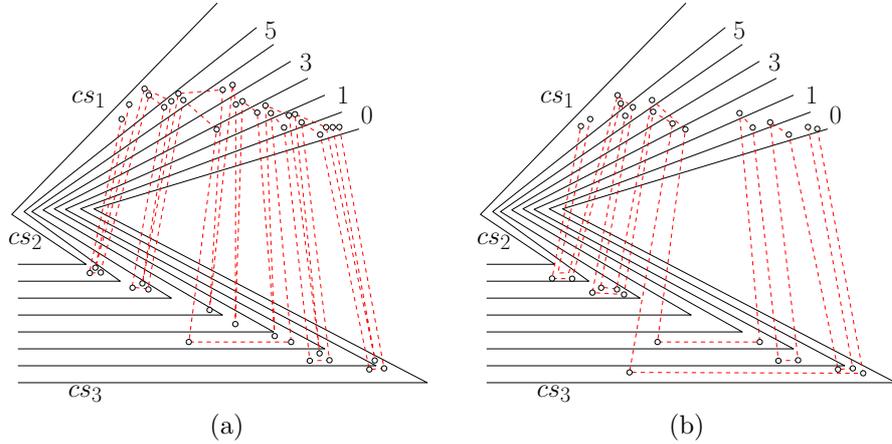


Figure 31: (a) A repetition with a double defect at  $ch_2$  is considered. (b) A repetition with a double defect at  $ch_0$  is considered.

are needed to obtain a partition with such a property. In that case, however, an ordered set  $(1, 2, \dots, 5)^3$  could be obtained by simply considering a repetition of  $(1, 2, \dots, 5)$  in each of the sets.  $\square$

**Lemma 18.** *If channel segment  $cs_2$  is part of the convex hull, then  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.*

**Proof:** First observe that, with an argument analogous to the one used in Lemma 13, it is possible to show that there exists a nesting at intersection  $I_{(4, \{1, 2\})}$ . Then, by Property 3, every vertex that is placed in  $cs_4$  is connected to two vertices that are placed either in  $cs_1$  or in  $cs_2$ . Hence, the continuous path-route connecting to a vertex placed in  $cs_4$  creates an open triangle, having one corner in  $cs_4$  and two corners either in  $cs_1$  or in its elongation, which cuts  $cs_4$  into its inner and outer area.

By Lemma 11, not all of these triangles can be placed in the bending area  $b(3, 4)$ . Hence, every extended formation, starting from the second of the sequence, has to place their vertices in both the inner and the outer area of the triangle created by the first one.

Observe that, in order to connect the inner area to the outer area, the extended formations can only use 1-side connections. Namely,  $cs_1$  creates a 1-side connection. Channel segment  $cs_2$  is on the convex hull. Since, by Property 3, every vertex that is placed in  $cs_3$  is connected to two vertices that are placed either in  $cs_1$  or in  $cs_2$ , also  $cs_3$  creates a 1-side connection. Hence, by Lemma 12,  $\mathcal{T}$  and  $\mathcal{P}$  do not admit any geometric simultaneous embedding.  $\square$

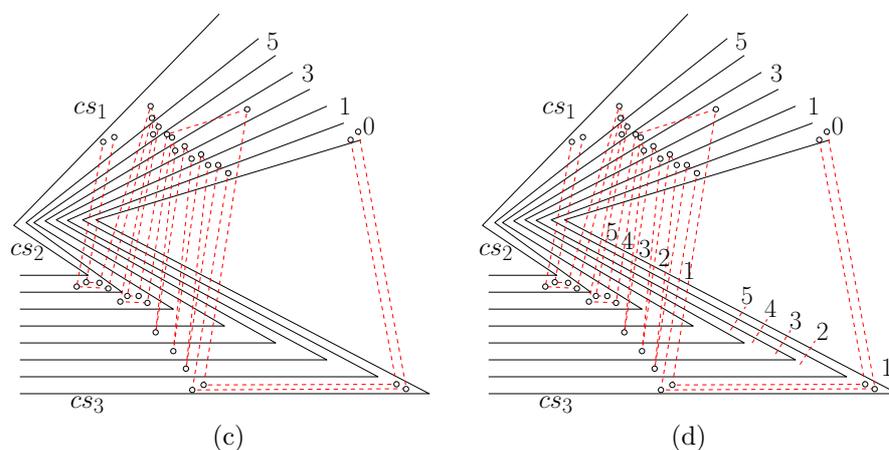


Figure 32: (a) A repetition without any double defect at channels  $ch_1, \dots, ch_5$  is considered. (b) An ordered set  $(1, \dots, 5)$  is obtained.

## 6 Geometric Simultaneous Embedding of a Tree of Height 2 and a Path

In this section we describe an algorithm for constructing a geometric simultaneous embedding of any tree  $\mathcal{T}$  of height 2 and any path  $\mathcal{P}$ . Refer to Fig. 33.

Draw the root  $r$  of  $\mathcal{T}$  at the origin of a coordinate system. Choose a ray  $R_1$  emanating from the origin and entering the first quadrant, and a ray  $R_2$  emanating from the origin and entering the fourth quadrant. Consider the wedge  $W$  delimited by  $R_1$  and  $R_2$  and containing the positive  $x$ -axis. Split  $W$  into  $t$  wedges  $W_1, \dots, W_t$ , in this clockwise order around the origin, where  $t$  is the number of vertices adjacent to  $r$  in  $\mathcal{T}$ , by emanating  $t - 1$  equispaced rays from the origin.

Then, consider the two subpaths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $\mathcal{P}$  starting at  $r$ . Assign an orientation to  $\mathcal{P}_1$  and  $\mathcal{P}_2$  such that the two edges  $(r, u) \in \mathcal{P}_1$  and  $(r, v) \in \mathcal{P}_2$  incident to  $r$  in  $\mathcal{P}$  are exiting  $r$ .

Finally, consider the  $t$  subtrees  $\mathcal{T}_1, \dots, \mathcal{T}_t$  of  $\mathcal{T}$  rooted at a node adjacent to  $r$ , such that  $u \in \mathcal{T}_1$  and  $v \in \mathcal{T}_t$ .

The vertices of each subtree  $\mathcal{T}_i$  are drawn inside wedge  $W_i$ , in such a way that:

1. vertex  $u$  is the vertex with the lowest  $x$ -coordinate in the drawing, except for  $r$ ;
2. vertices belonging to  $\mathcal{P}_1$  are placed in increasing order of  $x$ -coordinate according to the orientation of  $\mathcal{P}_1$ ;
3. vertex  $v$  is placed on  $R_2$  and is the vertex with the highest  $x$ -coordinate in the drawing;

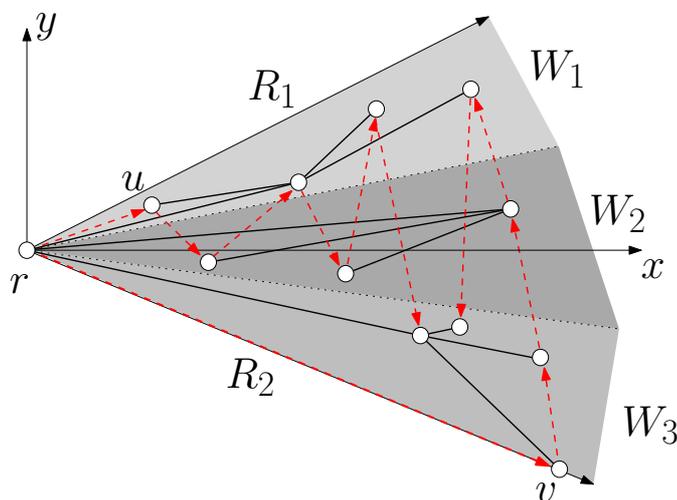


Figure 33: Construction of a geometric simultaneous embedding of a tree with height 2 and a path.

4. vertices belonging to  $\mathcal{P}_2 \setminus r$  are placed in decreasing order of  $x$ -coordinate according to the orientation of  $\mathcal{P}_2$ , in such a way that the leftmost vertex of  $\mathcal{P}_2 \setminus r$  is to the right of the rightmost vertex of  $\mathcal{P}_1$ ;
5. the drawing of each  $\mathcal{T}_i$  is planar.

Since  $\mathcal{T}$  has height 2, each subtree  $\mathcal{T}_i$ , with  $i = 1 \dots, t$ , is a star. Hence, it can be drawn inside its own wedge  $W_i$  without creating any intersection among tree-edges.

Since  $\mathcal{P}_1$  and  $\mathcal{P}_2 \setminus \{r\}$  are drawn in monotonic order of  $x$ -coordinate and are separated from each other, and edge  $(r, v)$  connecting such two paths is on the convex hull of the point-set, no intersection among path-edges is created.

From the discussion above, we have the following theorem.

**Theorem 2** *A tree of height 2 and a path always admit a geometric simultaneous embedding.*

## 7 Conclusions

In this paper we have shown that there exist a tree  $\mathcal{T}$  and a path  $\mathcal{P}$  on the same set of vertices that do not admit any geometric simultaneous embedding, which means that there exists no set of points in the plane allowing a planar embedding of both  $\mathcal{T}$  and  $\mathcal{P}$ .

We obtained this result by extending the concept of level nonplanar trees [9] to the one of region-level nonplanar trees. Namely, we showed that there exist

trees that do not admit any planar embedding if the vertices are forced to lie inside particularly defined regions according to a prescribed ordering. Then, we constructed  $\mathcal{T}$  and  $\mathcal{P}$  so that the path creates these particular regions and at least one of the many region-level nonplanar trees composing  $\mathcal{T}$  has its vertices forced to lie inside them in the desired order. Observe that our result also implies that there exist two edge-disjoint trees that do not admit any geometric simultaneous embedding, which answers an open question posed in [13], where the case of two non-edge-disjoint trees was solved.

It is important to note that, even if our counterexample consists of a huge number of vertices, it can also be considered as “simple”, in the sense that the height of the tree is just 4. In this direction, we proved that if the tree has height 2, then it admits a geometric simultaneous embedding with any path. This gives rise to an intriguing open question about whether a tree of height 3 and a path always admit a geometric simultaneous embedding or not.

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