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Cubic graphs have bounded slope parameter

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Abstract

We show that every finite connected graph G with maximum degree *three* and with at least one vertex of degree smaller than *three* has a straight-line drawing in the plane satisfying the following conditions. No three vertices are collinear, and a pair of vertices form an edge in G if and only if the segment connecting them is parallel to one of the sides of a previously fixed regular pentagon. It is also proved that every finite graph with maximum degree *three* permits a straight-line drawing with the above properties using at most *seven* different edge slopes.

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1 Introduction

A *drawing* of a graph G is a representation of its vertices by distinct points in the plane and its edges by continuous arcs connecting the corresponding endpoints, not passing through any other point corresponding to a vertex. In a *straight-line drawing* [8], the edges are represented by (possibly crossing) segments. If it leads to no confusion, we make no notational or terminological distinction between the vertices (edges) of G and the points (arcs) representing them.

The *thickness* of a graph G is the smallest number of its planar subgraphs whose union is G [14]. It is one of the several widely known graph parameters that measures how far is G from being planar. The *geometric thickness* of G is the smallest number of *crossing-free* subgraphs of a straight-line drawing of G, whose union is G [11]. The *slope number* of G is the minimum number of distinct edge slopes in a straight-line drawing of G [16]. It follows directly from the definitions that the thickness of any graph is at most as large as its geometric thickness, which, in turn, cannot exceed its slope number. For many interesting results about these parameters, consult [3, 6, 4, 5, 7, 9, 12, 15].

The *slope parameter* of a graph was defined by Ambrus, Barát, and P. Hajnal [1], as follows. By abusing the usual terminology, we say that the *slope* of a line ℓ in the xy-plane is the smallest angle $\alpha \in [0, \pi)$ such that ℓ can be rotated into a position parallel to the x-axis by a clockwise turn through α. Given a set P of points in the plane and a set Σ of slopes, define $G(P, \Sigma)$ as the graph on the vertex set P, in which two vertices $p, q \in P$ are connected by an edge if and only if the slope of the line pq belongs to Σ. The *slope parameter* $s(G)$ of G is the size of the smallest set Σ of slopes such that G is isomorphic to $G(P, \Sigma)$ for a suitable set of points P in the plane. This definition was motivated by the fact that all connections (edges) in an electrical circuit (graph) G can be easily realized by the overlay of $s(G)$ finely striped electrically conductive layers.

The slope parameter, $s(G)$, is closely related to the three other graph parameters mentioned before. For instance, for triangle-free graphs, $s(G)$ is at least as large as the slope number of G, the largest of the three quantities above. Indeed, in the drawing realizing the slope parameter, there are no three points on a line, so this drawing proves that the slope number is smaller or equal to the slope parameter.

On the other hand, it sharply differs from them in the sense that the slope parameter of a complete graph on n vertices is *one*, while the thickness, the geometric thickness, and the slope number of K_n tend to infinity as $n \to \infty$. Jamison [10] proved that the slope number of K_n is n.

Any graph G of maximum degree *two* splits into vertex-disjoint cycles, paths, and possibly isolated vertices. Hence, for such graphs we have $s(G) \leq 3$. In contrast, as was shown by Barát *et al.* [2], for any $d \geq 5$, there exist graphs of maximum degree d, whose slope parameters are arbitrarily large.

A graph is said to be *cubic* if the degree of each of its vertices is at most *three*. A cubic graph is *subcubic* if each of its connected components has a vertex of degree smaller than *three*.

The aim of this note is to prove the following result.

Theorem 1. *Every cubic graph has slope parameter at most* seven*.*

This theorem is not likely to be tight. The best lower bound we are aware of is *four*. This bound is attained, for example, for the 8-vertex subcubic graph that can be obtained from the graph formed by the edges of a 3-dimensional cube by deleting one of its edges.

We will refer to the angles $i\pi/5$, $0 \le i \le 4$, as the *five basic slopes*. In Section 2, we prove the following statement, which constitutes the first step of the proof of Theorem 1.

Theorem 2. *Every subcubic graph has slope parameter at most* five*. Moreover, this can be realized by a straight-line drawing such that no* three *vertices are on a line and each edge has one of the* five *basic slopes.*

Using the fact that in the drawing guaranteed by Theorem 2 no *three* vertices are collinear, we can also conclude that the slope *number* of every subcubic graph is at most *five*. In [12], however, it was shown that this number is at most *four* and for cubic graphs it is at most *five*. This was improved for connected cubic graphs in [13] to *four*.

2 Proof of Theorem 2

The proof is by induction on the number of vertices of the graph. Clearly, the statement holds for graphs with fewer than *three* vertices. Let n be fixed and suppose that we have already established the statement for graphs with fewer than n vertices. Let G be a subcubic graph of n vertices. We can assume that G is connected, otherwise we can draw each of its connected components separately and translate the resulting drawings through suitable vectors so that no two points in distinct components determine a line of basic slope.

To obtain a straight-line drawing of G, we have to find proper locations for its vertices. At each inductive step, we start with a drawing of a subgraph of G satisfying the conditions of Theorem 2 and extend it by adding a vertex. At a given stage of the procedure, for any vertex v that has already been added, consider the (basic) slopes of all edges adjacent to v that have already been drawn, and let $sl(v)$ denote the set of integers $0 \le i < 5$ for which $i\pi/5$ is such a slope. That is, at the beginning $sl(v)$ is undefined, then it gets defined, and later it may change (expand). Analogously, for any edge uv of G, denote by $sl(uv)$ the integer $0 \le i < 5$ for which the slope of uv is $i\pi/5$.

Case 1: *G has a vertex of degree* one. Assume without loss of generality, that v is a vertex of degree *one*, and let w denote its only neighbor. Deleting v from G, the degree of w in the resulting graph G' is at most *two*. Therefore, by the induction hypothesis, G' has a drawing meeting the requirements. As w has degree at most *two*, there is a basic slope σ such that no other vertex of G' lies on the line ℓ of slope σ that passes through w. Draw all *five* lines of basic slopes through each vertex of G'. These lines intersect ℓ in finitely many points. We can place v at any other point of ℓ, to obtain a proper drawing of G.

From now on, assume that G has no vertex of degree one.

Case 2: *G has no cycle that passes through a vertex of degree* two. Since G is subcubic, it contains a vertex w of degree *two* such that G is the union of two graphs, G_1 and G_2, having only vertex w in common. Both G_1 and G_2 are subcubic and have fewer than n vertices, so by the induction hypothesis both of them have a drawing satisfying the conditions. Translate the drawing of G_2 so that the points representing w in the two drawings coincide. Since w has degree *one* in both G_1 and G_2, by a possible rotation of G_2 about w through an angle that is a multiple of $\pi/5$, we can achieve that the two edges adjacent to w are not parallel. By scaling G_2 from w, if necessary, we can also achieve that the slope of no segment between a vertex of $G_1 \setminus w$ and a vertex of $G_2 \setminus w$ is a basic slope. Thus, the resulting drawing of G meets the requirements.

Case 3: *G has a cycle passing through a vertex of degree* two. If G itself is a cycle, we can easily draw it. If it is not the case, let C be a *shortest* cycle which contains a vertex of degree two. Let $u_0, u_1, \ldots, u_k$ denote the vertices of C, in this cyclic order, such that u_0 has degree *two* and u_1 has degree *three*. The indices are understood mod $k+1$, that is, for instance, $u_{k+1} = u_0$. It follows from the minimality of C that u_i and u_j are not connected by an edge of G whenever $|i-j| > 1$.

Since $G \setminus C$ is subcubic, by assumption, it admits a straight-line drawing satisfying the conditions. Each u_i has at most *one* neighbor in $G \setminus C$. Denote this neighbor by t_i, if it exists. For every i for which t_i exists, we place u_i on a line passing through t_i. We place the u_i's one by one, "very far" from $G \setminus C$, starting with u_1. Finally, we arrive at u_0, which has no neighbor in $G \setminus C$, so that it can be placed at the intersection of two lines of basic slope, through u_1 and u_k, respectively. We have to argue that our method does not create "unnecessary" edges, that is, we never place two independent vertices in such a way that the slope of the segment connecting them is a basic slope. In what follows, we make this argument precise.

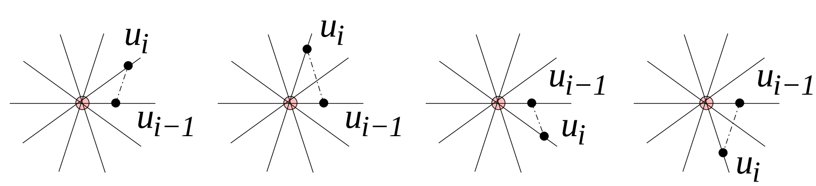

Figure 1: The four possible locations of u_i.

We determine the locations of the vertices $u_0, u_1, \ldots, u_k$ using the following Procedure(G, C, u_0, u_1, x), where G is the input subcubic graph, C is a shortest cycle passing through a vertex of degree *two*, u_0, that has a degree *three* neighbor, u_1, and x is a real parameter. Note that Procedure(G, C, u_0, u_1, x) is a *nondeterministic* algorithm, as we have more than one choice at certain steps. (However, it is very easy to make it deterministic.)

Procedure(G, C, u_0, u_1, x)

- Step 0. Since $G \setminus C$ is subcubic, it has a representation with the *five* basic slopes. Take such a representation, scaled and translated in such a way that t_1 (which exists since the degree of u_1 is three) is at the origin, and all other vertices are within unit distance from it.

  For any i, $2 \le i \le k$, for which u_i does not have a neighbor in $G \setminus C$, let t_i be any unoccupied point closer to the origin than 1, such that the slope of none of the lines connecting t_i to $t_1, t_2, \dots t_{i-1}$ or to any other already embedded point of $G \setminus C$ is a basic slope.

For any point p and for any $0 \le i \le 4$, let $\ell_i(p)$ denote the line with ith basic slope, $i\pi/5$, passing through p. Let ℓ_i stand for $\ell_i(O)$, where O denotes the origin.

We will place $u_1, \dots, u_k$ recursively, so that u_j is placed on $\ell_i(t_j)$, for a suitable i. Once u_j has been placed on some $\ell_i(t_j)$, define $ind(u_j)$, the *index* of u_j, to be i. (The indices are taken mod 5. Thus, for example, $|i - i'| \ge 2$ is equivalent to saying that $i \ne i'$ and $i \ne i' \pm 1 \mod 5$.) Start with u_1. The degree of t_1 in $G \setminus C$ is at most *two*, so that at the beginning the set $sl(t_1)$ (defined in the first paragraph of this section) has at most *two* elements. Let $l \notin sl(t_1)$. Direct the line $\ell_l(t_1)$ arbitrarily, and place u_1 on it at distance x from t_1 in the positive direction. (According to this rule, if $x < 0$, then u_1 is placed on $\ell_l(t_1)$ at distance $|x|$ from t_1 in the *negative* direction.)

Suppose that u_1, $u_2, \dots$, u_{i-1} have been already placed and that u_{i-1} lies on the line $\ell_l(t_{i-1})$, that is, we have $ind(u_{i-1}) = l$.

- Step i. We place u_i at one of the following four locations (see Figure 1):

  (1) the intersection of $\ell_{l+1}(t_i)$ and $\ell_{l+2}(u_{i-1})$;
  (2) the intersection of $\ell_{l+2}(t_i)$ and $\ell_{l+3}(u_{i-1})$;
  (3) the intersection of $\ell_{l-1}(t_i)$ and $\ell_{l-2}(u_{i-1})$;
  (4) the intersection of $\ell_{l-2}(t_i)$ and $\ell_{l-3}(u_{i-1})$.

  Choose from the above four possibilities so that the edge $u_i t_i$ is not parallel to any other edge already drawn and adjacent to t_i, i.e., before adding the edge $u_i t_i$ to the drawing, $sl(t_i)$ did not include $sl(u_i t_i)$.

It follows directly from (1)–(4) that $sl(u_{i-1}) \subset \{l, l-1, l+1 \mod 5\}$, while $sl(u_i u_{i-1}) \subset \{l-2, l+2 \mod 5\}$, that is, before adding the edge $u_i u_{i-1}$ to the drawing, we had $sl(u_i u_{i-1}) \notin sl(u_{i-1})$. Avoiding for $u_i t_i$ the slopes of the edges already incident to t_i, leaves available two of the choices (1), (2), (3), (4).

Let u'_{i-1} be the translation of u_{i-1} by the vector $\overrightarrow{t_{i-1}O}$, and similarly, let u'_i be the translation of u_i by the vector $\vec{t_iO}$. That is, $Ou'_{i-1}u_{i-1}t_{i-1}$ and $Ou'_iu_it_i$ are parallelograms. We have

$$\overline{Ou_{i-1}} - 1 < \overline{Ou'_{i-1}} < \overline{Ou_{i-1}} + 1,$$

$$\overline{Ou_i} - 1 < \overline{Ou'_i} < \overline{Ou_i} + 1,$$

and

$$2\cos\left(\frac{\pi}{5}\right)\overline{Ou'_{i-1}} = \overline{Ou'_i}.$$

Therefore, for any possible location of u_i, we have

$$1.6\overline{Ou_{i-1}}-4 < 2\cos\left(\frac{\pi}{5}\right)\overline{Ou_{i-1}}-4 < \overline{Ou_i} < 2\cos\left(\frac{\pi}{5}\right)\overline{Ou_{i-1}}+4 < 1.7\overline{Ou_{i-1}}+4.$$

Suppose that $|x| \geq 50$. Clearly, $|x| - 1 < \overline{Ou_1}$, and by the previous calculations it is easy to show by induction that $|x| - 1 < \overline{Ou_i}$ for all $i \leq k$. Therefore, $1.5\overline{Ou_{i-1}} < 1.6\overline{Ou_{i-1}} - 4$ so we obtain

$$1.5\overline{Ou_{i-1}} < \overline{Ou_i}. \quad (1)$$

We have to verify that the above procedure does not produce "unnecessary" edges, that is, the following statement is true.

Claim 1. *Suppose that* $|x| \geq 50$.

(i) *The slope of* u_iu_j *is not a basic slope, for any* $j < i - 1$.

(ii) *The slope of* u_iv *is not a basic slope, for any* $v \in V(G \setminus C)$, $v \neq t_i$.

Proof. (i) Suppose that the slope of u_iu_j is a basic slope for some $j < i-1$. By repeated application of inequality (1), we obtain that $\overline{Ou_i} > 1.5^{i-j}\overline{Ou_j} > 2\overline{Ou_j}$. On the other hand, if u_iu_j has a basic slope, then easy geometric calculations show that $\overline{Ou_i} < 2\cos\left(\frac{\pi}{5}\right)\overline{Ou_j} + 4 < 2\overline{Ou_j}$, a contradiction.

(ii) Suppose for simplicity that t_iu_i has slope 0, i.e., it is horizontal. By the construction, no vertex v of $G \setminus C$ determines a horizontal segment with t_i, but all of them are within distance 2 from t_i. As $\overline{Ou_i} > x-1$, segment vu_i is almost, but not exactly horizontal. That is, we have $0 < |\angle t_iu_iv| < \pi/5$, contradiction. □

Suppose that Step 0, Step 1, ..., Step k have already been completed. It remains to determine the position of u_0. We need some preparation. The notation $|x| \geq 2 \mod 5$ means that $x = 2$ or $x = 3 \mod 5$.

Claim 2. *There exist two integers* $0 \leq \alpha, \beta < 5$ *with* $|\alpha - \beta| \geq 2 \mod 5$ *such that starting the* Procedure *with* $ind(u_1) = \alpha$ *and with* $ind(u_1) = \beta$*, we can continue so that* $ind(u_2)$ *is the same in both cases.*

Proof. Suppose that the degrees of t_1 and t_2 in $G \setminus C$ are *two*, that is, there are two forbidden lines for both u_1 and u_2. In the other cases, when the degree of t_1 or the degree of t_2 is less than *two*, or when $t_1 = t_2$, the proof is similar, but simpler. We can place u_1 on $\ell_l(t_1)$ for any $l \notin sl(t_1)$. Therefore, we have three choices, two of which, $\ell_\alpha(t_1)$ and $\ell_\beta(t_1)$, are not consecutive, so that $|\alpha-\beta| \geq 2 \mod 5$.

The vertex u_2 cannot be placed on $\ell_m(t_2)$ for any $m \in sl(t_2)$, so there are three possible lines for u_2: $\ell_x(t_2)$, $\ell_y(t_2)$, $\ell_z(t_2)$, say. For any fixed location of u_1, we can place u_2 on four possible lines, so on at least two of the lines $\ell_x(t_2)$, $\ell_y(t_2)$, and $\ell_z(t_2)$. Therefore, at least one of them, say $\ell_x(t_2)$, can be used for both locations of u_1. □

Claim 3. *We can place the vertices $u_1, u_2, \ldots, u_k$ using the* Procedure *so that* $|ind(u_1) - ind(u_k)| \geq 2 \mod 5$.

Proof. By Claim 2, there are two placements of the vertices of $C \setminus \{u_0, u_k\}$, denoted by $u_1, u_2, \ldots, u_{k-1}$ and by $u'_1, u'_2, \ldots, u'_{k-1}$ such that $|ind(u_1) - ind(u'_1)| \geq 2 \mod 5$, and $ind(u_i) = ind(u'_i)$ for all $i \geq 2$. That is, we can start placing the vertices on two non-neighboring lines so that from the second step of the Procedure we use the same lines. We show that we can place u_k so that u_1 and u_k, or u'_1 and u_k are on non-neighboring lines. Having placed u_{k-1} (or u'_{k-1}), we have four choices for $ind(u_k)$. Two of them can be ruled out by the condition $ind(u_k) \notin sl(t_k)$. We still have two choices. Since u_1 and u'_1 are on non-neighboring lines, there is only one line which is a neighbor of both of them. Therefore, we still have at least one choice for $ind(u_k)$ such that $|ind(u_1) - ind(u_k)| \geq 2$ or $|ind(u'_1) - ind(u_k)| \geq 2$. □

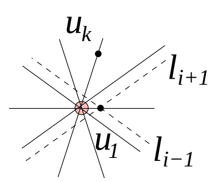

Figure 2: $\ell_{i+1}(u_1)$, does not separate the vertices of $G \setminus C$ from u_k, $\ell_{i-1}(u_1)$ does.

- Step $k+1$. Let $i = ind(u_1)$, $j = ind(u_k)$, and assume, by Claim 3, that $|i - j| \geq 2 \mod 5$. Consider the lines $\ell_{i-1}(u_1)$ and $\ell_{i+1}(u_1)$. One of them, $\ell_{i+1}(u_1)$, say, does not separate the vertices of $G \setminus C$ from u_k, the other one does. See Fig. 2.

  Place u_0 at the intersection of $\ell_{i+1}(u_1)$ and $\ell_i(u_k)$.

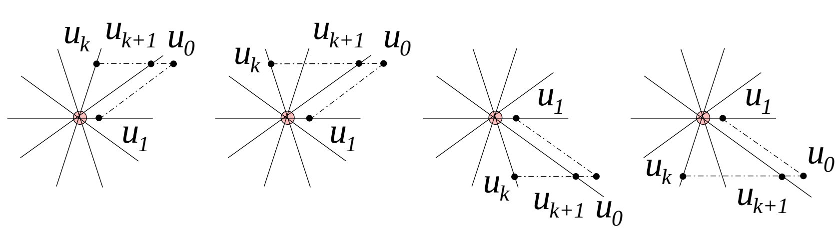

Figure 3: The four possible locations of u_0.

Claim 4. *Suppose that* $|x| \geq 50$.

(i) *The slope of u_0u_j is not a basic slope, for any $1 < j < k$.*

(ii) *The slope of u_0v is not a basic slope, for any $v \in V(G \setminus C)$.*

Proof. (i) Denote by u_{k+1} the intersection of $\ell_{i+1}(O)$ and $\ell_i(u_k)$. Suppose that the slope of u_0u_j is a basic slope for some $1 < j < k$. As in the proof of Claim 1, by repeated application of inequality 1, we obtain that $\overline{Ou_{k+1}} > 1.5^{k+1-j}\overline{Ou_j} > 2\overline{Ou_j}$. On the other hand, by an easy geometric argument, if the slope of u_0u_j is a basic slope, then $\overline{Ou_{k+1}} < 2\cos\left(\frac{\pi}{5}\right)\overline{Ou_j} + 4 < 2\overline{Ou_j}$, a contradiction, provided that $|x| \geq 50$.

(ii) For any vertex $v \in G \setminus C$, the slope of the segment u_0v is strictly between $i\pi/5$ and $(i+1)\pi/5$, therefore, it is not a basic slope. See Figure 3. This concludes the proof of the claim and hence Theorem 2. □

3 Proof of Theorem 1

First we note that if G is connected, then Theorem 1 is an easy corollary to Theorem 2. Indeed, delete any vertex, and then put it back using two extra directions. If G is not connected, the only problem that may arise is that these extra directions can differ for different components. We will define a family of drawings for each component G^i of G, depending on parameters ε_i, and then choose the values of these parameters in such a way that the extra directions will coincide.

Suppose that G is a cubic graph. If a connected component is not 3-regular then, by Theorem 2, it can be drawn using the *five* basic slopes. If a connected component is a complete graph K_4 on *four* vertices, then it can also be drawn using the basic slopes. For the sake of simplicity, suppose that we do not have such components, i. e. each connected component $G^1, \ldots, G^m$ of G is 3-regular and none of them is isomorphic to K_4.

First we concentrate on G^1. Let C be a shortest cycle in G^1. We distinguish two cases.

Case 1: *C is not a triangle.* Denote by $u_0, \ldots, u_k$ the vertices of C, and let t_0 be the neighbor of u_0 not belonging to C. Delete the edge u_0t_0, and let $\bar{G}$ be the resulting graph.

Case 2: *C is a triangle.* Every vertex of C has precisely *one* neighbor that does not belong to C. If all these neighbors coincide, then G^1 is a complete graph on *four* vertices, contradicting our assumption. So one vertex of C, u_0, say, has a neighbor t_0 which does not belong to C and which is not adjacent to the other two vertices, u_1 and u_2, of C. Delete the edge u_0t_0, and let $\bar{G}$ be the resulting graph.

Observe that in both cases, u_k and t_0 are not connected in G^1. Indeed, suppose for a contradiction that they are connected. In the first case, G^1 would contain the triangle $u_0u_kt_0$, contradicting the minimality of C. In the second case, the choice of u_0 would be violated.

There will be exactly two edges with extra directions, u_0u_k and u_0t_0. The slope of u_0u_k will be very close to a basic slope and the slope of u_0t_0 will be decided at the end, but we will show that almost any choice will do.

For any nonnegative ε and real x, ModifiedProcedure$(\bar{G}, C, u_0, u_1, x, \varepsilon)$ is defined as follows. Let Steps $0, 1, \ldots, k$ be identical to the corresponding

Steps of Procedure($\bar{G}, C, u_0, u_1, x$).

- Step $k+1$. If there is a segment, determined by the vertices of $G \setminus C$, of slope $i\pi/5 + \varepsilon$ or $i\pi/5 - \varepsilon$, for any $0 \le i < 5$, then Stop. In this case, we say that ε is *1-bad* for $\bar{G}$.

  Otherwise, when ε is *1-good*, let $i = ind(u_1)$ and $j = ind(u_k)$. We can assume by Claim 3 that $|i - j| \ge 2 \mod 5$. Consider the lines $\ell_{i-1}(u_1)$ and $\ell_{i+1}(u_1)$. One of them does not separate the vertices of $G \setminus C$ from u_k, the other one does.

  If $\ell_{i-1}(u_1)$ separates $G \setminus C$ from u_k, then place u_0 at the intersection of $\ell_{i+1}(u_1)$ and the line through u_k with slope $i\pi/5 + \varepsilon$. If $\ell_{i+1}(u_1)$ separates $G \setminus C$ from u_k, then place u_0 at the intersection of $\ell_{i-1}(u_1)$ and the line through u_k with slope $i\pi/5 - \varepsilon$.

Since Steps $0, 1, \ldots, k$ are identical in Procedure($\bar{G}, C, u_0, u_1, x$) and in ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$), Claims 1, 2, and 3 also hold for the ModifiedProcedure.

Moreover, it is easy to see that an analogue of Claim 4 also holds with an identical proof, provided that ε is sufficiently small: $0 < \varepsilon < 1/100$.

Claim 4'. *Suppose that $|x| \ge 50$ and $0 < \varepsilon < 1/100$.*

(i) *The slope of $u_0 u_j$ is not a basic slope, for any $1 < j < k$.*

(ii) *The slope of $u_0 v$ is not a basic slope, for any $v \in V(\bar{G} \setminus C)$.* □

Perform ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) for a fixed ε, and observe how the drawing changes as x varies. For any vertex u_i of C, let $u_i(x)$ denote the position of u_i, as a function of x. For every i, the function $u_i(x)$ is linear, that is, u_i moves along a line as x varies.

Claim 5. *With finitely many exceptions, for every value of x,* Modified-Procedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) *produces a proper drawing of $\bar{G}$, provided that ε is 1-good.*

Proof. Claims 1, 2, 3, and 4' imply Claim 5 for $|x| \ge 50$. Let u and v be two vertices of $\bar{G}$. Since $u(x)$ and $v(x)$ are linear functions, their difference, $\vec{uv}(x)$, is also linear.

If uv is an edge of $\bar{G}$, then the direction of $\vec{uv}(x)$ is the same for all $|x| \ge 50$. Therefore, it is the same for all values of x, with the possible exception of one value, for which $\vec{uv}(x) = 0$ holds.

If uv is not an edge of $\bar{G}$, then the slope of $\vec{uv}(x)$ is not a basic slope for any $|x| \ge 50$. Therefore, with the exception of at most *five* values of x, the slope of $\vec{uv}(x)$ is never a basic slope, nor does $\vec{uv}(x) = 0$ hold. □

Take a closer look at the relative position of the endpoints of the missing edge, $u_0(x)$ and $t_0(x)$. Since $t_0 \in \bar{G} \setminus C$, $t_0 = t_0(x)$ is the same for all values of x. The position of $u_0 = u_0(x)$ is a linear function of x. Let ℓ be the line determined by the function $u_0(x)$. If ℓ passes through t_0, then we say that ε is *2-bad* for $\bar{G}$. If ε is 1-good and it is not 2-bad for $\bar{G}$, then we say that it is *2-good* for $\bar{G}$. If ε is 2-good, then by varying x we can achieve almost any slope for the edge t_0u_0. This will turn out to be crucially important, because we want to attain that these slopes coincide in all components.

Claim 6. *Suppose that the values $\varepsilon \neq \delta$, $0 < \varepsilon, \delta < 1/100$, are 1-good for $\bar{G}$. Then at least one of them is 2-good for $\bar{G}$.*

Proof. Suppose, for simplicity, that $ind(u_1) = 0$, $ind(u_k) = 2$, and that u_1 and u_k are in the right half-plane (of the vertical line through the origin). The other cases can be settled analogously. To distinguish between ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) and ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \delta$), let $u_0^\varepsilon(x)$ denote the position of u_0 obtained by the first procedure and $u_0^\delta(x)$ its position obtained by the second. Let ℓ^ε and ℓ^δ denote the lines determined by the functions $u_0^\varepsilon(x)$ and $u_0^\delta(x)$. Suppose that x is very large. Since, by (1), we have $\overline{u_k(x)O} > 1.5\overline{u_1(x)O}$, both $u_0^\varepsilon(x)$ and $u_0^\delta(x)$ are on the line $\ell_1(u_1(x))$, very far in the positive direction. Therefore, both of them are above the line $\ell_{\pi/10}$. On the other hand, if $x < 0$ is very small (i.e., if $|x|$ is very big), both $u_0^\varepsilon(x)$ and $u_0^\delta(x)$ lie below the line $\ell_{\pi/10}$. It follows that the slopes of ℓ^ε and ℓ^δ are larger than $\pi/10$, but smaller than $\pi/5$.

Suppose that neither ε nor δ is 2-good. Then both ℓ^ε and ℓ^δ pass through t_0. That is, for a suitable value of x, we have $u_0^\varepsilon(x) = t_0$. We distinguish two cases.

Case 1: $u_0^\varepsilon(x) = t_0 = u_k(x)$. Then, as x varies, the line determined by $u_k(x)$ coincides with $\ell_2(t_0)$. Consequently, t_0 and u_k are connected in G^1, a contradiction.

Case 2: $u_0^\varepsilon(x) = t_0 \neq u_k(x)$. In order to get a contradiction, we try to determine the position of $u_0^\delta(x)$. If we consider Step $k+1$ in ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) and in ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \delta$), we can conclude that $u_1(x)$ lies on $\ell_1(u_0^\varepsilon) = \ell_1(t_0)$, $u_0^\delta(x)$ lies on $\ell_1(u_1(x))$, therefore, $u_0^\delta(x)$ lies on $\ell_1(t_0)$. On the other hand, $u_0^\delta(x)$ lies on ℓ^δ, and, by assumption, ℓ^δ passes through t_0. However, we have shown that ℓ^δ and $\ell_1(t_0)$ have different slopes, therefore, $u_0^\delta(x)$ must be at their intersection point, so we have $u_0^\delta(x) = u_0^\varepsilon(x) = t_0$.

Considering again Step $k+1$ in ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) and in ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \delta$), we can conclude that the point $u_0^\delta(x) = t_0 = u_0^\varepsilon(x)$ belongs to both $\ell_\varepsilon(u_k(x))$ and $\ell_\delta(u_k(x))$. This contradicts our assumption that $u_k(x)$ is different from $u_0^\delta(x) = t_0 = u_0^\varepsilon(x)$. □

By Claim 5, for every $\varepsilon < 1/100$ and with finitely many exceptions for every value of x, ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$) produces a proper drawing of $\bar{G}$. When we want to add the edge u_0t_0, the slope of $u_0(x)t_0$ may coincide with the slope of $u(x)u'(x)$, for some $u, u' \in \bar{G}$. The following statement guarantees that this does not happen "too often". We use $\alpha(\vec{u})$ to denote the *slope* of a vector $\vec{u}$.

Claim 7. *Let $\vec{u}(x)$ and $\vec{v}(x)$: $R \to R^2$ be two linear functions, and let $\ell(u)$ and $\ell(v)$ denote the lines determined by $\vec{u}(x)$ and $\vec{v}(x)$. Suppose that for some $x_1 < x_2 < x_3$, the vectors $\vec{u}, \vec{v}$ do not vanish and that their slopes coincide, that is, $\alpha(\vec{u}(x_1)) = \alpha(\vec{v}(x_1))$, $\alpha(\vec{u}(x_2)) = \alpha(\vec{v}(x_2))$, and $\alpha(\vec{u}(x_3)) = \alpha(\vec{v}(x_3))$. Then $\ell(u)$ and $\ell(v)$ must be parallel.*

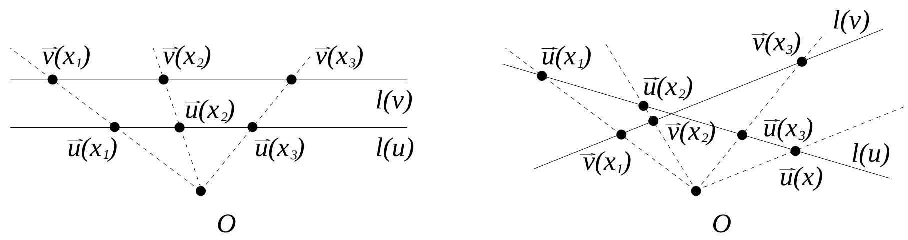

Figure 4: $\ell(u)$ and $\ell(v)$ must be parallel.

Proof. If $\ell(u)$ passes through the origin, then for every value of x, $\vec{u}(x)$ has the same slope. In particular, $\alpha(\vec{v}(x_1)) = \alpha(\vec{v}(x_2)) = \alpha(\vec{v}(x_3))$. Therefore, $\ell(v)$ also passes through the origin and is parallel to $\ell(u)$. (In fact, we have $\ell(u) = \ell(v)$.) We can argue analogously if $\ell(u)$ passes through the origin. Thus, in what follows, we can assume that neither $\ell(u)$ nor $\ell(v)$ passes through the origin.

Suppose that $\alpha(\vec{u}(x_1)) = \alpha(\vec{v}(x_1))$, $\alpha(\vec{u}(x_2)) = \alpha(\vec{v}(x_2))$, and $\alpha(\vec{u}(x_3)) = \alpha(\vec{v}(x_3))$. For any x, define $\vec{w}(x)$ as the intersection point of $\ell(v)$ and the line connecting the origin to $\vec{u}(x)$, provided that they intersect. Clearly, $\vec{v}(x) = \vec{w}(x)$ for $x = x_1, x_2, x_3$, and $\vec{u}(x)$ and $\vec{w}(x)$ have the same slope for every x. The transformation $\vec{u}(x) \to \vec{w}(x)$ is a projective transformation from $\ell(u)$ to $\ell(v)$, therefore, it preserves the cross ratio of any four points. That is, for any x, we have

$$(\vec{u}(x_1), \vec{u}(x_2); \vec{u}(x_3), \vec{u}(x)) = (\vec{w}(x_1), \vec{w}(x_2); \vec{w}(x_3), \vec{w}(x))\,.$$

Since both $\vec{u}(x)$ and $\vec{v}(x)$ are linear functions, we also have

$$(\vec{u}(x_1), \vec{u}(x_2); \vec{u}(x_3), \vec{u}(x)) = (\vec{v}(x_1), \vec{v}(x_2); \vec{v}(x_3), \vec{v}(x))\,.$$

Hence, we can conclude that $\vec{v}(x) = \vec{w}(x)$ for all x. However, this is impossible, unless $\ell(u)$ and $\ell(v)$ are parallel. Indeed, suppose that $\ell(u)$ and $\ell(v)$ are not parallel, and set x in such a way that $\vec{u}(x)$ is parallel to $\ell(v)$. Then $\vec{w}(x)$ cannot have the same slope as $\vec{u}(x)$, a contradiction. $\square$

Suppose that ε is 2-good and let us fix it. As above, let $u_0^{\varepsilon}(x)$ be the position of u_0 obtained by ModifiedProcedure($\bar{G}, C, u_0, u_1, x, \varepsilon$), and let ℓ^{ε} be the line determined by $u_0^{\varepsilon}(x)$.

Suppose also that there exist two independent vertices of $\bar{G}$, $u, u' \neq u_0$, such that the line determined by $\vec{uu'}(x)$ is parallel to ℓ^{ε}. Then we say that ε is *3-bad* for $\bar{G}$. If ε is 2-good and it is not 3-bad for $\bar{G}$, then we say that it is *3-good* for $\bar{G}$.

It is easy to see that, for any $0 < \varepsilon, \delta < 1/100$, ℓ^{ε} and ℓ^{δ} are not parallel, therefore, for any fixed u, u', there is at most one value of ε for which the line determined by $\vec{uu'}(x)$ is parallel to ℓ^{ε}. Thus, with finitely many exceptions, all values $0 < \varepsilon < 1/100$ are 3-good.

Summarizing, we have obtained the following.

Claim 8. *Suppose that ε is 3-good for $\bar{G}$. With finitely many exceptions, for every value of x,* ModifiedProcedure$(\bar{G}, C, u_0, u_1, x, \varepsilon)$ *gives a proper drawing of G^1.* $\Box$

Now we are in a position to complete the proof of Theorem 1. Proceed with each of the components as described above for G^1. For any fixed i, let $u_0^i v_0^i$ be the edge deleted from G^i, and denote the resulting graphs by $\bar{G}^1, \ldots, \bar{G}^m$. Let $0 < \varepsilon < 1/100$ be fixed in such a way that ε is 3-good for all graphs $\bar{G}^1, \ldots, \bar{G}^m$. This can be achieved, in view of the fact that there are only finitely many values of ε which are not 3-good. Perform ModifiedProcedure$(\bar{G}^i, C^i, u_0^i, u_1^i, x^i, \varepsilon)$. Now the line ℓ^i determined by all possible locations of u_0^i does not pass through t_0^i.

Notice that when ModifiedProcedure$(\bar{G}^i, C^i, u_0^i, u_1^i, x^i, \varepsilon)$ is executed, apart from edges with basic slopes, we use an edge with slope $r\pi/5 \pm \varepsilon$, for some integer $r \mod 5$. By using rotations through $\pi/5$ and a reflection, if necessary, we can achieve that each component $\bar{G}^i$ is drawn using the basic slopes and one edge of slope ε.

It remains to set the values of x_i and draw the missing edges $u_0^i v_0^i$. Since the line ℓ^i determined by the possible locations of u_0^i does not pass through t_0^i, by varying the value of x^i, we can attain any slope for the missing edge $t_0^i u_0^i$, except for the slope of ℓ^i. By Claim 8, with finitely many exceptions, all values of x^i produce a proper drawing of G^i. Therefore, we can choose $x^1, \ldots, x^m$ so that all segments $t_0^i u_0^i$ have the same slope and every component G^i is properly drawn using the same seven slopes. Translating the resulting drawings through suitable vectors gives a proper drawing of G, this completes the proof of Theorem 1.

4 Concluding Remarks

In the proof of Theorem 1, the slopes we use depend on the graph G. However, the proof shows that one can simultaneously embed all cubic graphs using only *seven* fixed slopes.

It is unnecessary to use $|x| \geq 50$, in every step, we could pick any x, with finitely many exceptions.

It seems to be only a technical problem that we needed *two* extra directions in the proof of Theorem 1. We believe that *one* extra direction would suffice.

The most interesting problem that remains open is to decide whether the number of slopes needed for graphs of maximum degree *four* is bounded.

Another not much investigated question is to estimate the complexity of computing the slope parameter of a graph. A related problem is to decide under what conditions a graph can be drawn on a polynomial sized grid using a fix number of slopes. Is it possible to draw all cubic graphs?

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