

## A First Order Logic Definition of Beyond-Planar Graphs

*Franz J. Brandenburg*

94030 Passau, Germany

### Abstract

Beyond-planarity is a collective term for classes of graphs that extend the planar graphs and are defined by drawings with restrictions on crossings. Examples are 1-planar, fan-planar, fan-crossing free, and quasi-planar graphs. We define these and other classes of beyond-planar graphs by simple first order formulas, using two predicates to express a crossing and an adjacency of two edges, and establish inclusion relationships between the so obtained graph classes.

Submitted: May 2017	Reviewed: August 2017	Revised: September 2017	Accepted: October 2017	Final: October 2017
------------------------	--------------------------	----------------------------	---------------------------	------------------------

Published:

Article type:  
Regular Paper

Communicated by:  
M. Bekos, M. Kaufmann, F. Montecchiani

## 1 Introduction

In the past few years, several classes of graphs have been introduced that extend the planar graphs and are defined by drawings with restrictions on crossings. These graphs are studied in Topological Graph Theory, Graph Drawing, and Computational Geometry. Particular examples, which are defined in Section 2, are 1-planar graphs [31, 32], fan-planar graphs [8, 10, 26], fan-crossing free graphs [15], quasi-planar graphs [3], and right angle crossing (RAC) graphs [21]. Moreover, there are specializations with all vertices in the outer face, such as outer 1-planar graphs [6, 24] and outer fan-planar graphs [8, 10].

These definitions are motivated by the need for classes of non-planar graphs from real world applications and a negative correlation between edge crossings and the readability of graph drawings by human users. The aforementioned graph classes aim to meet both requirements. Graphs belonging to these classes are called beyond-planar graphs [27, 29]. From the results obtained so far in this area, it turned out that most of these graphs have common graph properties, such as a linear density, an NP-hard recognition problem, and drawings using a planarization with a dummy vertex for each crossing point.

We aim at a complete list of classes of beyond-planar graphs. This goal is unreachable, since researchers are creative and there is no limit on restrictions of crossings and ways to extend the planar graphs. Here, we introduce a uniform framework with simple first-order logic formulas that captures the aforementioned classes of beyond-planar graphs and introduces some new ones. The formulas are defined in Section 3. In Section 4 we display inclusion relations in a hierarchy diagram.

## 2 Preliminaries

In this section, we define some classes of beyond-planar graphs.

A *graph*  $G = (V, E)$  consists of finite sets of vertices and edges. Two vertices  $u$  and  $v$  are *adjacent* if there is an edge  $e = \{u, v\}$  and two edges are *adjacent* if they share a common endvertex. We consider undirected graphs that are *simple* both in a graph theoretic and in a topological sense. Thus we do not admit multiple edges and self-loops, and exclude multiple crossings of two edges and crossings among adjacent edges. The subgraph induced by a subset  $U$  of vertices is denoted by  $G[U]$ .

A *drawing* of a graph  $G$  is a mapping of  $G$  into the plane such that the vertices are mapped to distinct points and each edge is mapped to a Jordan arc between the endpoints. A drawn graph is called a *topological graph*. In other words, a topological graph is called an *embedding*, which is the class of topologically equivalent drawings of a graph. A drawing may subdivide an edge at its crossing points. Non-crossed pieces are called *edge segments*, whose endpoints are vertices or crossing points. A drawn graph partitions the plane into topologically connected regions, called *faces*. The *boundary* of each face consists of a cyclic sequence of edge segments. It can be specified by the cyclic sequence

of vertices and crossing points of its edge segments. A vertex, crossing point or edge is said to be *incident* to a face if (a segment of) it is part of its boundary. The unbounded region is called the *outer face*. If all vertices are incident to the outer face, then a drawing is called an *outer drawing*. Outer drawings can also be specified by an additional vertex that is placed in the outer face and is connected to all other vertices by non-crossed edges, or by a Hamiltonian cycle  $H$  after an augmentation of the given graph by appropriate edges. The edges of  $H$  are non-crossed and all other edges are routed in the interior of  $H$ . For convenience, we identify vertices and edges of a graph, respectively, as points and Jordan arcs representing them in a drawing or an embedding.

A *fan* (or star or radial grid) is a set of edges with a single common endvertex. Edges are *independent* if they do not share a common endvertex. A set of edges is called a *tangle* if the edges mutually cross each other in a drawing, see Fig. 1(c). An edge  $e$  has a *fan-crossing* if the edges crossing  $e$  form a fan, see Fig. 1(a), and an *independent crossing* if the crossing edges are independent, see Fig. 1(b). Drawings with only independent crossings are commonly called *fan-crossing free* [15]. For the definition of fan-planar graphs, Kaufmann and Ueckerdt [26] imposed a special restriction. Let  $e, f$  and  $g$  be three edges in a drawing so that  $e$  is crossed by  $f$  and  $g$ , and  $f$  and  $g$  share a common vertex  $z$ . Then they form *configuration II* if one endvertex of  $e$  is inside a cycle through  $z$  and consisting of segments of  $e, f$  and  $g$ , and the other endvertex of  $e$  is outside this cycle, see Fig. 2(a). Then an edge may even cross a triangle, as illustrated in Fig. 2(b). Note that a triangle is the only configuration in which a crossed edge does not have a fan-crossing with all its crossed edges and, at the same time, does not cross any two independent edges.

A graph  $G$  is *planar* [23] if it admits a drawing so that edges do not cross, and it is *1-planar* [32] (*k-planar* [31]) if it admits a drawing so that each edge is crossed at most once (by at most  $k$  edges). If there are no  $k \geq 2$  pairwise crossing edges, then  $G$  is called *k-quasi-planar*, where 2-quasi-planar graphs are planar and 3-quasi-planar graphs are simply called *quasi-planar* [3]. Hence, quasi-planar graphs exclude tangles of three or more edges. Conversely, we call  $G$  a *tangle graph* if it admits a drawing such that for every edge  $e$ , the set of edges crossing  $e$  is a tangle. A graph  $G$  is *fan-crossing free* if there are no fan-crossings [15]. Then there are only crossings of independent edges. We call  $G$  a *grid-crossing graph* if it has a drawing that is simultaneously quasi-planar and fan-crossing free. Then the set of edges can be partitioned into non-crossed edges and sets of independent edges  $X_i$  and  $Y_i$  for some  $i \geq 0$  with the following properties: the edges of  $X_i$  cross the edges of  $Y_i$  and do not cross edges of  $X_i$  or  $Y_j$  for  $i \neq j$ , and accordingly for  $Y_i$ . Moreover, there is no edge that crosses both an edge of  $X_i$  and an edge of  $Y_i$ . In complement, grid-crossings are excluded in grid free graphs [1]. A graph  $G$  is *fan-planar* if it admits a drawing that avoids independent crossings and configuration II [26], see Fig. 2. Then there are fan-crossings, but only if the crossing edges have a common endvertex on the same side of the crossed edge. We call a graph *fan-crossing* if it admits a drawing in which every crossing is a fan-crossing, and *adjacency-crossing* if it can be drawn so that each edge is crossed by edges that are pairwise adjacent,

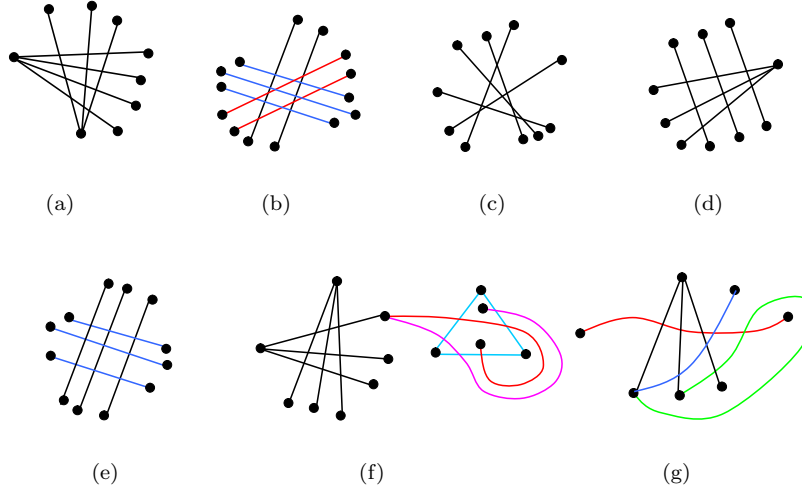


Figure 1: (a) fan-crossing, (b) fan-crossing free with  $k \geq 3$  sets of independent edges, (c) tangle crossing = pairwise crossing edges, (d) quasi-planar = no three mutually crossing edges, (e) grid-crossing with two sets of crossing edges, (f) adjacency-crossing with a fan (left) and a crossed triangle (right), and (g) adjacency-tangle crossings.

as shown in Fig. 1(f). Note that fan-crossing graphs admit configuration II and exclude edges crossing a triangle, and adjacency crossing graphs only exclude independent crossings. Finally, a graph  $G$  is called an *adjacency-tangle crossing* graph if edges  $f_1$  and  $f_2$  crossing edge  $e$  in a drawing of  $G$  are adjacent or cross each other, see Fig. 1(g).

From the aforementioned graph classes, the quasi-planar graphs were studied first [3]. It has been shown that they have at most  $6.5n - 20$  edges [2]. Fan-crossings and fan-crossing free graphs were introduced by Cheong et al. [15], who showed that fan-crossing free graphs have at most  $4n - 8$  edges. Graphs with fan-crossings were studied first by Kaufmann and Ueckerdt [26], who proved that fan-planar graphs have at most  $5n - 10$  edges. However, the relationship between fan-planar, fan-crossing, and adjacency-crossing graphs is unclear. Kaufmann and Ueckerdt observed that configuration II cannot occur in straight-line drawings, so that every straight-line adjacency crossing drawing is fan-planar. They posed the density of adjacency-crossing graphs as an open problem.

Note that adjacency-crossing and fan-crossing free, as well as quasi-planar and tangle are complementary pairs with respect to their defining properties, and 1-planar is a specialization of each of them. Adjacency-tangle crossings combine adjacency and tangle crossings by a disjunction, and grid-crossing is the intersection of quasi-planar and fan-crossing free. For an illustration see Fig. 1 and consider Fig. 5 for inclusion relations between the graph classes.

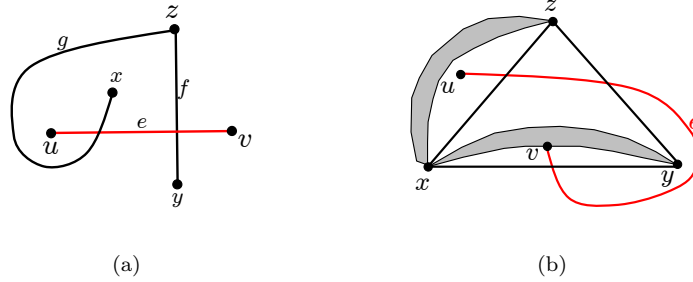


Figure 2: (a) configuration II and (b) a triangle crossing. The shaded regions in (b) represent subgraphs that prevent a rerouting of edge  $e$ . Such subgraphs must also exist in (a).

We use names in capital letters for graph classes (with shortcuts) and denote the aforementioned classes of simple (graph theoretic and topological) graphs by PLANAR, 1-PLANAR, FAN-PLANAR, FAN-CROSSING, ADJ-CROSSING, QUASI-PLANAR, FAN-FREE, GRID-CROSSING, TANGLE, and ADJ-TANGLE, respectively. We add the prefix “outer” for the respective outer classes and denote the class of all undirected graphs by GRAPHS. The classes of adjacency-crossing, fan-crossing, grid-crossing, tangle and adjacency-tangle crossing graphs have not been studied before.

One may also consider non-simple topological graphs with multiple edge crossings and crossings among adjacent edges. Clearly, there is no difference between simple and non-simple planar (1-planar, RAC) graphs. However, there are non-simple quasi-planar graphs with  $7n - O(1)$  edges whereas simple quasi-planar graphs have at most  $6.5n - 20$  edges [2]. Non-simple graphs have not yet been studied in the other cases.

There are further classes of graphs that are beyond-planar in some sense and are related to the aforementioned graph classes, such as right angle crossing graphs (RAC) [21], map graphs [14] and several graph classes that are defined by visibility representations [9, 11, 20, 25, 30]. Also, topological graphs that avoid other crossing patterns, such as grids, are related [1].

### 3 First-Order Logic Formulas

In algebra and logic, a graph  $G$  is a structure with a set of vertices and a binary relation for the edges such that  $adj(u, v)$  if and only if there is an edge  $e = \{u, v\}$ . An undirected graph satisfies the formula  $\forall u, v adj(u, v) \Rightarrow adj(v, u)$  and it is simple if also  $\forall v \neg adj(v, v)$  holds. Multiple edges cannot exist in this structure. Richer structures with sets of vertices and edges and the incidence relation are used for the expression of graph properties in monadic second order logic [16, 19]. In a further generalization, Courcelle [17] introduced predicates

for a rotation system, i.e., the cyclic ordering of edges at a vertex, and showed that the unique embedding of 3-connected planar graphs can be specified in monadic second order logic. Furthermore, he introduced predicates to express the order in which a directed edge is crossed by other edges and crossings from left to right [18].

For the definition of some classes of beyond-planar graphs, we use predicates for adjacency and edge crossings.

**Definition 1** *For a topological graph  $G$  and edges  $e, f$ , we say that  $e$  and  $f$  are adjacent, denoted  $\alpha(e, f)$ , if they share a common endpoint. Let  $\chi(e, f)$  if  $e$  and  $f$  cross exactly once in a drawing of  $G$  and  $\chi^*(e, f)$  if  $e$  and  $f$  cross at least once.*

The adjacency relation  $\alpha(e, f)$  is defined on graphs whereas the crossing relation needs a drawing or a topological graph. The relations can be used to describe simple and non-simple topological graphs.

We consider universally quantified first-order formulas with three variables  $e, f$  and  $g$  for edges and the predicates  $\alpha$  and  $\chi$ . For convenience, we use the same letters for edges and variables for edges. Let  $\Pi = \{\chi(e, f), \chi(e, g), \chi(f, g), \alpha(e, f), \alpha(e, g), \alpha(f, g)\}$ . A predicate  $\pi \in \Pi$  can be regarded as a boolean variable so that each formula corresponds to a boolean function over six variables. In total, there are  $2^{2^6}$  boolean formulas over  $\Pi$ . We wish to define classes of beyond-planar graphs and therefore restrict ourselves to formulas of the form  $\forall e, f, g (\beta \Rightarrow \gamma)$ , where the subformula  $\beta$  expresses edge crossings.

**Definition 2** *For simple topological graphs and three variables for distinct edges  $e, f$  and  $g$  we define:*

1.  $\varphi_1 = \alpha(e, f)$ .
2.  $\varphi_2 = \neg\alpha(e, f)$ .
3.  $\varphi_3 = \chi(e, f)$ .
4.  $\varphi_4 = \neg\chi(e, f)$ .
5.  $\varphi_5 = \chi(e, f) \Rightarrow \chi(e, g)$ .
6.  $\varphi_6 = \chi(e, f) \Rightarrow \neg\chi(e, g)$ .
7.  $\varphi_7 = \chi(e, f) \wedge \chi(e, g) \Rightarrow \alpha(f, g)$ .
8.  $\varphi_8 = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\alpha(f, g)$ .
9.  $\varphi_9 = \chi(e, f) \wedge \chi(e, g) \Rightarrow \chi(f, g)$ .
10.  $\varphi_{10} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\chi(f, g)$ .
11.  $\varphi_{11} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \alpha(f, g) \vee \chi(f, g)$ .

12.  $\varphi_{12} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\alpha(f, g) \wedge \neg\chi(f, g)$ .
13.  $\varphi_{13} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\alpha(f, g) \vee \chi(f, g)$ .
14.  $\varphi_{14} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \alpha(f, g) \wedge \neg\chi(f, g)$ .
15.  $\varphi_{15} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \alpha(f, g) \vee \neg\chi(f, g)$ .
16.  $\varphi_{16} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\alpha(f, g) \wedge \chi(f, g)$ .
17.  $\varphi_{17} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \neg\alpha(f, g) \vee \neg\chi(f, g)$ .
18.  $\varphi_{18} = \chi(e, f) \wedge \chi(e, g) \Rightarrow \alpha(f, g) \wedge \chi(f, g)$ .

Formulas  $\varphi_1$  to  $\varphi_4$  are the basic ones and express that two edges are adjacent or cross or not.  $\varphi_5$  and  $\varphi_6$  admit that two edges do or do not cross and if they cross whether or not a third edge crosses each of them, since  $e$  and  $f$  can be exchanged. Finally,  $\varphi_7$  to  $\varphi_{18}$  is an exhaustive list of consequents (to the right of “ $\Rightarrow$ ”) with predicates from the set  $\Pi$  and three variables for edges so that the antecedent (to the left of “ $\Rightarrow$ ”) expresses that an edge is crossed at least twice.

For each formula  $\varphi$  from above there is a set of simple topological graphs  $\mathcal{G}$  satisfying the extension of  $\varphi$  to a first-order formula  $\Phi$  so that  $\mathcal{G} \models \Phi$ . The first order formula expresses that edges are distinct and three edges may cross or are adjacent, where adjacent edges do not cross. In other words,  $\mathcal{G}$  is a *model* of  $\Phi$  or simply of  $\varphi$  [33].

**Theorem 1** *Consider first order logic formulas with variables for edges in a topological graph of the form*

$$\Phi_i = \forall e, f, g \quad \eta(\varphi_i),$$

where  $\eta(\varphi_i) = (e \neq f \wedge e \neq g \wedge f \neq g) \Rightarrow (\varphi_i \wedge \beta)$

and  $\beta = (\alpha(e, f) \Rightarrow \neg\chi(e, f)) \wedge (\alpha(e, g) \Rightarrow \neg\chi(e, g)) \wedge (\alpha(f, g) \Rightarrow \neg\chi(f, g))$ .  
Alternatively, one can write:

$$\Phi_i = \forall e, f (e \neq f \Rightarrow \neg(\alpha(e, f) \wedge \chi(e, f))) \wedge \forall e, f, g (e \neq f \neq g \neq e \Rightarrow \varphi_i).$$

Then we obtain the following characterizations for graphs  $G$ :

1.  $G \models \Phi_1$  if and only if  $G$  is a fan or a triangle and may have isolated vertices.
2.  $G \models \Phi_2$  if and only if  $G$  consists of isolated edges and vertices.
3.  $G \models \Phi_3$  if and only if  $G$  consists of a tangle and isolated vertices.
4.  $G \models \Phi_4$  if and only if  $G$  is planar.
5.  $G \models \Phi_5$  if and only if  $G$  is planar or consists of a single tangle together with isolated vertices.
6.  $G \models \Phi_6$  if and only if  $G$  is 1-planar.
7.  $G \models \Phi_7$  if and only if  $G$  is adjacency-crossing.

8.  $G \models \Phi_8$  if and only if  $G$  is fan-crossing free.
9.  $G \models \Phi_9$  if and only if  $G$  is a tangle graph.
10.  $G \models \Phi_{10}$  if and only if  $G$  is quasi-planar.
11.  $G \models \Phi_{11}$  if and only if  $G$  is an adjacency-tangle graph.
12.  $G \models \Phi_{12}$  if and only if  $G$  is grid-crossing.
13.  $G \models \Phi_{13}$  if and only if  $G$  is fan-crossing free.
14.  $G \models \Phi_{14}$  if and only if  $G$  is adjacency-crossing.
15.  $G \models \Phi_{15}$  if and only if  $G$  is quasi-planar.
16.  $G \models \Phi_{16}$  if and only if  $G$  a tangle graph.
17.  $G \models \Phi_{17}$  if and only if  $G$  is a simple topological graph.
18.  $G \models \Phi_{18}$  if and only if  $G$  is 1-planar.

**Proof:** The characterization for  $\Phi_1, \dots, \Phi_4$  and  $\Phi_6$  are obvious. If there is a pair of crossing edges, then all edges must cross if  $\Phi_5$  holds. A graph  $G$  satisfying  $\Phi_i$  for  $i = 7, \dots, 18$  has the following properties: if an edge  $e$  is crossed by edges from a set  $F = \{f_1, \dots, f_k\}$  with  $k \geq 2$ , then every pair of edges in the set  $F$  must satisfy the consequent of  $\varphi_i$ . Then two edges cross at most once and adjacent edges do not cross so that  $G$  is simple. Now suppose that  $e$  is crossed by the edges of  $F$ . If  $G$  satisfies  $\Phi_7$ , then the crossing edges  $f_1, \dots, f_k$  are adjacent and  $G$  is adjacency-crossing. For  $i = 8, 10$  and  $12$ , the graphs are fan-crossing free, quasi-planar and grid-crossing, respectively. The edges of  $F$  cross pairwise and form a tangle together with  $e$  if  $i = 9$ . There may be several tangles. If two edges from two tangles cross, then all edges of the tangles must cross. Hence, distinct tangles are edge disjoint and  $G$  is a tangle graph. For  $i = 11$  the edges of  $F$  cross or are adjacent so that  $G$  is an adjacency-tangle graph. Since adjacent edges do not cross, the formula  $\alpha(e, f) \Rightarrow \neg\chi(e, f)$  leads to a simplification of the consequent of  $\Phi_i$  for  $i = 13, 14, 15, 16, 18$  which can be replaced by  $\neg\alpha(f, g), \alpha(f, g), \neg\chi(f, g), \chi(f, g)$ , and *true*, respectively so that the model coincides with the specified graph class. In other words,  $\Phi_{13} \equiv \Phi_8$ ,  $\Phi_{14} \equiv \Phi_7$ ,  $\Phi_{15} \equiv \Phi_{10}$ ,  $\Phi_{16} \equiv \Phi_9$ , and  $\Phi_{18} \equiv \Phi_6$ , where “ $\equiv$ ” means equivalence [33]. In particular, if  $G$  satisfies  $\Phi_{14}$ , then the edges of  $F$  crossing  $e$  are adjacent and do not cross each other. However, adjacent edges do not cross so that  $G$  is adjacency-crossing. Finally,  $\varphi_{18}$  is false, since adjacent edges do not cross. Hence, there cannot be two such edges so that each edge is crossed at most once. In complement,  $\varphi_{17}$  is true so that  $\Phi_{17}$  describes every simple topological graph. This concludes the proof of the only-if direction.

The if-direction is clear from the definitions. □

Note that the consequents of  $\varphi_i$  and  $\varphi_{i+1}$  are complementary for  $i = 2j - 1$  and  $j = 1, \dots, 9$  so that  $\varphi_{i+1} = \gamma \Rightarrow \neg\eta$  if  $\varphi_i = \gamma \Rightarrow \eta$ . The defining property



of the graphs satisfying  $\Phi_{i+1}$  is the negation of the defining property of the graphs satisfying  $\Phi_i$ . However, the graph classes are not complementary and each includes the 1-planar graphs.

There are no more formulas of the form  $\Phi = \eta(\varphi)$  with  $\varphi = \chi(e, f) \wedge \chi(e, g) \Rightarrow \beta$  for some boolean formula  $\beta$  over  $\Pi$ , since  $\neg(\chi(e, f) \wedge \alpha(e, f))$  is assumed for any two edges  $e$  and  $f$ . However, there are more formulas of the form  $\varphi = \chi(e, f) \Rightarrow \beta$ . For example, let  $\beta = \chi(e, g) \vee \alpha(e, g)$ . If a graph  $G$  satisfying  $\Phi$  with  $\Phi = \eta(\varphi)$  has edges  $e_1, \dots, e_k$  for some  $k \geq 3$  and  $e_1$  and  $e_2$  cross, then the remaining edges  $e_3, \dots, e_k$  must cross  $e_1$  and  $e_2$  or they must be adjacent. It is unclear which of these formulas is meaningful and leads to the definition of a useful class of beyond-planar graphs. Non-crossed edges are possible by the implication.

Also, note that the use of  $\chi(e, f)$  and  $\chi^*(e, f)$  distinguishes between simple and non-simple topological graphs, which is relevant, in particular, for quasi-planar graphs [2].

The graph classes ADJ-CROSSING, FAN-CROSSING, GRID-CROSSING, TANGLE and ADJ-TANGLE are new. Each class contains the 1-planar graphs. The classes ADJ-CROSSING and FAN-CROSSING are an extension of the class of fan-planar graphs introduced by Kaufmann and Ueckerdt [26], since triangle crossings and configuration II are allowed. The set of all simple topological graphs is the model of  $\Phi_{17}$ , since the restrictions on crossings are vacuous. However, we do not consider the class GRAPHS to be beyond-planar.

A topological graph  $G = (V, E)$  is *outer- $\gamma$*  for a graph property  $\gamma$  if  $G$  admits a drawing with all vertices in the outer face so that the drawing satisfies  $\gamma$ . The outer restriction can be expressed by a formula with variables for vertices and edges and the relations  $\alpha(e, f)$ ,  $\chi(e, f)$  and  $inc(e, v)$ , where  $inc(e, v)$  describes the incidence between an edge  $e$  and an endvertex  $v$ . There must be a new vertex  $w \notin V$  in the outer face that is adjacent to all other vertices by non-crossed edges. However, structures with variables for edges and vertices and three predicates are richer than the ones used in our framework. Another description of outer- $\gamma$  graphs is given by Courcelle [18] using a Hamiltonian cycle.

The formulas in Theorem 1 with three or  $k+2$  universally quantified variables for edges describe the case  $k = 1$ . It is the first layer of classes of beyond-planar graphs. Formulas with  $(k + 2)$  variables are more powerful and admit the expression of  $k$ -planar [31],  $k + 2$ -quasi-planar [3], and  $k$ -fan-crossing free graphs [15] on the  $k$ -th layer of classes of beyond-planar graphs. In this way, many new classes of beyond-planar graphs can be defined.

## 4 Classification

The 1-planar graphs are the best known class of beyond-planar graphs. They were introduced by Ringel [32] and have intensively been studied since then.

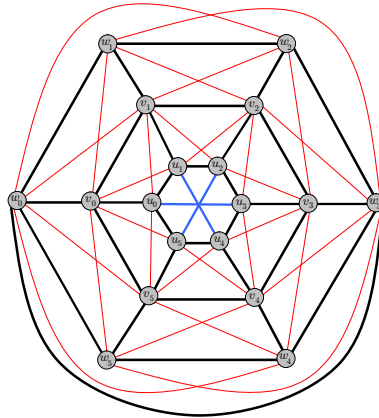


Figure 3: A tangle graph that is not 1-planar

In an annotated bibliography, Kobourov et al. [27] summarize the state on 1-planar graphs. The listed problems can be used as a guideline for a study of beyond-planar graphs. We restrict ourselves to inclusion relations between the aforementioned and some graph classes and leave typical problems of beyond-planar graphs, such as upper and lower bounds on the number of edges of maximal graphs (density and sparsity), the recognition problem, unique embeddings, alternative drawings and other representations, graph parameters, and the closure under graph operations, for further studies.

There are obvious inclusion relations between classes of beyond-planar graphs by the definition. Non-inclusions can be derived from the density bounds and from counter-examples, which are often hard to obtain, and results are rare. Some other graph classes come into play.

A graph is called *right-angle crossing* (RAC) if it admits a straight-line drawing in the plane so that edges may cross at a right angle [21]. RAC graphs are defined by a geometric property and not by a topological one. A graph is *1-bend-RAC* [22] if it admits a polyline drawing with at most one bend per edge and segments may cross at a right angle. A 1-planar graph is *IC-planar* [5, 12] (*NIC-planar* [7, 34]) if each vertex is incident to at most one crossing edge (each pair of vertices is incident to at most one pair of crossing edges).

Before we display our hierarchy diagram we need a technical result.

**Lemma 1** *Graph  $N$  from Fig. 3 is a tangle graph and not 1-planar.*

**Proof:** Graph  $N = (V, E)$  consists of an inner hexagon with vertices  $u_0, \dots, u_5$ , two rings of six  $K_4$  with vertices  $u_i, u_{i+1}, v_i, v_{i+1}$  and  $v_i, v_{i+1}, w_i, w_{i+1}$  for  $i = 0, \dots, 5$ , and two outer  $K_4$  with vertices  $w_0, w_1, w_2, w_3$  and  $w_0, w_5, w_4, w_3$ . All indices are modulo 6. The drawing proves that  $N$  is a tangle graph and 1-planar except for the triple of crossing edges  $\{u_i, u_{i+3}\}$  for  $i = 0, 1, 2$ . We shall show that the embedding of  $N$  is almost unique if it shall be 1-planar.

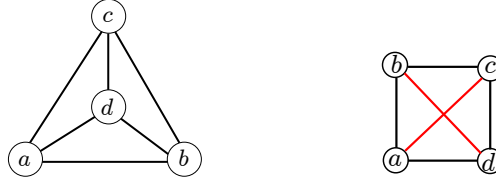


Figure 4: The two possible embeddings of  $K_4$  as a (planar) tetrahedron or as a kite

There are two embeddings or topological graphs of  $K_4$  [28], as a tetrahedron (where the non-crossed planar edges may be crossed by other edges) and as a kite (augmented X-configuration [4]) with one pair of crossing edges as shown in in Fig. 4. Then the other edges are non-crossed, since  $N$  is 3-connected and the embedding can be transformed into normal form [4].

We claim that each  $K_4$  with vertices  $v_i, v_{i+1}$  for  $i = 0, \dots, 5$  must be embedded as a kite. Otherwise, there is no 1-planar embedding of  $N$ . Let  $\mathcal{E}(N)$  be any 1-planar embedding. Suppose that the  $K_4$  subgraph induced by  $v_2, v_3, u_2, u_3$  is embedded as a tetrahedron  $T$  with  $u_2$  in the interior of the triangle of  $u_3, v_2, v_3$ . All other cases are similar, since  $N$  has many symmetries. Every proper subgraph  $N[U]$  with  $U \subset V$  has at least six neighbors in  $N[V - U]$  and it has at least seven neighbors if  $U \neq \{u_i\}$  or  $V - U \neq \{u_i\}$  for  $i = 0, \dots, 5$ . Hence, the remaining vertices must be placed in the outer face of  $\mathcal{E}(N)$  or in the triangle  $(u_2, v_2, v_3)$  of  $T$ , since an edge of  $T$  would be crossed at least twice, otherwise. Suppose the vertices are placed in the outer face of  $\mathcal{E}(N)$ , otherwise exchange the roles of  $u_2$  and  $u_3$ . Since  $u_2$  has degree six, each outer edge of  $T$  is crossed by one edge incident to  $u_2$  in  $\mathcal{E}(N)$ . The  $K_4$  subgraph  $N[v_2, v_3, w_2, w_3]$  must be embedded as a kite, since  $w_2$  (or symmetrically  $w_3$ ), as a center of a tetrahedron, cannot be connected to its neighbors in  $\mathcal{E}(N)$ . This implies that the edge  $\{v_2, v_3\}$  is non-crossed, but it must be crossed by an edge from  $\{u_2, u_1\}$ ,  $\{u_2, v_1\}$  or  $\{u_2, u_5\}$ , a contradiction.

Hence, the rings of  $K_4$  of  $N$  have a unique 1-planar embedding so that the edges  $\{u_i, u_{i+1}\}$ ,  $\{v_i, v_{i+1}\}$ ,  $\{w_i, w_{i+1}\}$  and  $\{u_i, v_i\}$ ,  $\{v_i, w_i\}$  are non-crossed for  $i = 0, \dots, 5 \pmod{6}$ . Then  $\mathcal{E}(N)$  is unique up to the choice of the outer face. Now, the edges  $\{u_i, u_{i+3}\}$  of the inner crossed hexagon cross pairwise in  $\mathcal{E}(N)$  so that  $\mathcal{E}(N)$  is not 1-planar.  $\square$

Note that there is a quasi-planar drawing of  $N$  by routing, e.g., edge  $\{u_0, u_3\}$  around  $u_1$  and thereby avoiding a crossing with  $\{u_1, u_4\}$ .

We now display the inclusion relationship of the aforementioned classes of beyond-planar graphs.

**Theorem 2** *A graph class  $\mathcal{G}$  is contained in a graph class  $\mathcal{G}'$ ,  $\mathcal{G} \subseteq \mathcal{G}'$ , if there is an arrow from  $\mathcal{G}$  to  $\mathcal{G}'$  in Fig. 5, and the inclusion is proper,  $\mathcal{G} \subset \mathcal{G}'$ , if there is a thick arrow. There is an incomparability, if there is a dotted line.*

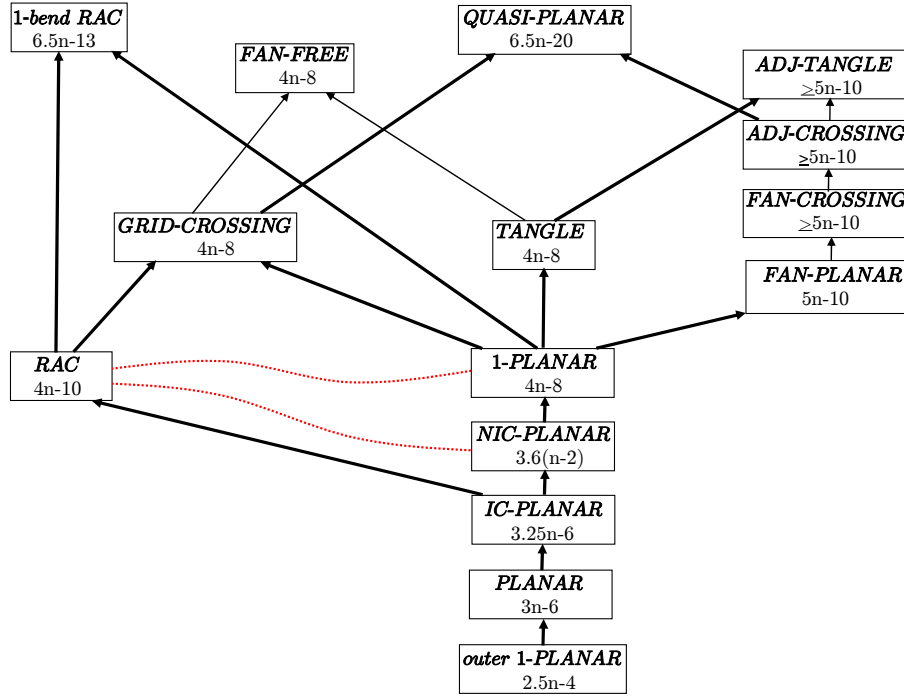


Figure 5: A hierarchy of classes of beyond-planar graphs and their density. A (thick) arrow indicates a (proper) inclusion and red, dotted lines an incomparability between the graph classes.

**Proof:** We scan the diagram from bottom to top. Auer et al [6] have shown that every outer 1-planar graph is planar. The hierarchy from planar to tangle is obvious by definition and it is proper due to the density and by Lemma 1.

Since our graphs are simple topological graphs, a tangle consists of independent edges. Hence, every tangle graph is fan-crossing free, and thereby tangle graphs of size  $n$  have at most  $4n - 8$  edges [15]. Clearly,  $TANGLE \subset ADJ-TANGLE$ , where the proper inclusion is due to the density. Every RAC-drawing is simultaneously fan-crossing free and quasi-planar, and therefore  $RAC \subset GRID-CROSSING$  holds. The inclusion is proper, since every 1-planar graph is grid-crossing and 1-planar and RAC graphs are incomparable [21]. The incomparability has been extended to NIC and RAC by Bachmaier et al. [7] who showed that there are mutual counterexamples. On the other hand, every IC-planar graph admits a RAC-drawing [12], and every 1-planar graph admits a 1-bend RAC drawing [22], and the properness of the inclusions is due to the density. Clearly, every 1-planar graph is grid-crossing but not conversely. Since grid-crossing graphs are fan-crossing free, their density is  $4n - 8$ , which is less than the density of quasi-planar graphs. The inclusion relations from 1-PLANAR to ADJ-TANGLE are obvious and every adjacency-crossing graph

is quasi-planar, since three edges do not cross pairwise in an adjacency-crossing of a simple topological graph.  $\square$

The displayed hierarchy diagram can be extended by outer graph classes, such as outer-RAC, outer FAN-PLANAR, etc. Such classes are candidates for a recognition in polynomial time, as shown for outer 1-planar [6, 24] and maximal outer fan-planar graphs [8].

We conjecture that the diagram is complete in the sense that there is a proper inclusion between graph classes  $\mathcal{G}$  and  $\mathcal{G}'$  if there is a path from  $\mathcal{G}$  to  $\mathcal{G}'$  and an incomparability otherwise. For proofs of an incomparability we wish to use path-additions [13] to distinguish beyond-planar graph classes and to construct counterexamples and graphs with an almost unique embedding.

## 5 Conclusion

We have defined some classes of beyond-planar graphs by a uniform framework of simple first-order logic formulas and have established inclusion relationships among the defined graph classes. The approach can be extended to introduce many more classes of beyond-planar graphs, including classes of outer graphs and graph classes on larger layers with  $k \geq 2$ . It opens a very broad field for studies of problems that are typical for beyond-planar graphs.

## 6 Acknowledgements

I wish to thank the anonymous reviewers for their valuable suggestions.

## References

- [1] E. Ackerman, J. Fox, J. Pach, and A. Suk. On grids in topological graphs. *Comput. Geom.*, 47(7):710–723, 2014. doi:10.1016/j.comgeo.2014.02.003.
- [2] E. Ackerman and G. Tardos. On the maximum number of edges in quasi-planar graphs. *J. Comb. Theory, Ser. A*, 114(3):563–571, 2007. doi:10.1016/j.jcta.2006.08.002.
- [3] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997. doi:10.1007/BF01196127.
- [4] M. J. Alam, F. J. Brandenburg, and S. G. Kobourov. Straight-line drawings of 3-connected 1-planar graphs. In S. Wismath and A. Wolff, editors, *GD 2013*, volume 8242 of *LNCS*, pages 83–94. Springer, 2013. doi:10.1007/978-3-319-03841-4\_8.
- [5] M. Albertson. Chromatic number, independence ratio, and crossing number. *Ars Math. Contemp.*, 1(1):1–6, 2008.
- [6] C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, and J. Reislhuber. Outer 1-planar graphs. *Algorithmica*, 74(4):1293–1320, 2016. doi:10.1007/s00453-015-0002-1.
- [7] C. Bachmaier, F. J. Brandenburg, K. Hanauer, D. Neuwirth, and J. Reislhuber. NIC-planar graphs. *Discrete Appl. Math.*, 232:23–40, 2017. doi:10.1016/j.dam.2017.08.015.
- [8] M. A. Bekos, S. Cornelsen, L. Grilli, S. Hong, and M. Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. *Algorithmica*, 79(2):401–427, 2017. doi:10.1007/s00453-016-0200-5.
- [9] T. C. Biedl, G. Liotta, and F. Montecchiani. On visibility representations of non-planar graphs. In S. P. Fekete and A. Lubiw, editors, *SoCG 2016*, volume 51 of *LIPICs*, pages 19:1–19:16. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPICs.SoCG.2016.19.
- [10] C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, A. Symvonis, and I. G. Tollis. Fan-planarity: Properties and complexity. *Theor. Comput. Sci.*, 589:76–86, 2015. doi:10.1016/j.tcs.2015.04.020.
- [11] F. J. Brandenburg. T-shape visibility representations of 1-planar graphs. *CoRR*, abs/1702.05265, 2017. URL: <http://arxiv.org/abs/1702.05265>.
- [12] F. J. Brandenburg, W. Didimo, W. S. Evans, P. Kindermann, G. Liotta, and F. Montecchiani. Recognizing and drawing IC-planar graphs. *Theor. Comput. Sci.*, 636:1–16, 2016. doi:10.1016/j.tcs.2016.04.026.

- [13] F. J. Brandenburg, A. Esch, and D. Neuwirth. Path-additions. *CoRR*, abs/1605.02891, 2016. URL: <http://arxiv.org/abs/1605.02891>.
- [14] Z. Chen, M. Grigni, and C. H. Papadimitriou. Map graphs. *J. ACM*, 49(2):127–138, 2002. doi:10.1145/506147.506148.
- [15] O. Cheong, S. Har-Peled, H. Kim, and H. Kim. On the number of edges of fan-crossing free graphs. *Algorithmica*, 73(4):673–695, 2015. doi:10.1007/s00453-014-9935-z.
- [16] B. Courcelle. Graph rewriting: An algebraic and logic approach. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics (B)*, pages 193–242. Elsevier, 1990.
- [17] B. Courcelle. The monadic second-order logic of graphs XII: planar graphs and planar maps. *Theor. Comput. Sci.*, 237(1-2):1–32, 2000. doi:10.1016/S0304-3975(99)00305-9.
- [18] B. Courcelle. The monadic second-order logic of graphs XIII: graph drawings with edge crossings. *Theor. Comput. Sci.*, 244(1-2):63–94, 2000. doi:10.1016/S0304-3975(00)00180-8.
- [19] B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*, volume 138 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2012.
- [20] A. M. Dean, W. Evans, E. Gethner, J. D. Laison, M. A. Safari, and W. T. Trotter. Bar k-visibility graphs. *J. Graph Algorithms Appl.*, 11(1):45–59, 2007. doi:10.7155/jgaa.00136.
- [21] W. Didimo, P. Eades, and G. Liotta. Drawing graphs with right angle crossings. *Theor. Comput. Sci.*, 412(39):5156–5166, 2011. doi:10.1016/j.tcs.2011.05.025.
- [22] W. Didimo, G. Liotta, S. Mehrabi, and F. Montecchiani. 1-bend RAC drawings of 1-planar graphs. In Y. Hu and M. Nöllenburg, editors, *GD 2016*, volume 9801 of *Lecture Notes in Computer Science*, pages 335–343. Springer, 2016. doi:10.1007/978-3-319-50106-2\_26.
- [23] R. Diestel. *Graph Theory*. Springer, 2000.
- [24] S. Hong, P. Eades, N. Katoh, G. Liotta, P. Schweitzer, and Y. Suzuki. A linear-time algorithm for testing outer-1-planarity. *Algorithmica*, 72(4):1033–1054, 2015. doi:10.1007/s00453-014-9890-8.
- [25] J. P. Hutchinson, T. Shermer, and A. Vince. On representations of some thickness-two graphs. *Computational Geometry*, 13:161–171, 1999. doi:10.1016/S0925-7721(99)00018-8.

- [26] M. Kaufmann and T. Ueckerdt. The density of fan-planar graphs. Technical Report arXiv:1403.6184 [cs.DM], Computing Research Repository (CoRR), March 2014. <http://arxiv.org/abs/1403.6184>.
- [27] S. G. Kobourov, G. Liotta, and F. Montecchiani. An annotated bibliography on 1-planarity. *Computer Science Review*, 2017. doi:<http://dx.doi.org/10.1016/j.cosrev.2017.06.002>.
- [28] J. Kyncl. Enumeration of simple complete topological graphs. *Eur. J. Comb.*, 30(7):1676–1685, 2009. doi:10.1016/j.ejc.2009.03.005.
- [29] G. Liotta. Graph drawing beyond planarity: some results and open problems. In S. Bistarelli and A. Formisano, editors, *Proc. 15th Italian Conference on Theoretical Computer Science*, volume 1231 of *CEUR Workshop Proceedings*, pages 3–8. CEUR-WS.org, 2014. URL: <http://ceur-ws.org/Vol-1231>.
- [30] G. Liotta and F. Montecchiani. L-visibility drawings of IC-planar graphs. *Inf. Process. Lett.*, 116(3):217–222, 2016. doi:10.1016/j.ipl.2015.11.011.
- [31] J. Pach and G. Tóth. Graphs drawn with a few crossings per edge. *Combinatorica*, 17:427–439, 1997. doi:10.1007/BF01215922.
- [32] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. aus dem Math. Seminar der Univ. Hamburg*, 29:107–117, 1965. doi:10.1007/bf02996313.
- [33] J. R. Schoenfeld. *Mathematical Logic*. Addison Wesley, 1967.
- [34] X. Zhang and G. Liu. The structure of plane graphs with independent crossings and its application to coloring problems. *Central Europ. J. Math.*, 11(2):308–321, 2013.