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The Effect of Planarization on Width

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Abstract

We study the effects on graph width parameters of planarization, the construction of a planar diagram from a non-planar graph drawing by replacing each crossing with a new vertex. We show that for treewidth, pathwidth, branchwidth, clique-width, and tree-depth there exists a family of n-vertex graphs with bounded parameter value, all of whose planarizations have parameter value $\Omega(n)$. However, for bandwidth, cutwidth, and carving width, every graph with bounded parameter value has a planarization of linear size whose parameter value remains bounded. The same is true for the treewidth, pathwidth, and branchwidth of graphs of bounded degree. To show our lower bounds on the width of planarizations, we prove that arrangements of curves with many crossing pairs of curves must generate planar graphs of high width.

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1 Introduction

Planarization is a graph transformation, standard in graph drawing, in which a given graph G is drawn in the plane with simple crossings of pairs of edges, and then each crossing of two edges in the drawing is replaced by a new dummy vertex, subdividing the two edges [2, 7, 14, 20]. This should be distinguished from a different problem, also called planarization, in which we try to find a large planar subgraph of a nonplanar graph [1, 4, 5, 28]. A given graph G may have many different planarizations, with different properties; see Figure 1. Although the size of the planarization (equivalently the crossing number of G) is of primary importance in graph drawing, it is natural to ask what other properties can be transferred from G to its planarizations.

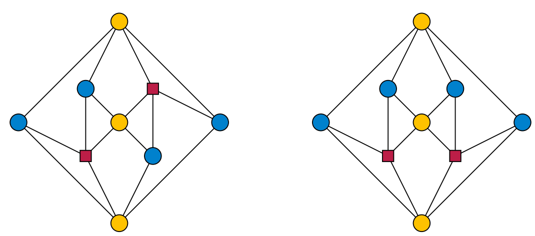

Figure 1: Two different planarizations of $K_{3,4}$. Both have the minimum number of added crossing vertices (the two small red squares) among planarizations of this graph.

One problem of this type arose in the work of Jansen and Wulms on the fixed-parameter tractability of graph optimization problems on graphs of bounded pathwidth [16]. One of their constructions involved the planarization of a nonplanar graph of bounded pathwidth, and they observed that the planarization maintained the low pathwidth of their graph. Following this observation, Jansen asked on cstheory.stackexchange.com whether planarization preserves the property of having bounded pathwidth, and in particular whether $K_{3,n}$ (a graph of bounded pathwidth) has a bounded-pathwidth planarization.[1] This paper represents an extended response to this problem. We provide a negative answer to Jansen's question: planarizations of $K_{3,n}$ do not have bounded pathwidth. However, for bounded-degree graphs of bounded pathwidth, there always exists a planarization that maintains bounded pathwidth. More generally we study similar questions for many other standard graph width parameters.

Our work should be distinguished from a much earlier line of research on planarization and width, in which constraints on the width of planar graphs are transferred in the other direction, to information about the graph being planarized. In particular, Leighton [20] used the facts that planar graphs have width at most proportional to the square root of their size, and that (for certain width parameters) planarization cannot decrease width, to show that when the

[1] See `https://cstheory.stackexchange.com/q/35974/95`.

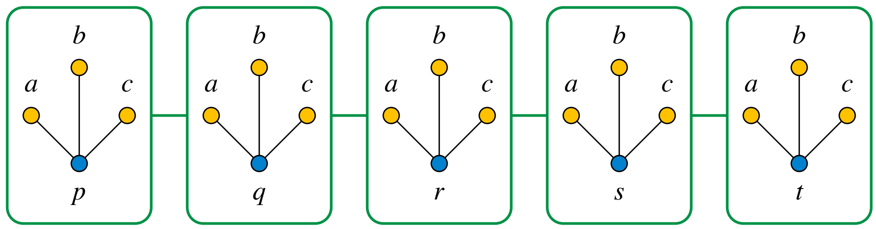

Figure 2: A tree-decomposition and path-decomposition of $K_{3,5}$, with width three. Vertices a, b, and c (on one side of the bipartition) belong to all bags; vertices p, q, r, s, and t (on the other side) are each in only one bag.

original graph has high width it must have crossing number quadratic in its width. In our work, in contrast, we are assuming that the original graph has low width and we derive properties of its planarization from that assumption.

1.1 Width parameters in graphs

There has been a significant amount of research on graph width parameters and their algorithmic implications; see Nešetřil and Ossona de Mendez [22] for a more detailed survey. We briefly describe the parameters that we use here.

Treewidth. Treewidth has many equivalent definitions; the one we use is that the treewidth of a graph G is the minimum width of a tree-decomposition of G [22]. Here, a tree-decomposition is a tree T whose nodes, called *bags*, are labeled by sets of vertices of G. Each vertex of G must belong to the bags of a contiguous subtree of T, and for each edge of G there must exist a bag containing both endpoints of the edge. The width of the decomposition is one less than the maximum cardinality of the bags. Figure 2 shows such a decomposition for $K_{3,5}$.

Pathwidth. The pathwidth of a graph G is the minimum width of a tree-decomposition of G whose tree is a path [22], as it is in Figure 2. Equivalently the pathwidth equals the minimum vertex separation number of a linear arrangement of the vertices of G (an arrangement of the vertices into a linear sequence) [17]. Every linear arrangement of an n-vertex graph defines $n-1$ cuts, that is, $n-1$ partitions of the vertices into a prefix of the sequence and a disjoint suffix of the sequence. The vertex separation number of a linear arrangement is the maximum, over these cuts, of the number of vertices in the prefix that have a neighbor in the suffix. From a linear arrangement one can construct a tree-decomposition in the form of a path, where the first bag on the path for each vertex v contains v together with all vertices that are earlier than v in the arrangement but that have v or a later vertex as a neighbor.

Cutwidth. The cutwidth of a graph G equals the minimum edge separation

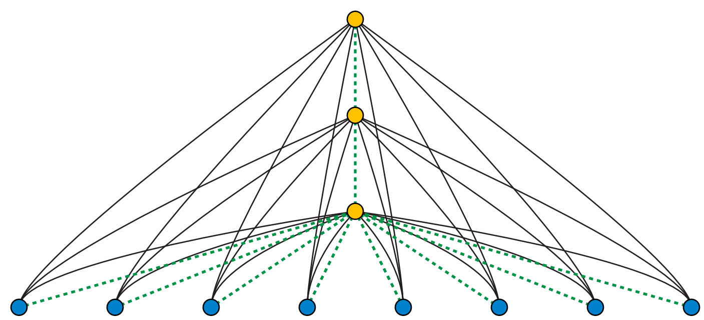

Figure 3: $K_{3,8}$ has tree-depth three: The depth-three tree shown by the green dashed edges forms a depth-first search tree of a supergraph of $K_{3,8}$.

number of a linear arrangement of the vertices of G [3]. The edge separation number of a linear arrangement is the maximum, over the prefix–suffix cuts of the arrangement, of the number of edges that cross the cut.

Bandwidth. The bandwidth of a graph G equals the minimum span of a linear arrangement of the vertices of G [3]. The span of a linear arrangement is the maximum, over the edges of G, of the number of vertices between the endpoints of the edge (plus one).

Branchwidth. A branch-decomposition of a graph G is an undirected tree T, with leaves labeled by the edges of G, and with every interior vertex of T having degree three. Removing any edge e from T partitions T into two subtrees; these subtrees partition the leaves of T into two sets, and correspondingly partition the edges of G into two subgraphs. The width of the decomposition is the maximum, over all edges e of T, of the number of vertices that belong to both subgraphs. The branchwidth of G is the minimum width of any branch-decomposition [27].

Carving width. A carving decomposition of a graph G is an undirected tree T, with leaves labeled by the vertices of G, and with every interior vertex of T having degree three. Removing any edge e from T partitions T into two subtrees; these subtrees partition the leaves of T into two sets, and correspondingly partition the vertices of G into two induced subgraphs. The width of the decomposition is the maximum, over all edges e of T, of the number of edges of G that connect one of these subgraphs to the other. The carving width of G is the minimum width of any carving decomposition [27]. For instance, Figure 7 depicts a carving decomposition of $K_{3,3}$ with width four, the minimum possible for this graph.

Tree-depth. The tree-depth of G is the minimum depth of a depth-first-search tree T of a supergraph of G (Figure 3). Such a tree can be characterized more simply by the property that every edge of G connects an ancestor–descendant pair in T [22].

Clique-width. A clique-construction of a graph G is a process that constructs a vertex-colored copy of G from smaller vertex-colored graphs by steps that create a new colored vertex, take the disjoint union of two colored graphs, add all edges from vertices of one color to vertices of another, or assigning a new color to vertices of a given color. The width of a clique-construction is the number of distinct colors it uses, and the clique-width of a graph is the minimum width of a clique-construction [6].

1.2 New results

In this paper, we consider for each of the depth parameters listed above how the parameter can change from a graph to its planarization, when the planarization is chosen to minimize the parameter value. We show that for treewidth, pathwidth, branchwidth, tree-depth, and clique-width there exists an n-vertex graph with bounded parameter value, all of whose planarizations have parameter value $\Omega(n)$. In each of these cases, the graph can be chosen as a complete bipartite graph $K_{3,n}$. However, for bandwidth, cutwidth, and carving width, every graph with bounded parameter value has a planarization of linear size[2] whose parameter value remains bounded. The same is true for the treewidth, pathwidth, branchwidth, and clique-width of graphs of bounded degree. (In graphs of bounded degree and bounded tree-depth every connected component has bounded size, so this final case is not interesting.)

Our proof that the planarizations of $K_{3,n}$ have high width combines two ideas:

- **High crossing number**. It was known that the planarizations of $K_{3,n}$ have quadratic size [32]; that is, its crossing number is $\Omega(n^2)$. In Section 2 we adapt a proof of this result to show that, in addition, the number of pairs of edges that cross is $\Omega(n^2)$. That is, the high crossing number of $K_{3,n}$ cannot be obtained by drawings in which only a small number of pairs of edges each cross many times; instead, many distinct pairs must cross. In fact in $K_{3,n}$ the crossing number and the minimum number of pairs of edges that cross are equal; it is unknown whether they can be unequal in any other graph.

- **Width of curve arrangements**. In Section 3 we prove that, if a collection of n curves has only simple crossings (no three curves cross at the same point) and has m crossing pairs of curves, then the planarization of the curve arrangement must have treewidth $\Omega(m/(n \log(n^2/m)))$. This result can be seen as a partial converse to known results that curve arrangements with few crossing pairs have intersection graphs of low width [11, 12, 19, 21].

Since the edges of any simple drawing of $K_{3,n}$ form a simple arrangement of curves with many crossing pairs, the corresponding planarization must have high width.

[2] For n-vertex graphs of bounded width, the number of edges is $O(n)$, so "linear size" means that the planarization also has $O(n)$ vertices and edges.

After these two sections, the rest of the paper is organized as follows. In Section 4 we return to the width parameters of graphs. We observe that for treewidth, branchwidth, pathwidth, tree-depth, and clique-width, the graphs $K_{3,n}$ have bounded width but (by the above argument, and known relations between these width parameters) their planarizations have high width. However, in Section 5, Section 6, and Section 7, we show that cutwidth, bandwidth, and carving width (respectively) are better-behaved, remaining bounded when we planarize a graph of bounded width. The same is true for the pathwidth and treewidth of bounded-degree graphs. We conclude in Section 8.

2 Crossing pairs of edges in $K_{3,n}$

We begin by determining a formula for the crossing number of $K_{3,n}$. This is a special case of Turán's brick factory problem of determining the crossing number of all complete bipartite graphs. For our results we need a variant of the crossing number:

Definition 1 *For a given graph G, a* drawing *of G is a mapping from vertices of G to distinct points in the plane and edges of G to open curves in the plane, such that no vertex belongs to any edge curve and such that the endpoints of each edge are mapped to the endpoints of the corresponding curve. A drawing is* simple *if each intersection of multiple curves is a point where exactly two curves cross each other; that is, this point has a neighborhood within which the curves are homeomorphic to two crossing lines. (In particular, curves that touch each other without crossing are not allowed.)*

Definition 2 *The* crossing number $\mathrm{cr}(G)$ *is the minimum number of crossing points in any simple drawing of G; any two edges may cross each other multiple times, but each crossing point counts towards the crossing number. We define the* pair-crossing number $\mathrm{cr}_{\mathrm{pair}}(G)$ *to be the minimum number of pairs of crossing edges in any drawing of G. This variation of the crossing number again allows edges to cross each other multiple times in the drawing, but we only count a single crossing in each such case.*

For more on the relation between cr and $\mathrm{cr}_{\mathrm{pair}}$ see Pach and Tóth [24, 29] and Schaefer [26]. For every graph G, $\mathrm{cr}_{\mathrm{pair}}(G) \leq \mathrm{cr}(G)$; however, it is unknown whether there exist graphs for which these two numbers are different [24, 26, 29]. The exact value of $\mathrm{cr}(K_{3,n})$ was proven by Zarankiewicz [32]. We follow the same argument, which we adapt from Kleitman [18], to prove that $\mathrm{cr}_{\mathrm{pair}}(K_{3,n})$ has the same value.

Lemma 1

$$\mathrm{cr}_{\mathrm{pair}}(K_{3,n}) = \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2} = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

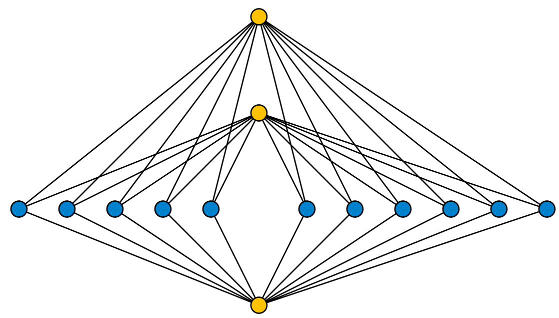

Figure 4: A drawing of $K_{3,11}$ with 25 crossings, the minimum possible for this graph.

Proof: To show that a drawing with this many crossing pairs exists, place the n vertices on one side of the bipartition of $K_{3,n}$ along the x-axis, with $\lfloor n/2 \rfloor$ on one side of the origin and $\lceil n/2 \rceil$ on the other. Place the three vertices on the other side of the bipartition along the y-axis, with two points on one side of the origin and one on the other. Connect all of the pairs of points that have one point on each axis by a straight line segment, as shown in Figure 4. This layout, with the two sides of the bipartition on the two coordinate axes, each bisected by the origin, is the construction conjectured to give the minimum number of crossings for all complete bipartite graphs. The numbers of crossings in the top left and top right quadrants of the drawing are the two binomial coefficients in the formula of the lemma; there are no crossings in the lower left and lower right quadrants. Straightforward algebraic simplification shows that these two binomial coefficients have the claimed sum.

In the other direction, we know as base cases that $\mathrm{cr}_{\mathrm{pair}}(K_{3,2}) = 0$ (because $K_{3,2}$ is a planar graph) and $\mathrm{cr}_{\mathrm{pair}}(K_{3,3}) = 1$ (as one of the two Kuratowski graphs, $K_{3,3}$ is nonplanar, but the drawing described above gives it only one crossing). For any larger n, let the vertices of the n-vertex side of the bipartition of $K_{3,n}$ be $v_1, v_2, \ldots v_n$, and consider any fixed drawing that minimizes the number of crossing pairs of edges. If every pair v_i, v_j form the endpoints of at least one pair of crossing edges in this drawing, then each such crossing pair would be distinct from each other pair, so the total number of crossings would be at least $\binom{n}{2}$, an impossibility since we already know there is a drawing with fewer crossing pairs. Therefore, some two vertices v_i and v_j are not incident to any crossing pair of edges. We may renumber the vertices so that these two vertices are last in the numbering. That is, we may assume without loss of generality that v_{n-1} and v_n do not form the endpoints of any pair of crossing edges.

Then, in the chosen optimal drawing of $K_{3,n}$, the $K_{3,n-2}$ subgraph formed by deleting v_{n-1} and v_n has at least $\mathrm{cr}_{\mathrm{pair}}(K_{3,n-2})$ crossing pairs of edges. Each of the $n-2$ $K_{3,3}$ subgraphs induced by v_{n-1}, v_n, exactly one other v_i, and the three vertices on the other side of the bipartition also includes at least one crossing, because $\mathrm{cr}_{\mathrm{pair}}(K_{3,3}) = 1$. None of these $K_{3,n-2}$ or $K_{3,3}$ subgraphs share any crossings, because the crossings in the $K_{3,n-2}$ subgraph involve neither v_{n-1} nor v_n, while the crossings in each $K_{3,3}$ subgraph involve exactly one of these two vertices and the one other vertex v_i included in the subgraph. Therefore, we have that

$$\mathrm{cr}_{\mathrm{pair}}(K_{3,n}) \geq \mathrm{cr}_{\mathrm{pair}}(K_{3,n-2}) + (n-2)\,\mathrm{cr}_{\mathrm{pair}}(K_{3,3}).$$

The result follows by induction on n. □

3 Width of curve arrangements

A finite set of curves in the plane is called an *arrangement.* We can define simplicity and pair crossings for arrangements, by analogy to the same concepts for graph drawings:

Definition 3 *An arrangement of curves is* simple *if, at every point where two or more curves intersect, exactly two curves cross. The* pair-crossing number $\mathrm{cr}_{\mathrm{pair}}(A)$ *of an arrangement of curves is the number of pairs of curves in A that have a point of intersection. The* string graph *of an arrangement of curves is a graph having a vertex for each curve and an edge for each pair of curves that intersect; the pair-crossing number of the arrangement is just the number of edges in the string graph. The* planarization *of a simple arrangement of curves is the planar graph whose vertices are crossing points of pairs of curves, and whose edges are pairs of vertices that are connected by a contiguous arc of one of the curves that is not crossed by any other curve.*

To prove that planarizations of curve arrangements have high treewidth, we need to find subarrangements (subsets of the curves) with greater densities of crossings than the given arrangement. To do so, we use the following "densification lemma", which we will apply to the string graph of the arrangement.

Lemma 2 *Let G be a disconnected graph with n vertices and m edges, number the connected components of G arbitrarily, and let n_i and m_i denote the number of vertices of edges of connected component i. Then there exists i such that $m_i/n_i \geq m/n$.*

Proof: We can represent m/n as a convex combination of the corresponding quantities in the subgraphs:

$$\frac{m}{n} = \sum_i \frac{n_i}{n} \cdot \frac{m_i}{n_i}.$$

The result follows from the fact that a convex combination of numbers cannot exceed the maximum of the numbers. □

The following fact about separators in graphs of low treewidth is standard (see, e.g., [25, Paragraph (2.5)]) and can be proven by orienting each edge of a tree-decomposition towards the subtree whose induced subgraph of the given graph is larger. The resulting oriented tree necessarily has a sink, whose bag provides the desired separator.

Lemma 3 *Let graph G have n vertices and treewidth w. Then there exists a set S of at most $w+1$ vertices (one of the bags of a tree-decomposition of G) such that each connected component of the graph formed from G by removing all vertices in S has at most $n/2$ vertices.*

However, we need a variant of this lemma that applies more directly to planarizations of curves. We adapt the usual proof of Lemma 3 to prove this variant:

Lemma 4 *Let A be an arrangement of n curves whose planarization G has treewidth w. Then there exists a set B of at most $2(w+1)$ curves (the curves whose crossings are vertices in one of the bags of a tree-decomposition of G) such that each connected component of the arrangement formed from A by removing all curves in B has at most $n/2$ curves.*

Proof: We perform the following steps to find B:

1. Construct an optimal tree decomposition T of G, and initialize S to be any bag of T.
2. Let B be the set of curves whose crossings are vertices in S. While the subarrangement A' formed by removing the curves in B from A has a large component, perform the following steps:
   (a) Find the subtrees T_i of T that would be formed by removing B from T.
   (b) Each component of A' must have all of its crossings in a single one of these subtrees, because any two crossings on the same curve C of A' are connected by a path in G none of whose vertices can belong to bag S (or else C would have been removed when forming A'). Let T' be the subtree containing the large component of A'.
   (c) Replace S by the neighbor of S in T', and B by the set of curves whose crossings are vertices in the new choice of S.

Each step of this process reduces the number of curves having crossings within the subtree containing the large component. The other subtrees, after each step, have fewer than $n/2$ curves, because none of their curves come from the large component. It is not possible to reduce the integer size of the large component infinitely often, so this process eventually terminates, which can only happen when no component has more than $n/2$ curves. $\square$

By repeatedly applying this separator lemma to the arrangement and then choosing the most dense remaining component we obtain that the planarization of an arrangement with many crossings has treewidth nearly as large as the average number of crossings per curve. If it had smaller treewidth, we could recursively partition the arrangement into pieces one of which would have more crossing pairs of curves than it had total pairs of curves, a contradiction. In more detail, we have the following lemma.

Lemma 5 *Let A be an arrangement of n curves, with $\mathrm{cr}_{\mathrm{pair}}(A) = m$. Then the treewidth of the planarization of A is*

$$\Omega\left(\frac{m}{n}\Big/\log\frac{n^2}{m}\right).$$

Proof: Let the treewidth of the planarization of A be w. Letting n' and m' denote the number of curves and crossing pairs in the current arrangement (initially arrangement A), repeat the following steps as long as $n' > m/n$:

- Apply Lemma 4 to find a set B of at most $2(w+1)$ curves, the removal of which partitions the arrangement into subarrangements with at most half as many curves as before.
- Replace the arrangement A by the subarrangement whose number n_i of curves and whose pair-crossing number m_i maximizes m_i/n_i.

At the end of this process, the remaining arrangement has at most m/n curves in it. Therefore, its ratio m'/n' of crossings to curves is at most $(m/n-1)/2$, the maximum possible for an arrangement with this many curves, achieved when every curve crosses every other curve. Each step reduces the number of curves to at most half its previous value, so the number of steps is at most $\log_2(n^2/m)$.

In a given step, suppose that $w < \epsilon m'/n'$ for some $\epsilon < 1$. Then, the $2(w+1)$ curves in B have at most

$$2(w+1)(n'-1) \le 2\epsilon m'$$

crossings in them. The removal of these crossings will reduce m' by a factor of $(1-2\epsilon)$ or larger. The removal of the $2(w+1)$ curves also affects n', but in the opposite direction, causing m'/n' to increase, as does the subsequent choice of one component of the subarrangement. Therefore, in such a step, the total factor by which m'/n' decreases is $(1-2\epsilon)$ or larger.

Suppose for a contradiction that the original planarization could have width

$$w < \frac{m\ln 2}{4n\log_2(n^2/m)}$$

then, as long as the density m'/n' remains at least $m/2n$, we would have $w < \epsilon m'/n'$ for

$$\epsilon < \frac{\ln 2}{2n\log_2(n^2/m)},$$

and we would reduce m'/n' in each step by a factor of

$$1 - \frac{\ln 2}{n\log_2(n^2/m)}$$

or larger. We can do this at least $\log_2(n^2/m)$ times before reducing m'/n' to $m/2n$, which is at least the number of iterations in the process above. Therefore, when the process terminates, $m'/n' \ge m/2n$. But this contradicts the inequality $m'/n' < m/2n$ derived earlier from the number of curves in the remaining arrangement. This contradiction shows that w cannot be too small. In particular, $w = \Omega(m/(n\log_2(n^2/m))$, the bound of the lemma. □

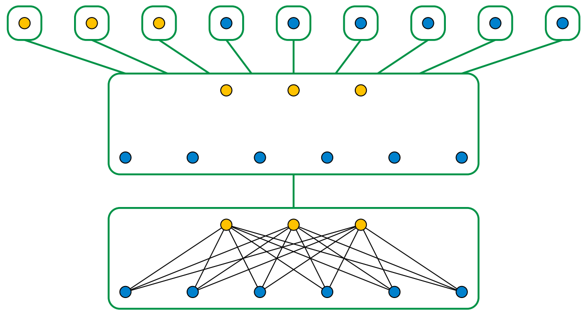

Figure 5: Clique-width 2 construction of $K_{3,6}$ by a disjoint union of colored single vertices, followed by an operation that adds an edge between each bichromatic pair of vertices.

4 Treewidth, branchwidth, pathwidth, tree-depth, and clique-width

Combining Lemma 1 and Lemma 5 shows that all planarizations of $K_{3,n}$ have high width:

Theorem 1 *Every planarization of $K_{3,n}$ has treewidth $\Omega(n)$.*

Proof: Let D be a drawing of $K_{3,n}$ whose planarization has the minimum possible treewidth, and let A be the simple arrangement of $3n$ curves formed by the edges of D. By Lemma 1, A has pair-crossing number $\Omega(n^2)$. By Lemma 5, the planarization of A has treewidth $\Omega(n)$. But the planarization of A is a subgraph of the planarization of the drawing D (obtained from the drawing by removing the vertices of $K_{3,n}$), and taking subgraphs can only reduce the treewidth. So the planarization of D also has treewidth $\Omega(n)$. □

Corollary 1 *For every planarization of $K_{3,n}$, and every parameter in {treewidth, branchwidth, pathwidth, tree-depth, clique-width}, the value of the parameter on $K_{3,n}$ is $O(1)$ but the value of the parameter on the planarization is $\Omega(n)$. Therefore, there exists a family of graphs for which each of these parameters is bounded but for each planarization has linear parameter value.*

Proof: All of these parameters except clique-width are bounded from below by a linear function of the treewidth, which is $\Omega(n)$ by Theorem 1. The $\Omega(n)$ lower bound on clique-width follows from the facts that (as a planar graph) any planarization has no $K_{3,3}$ subgraph and that, for graphs with no $K_{t,t}$ subgraph, the treewidth is upper-bounded by a constant factor (depending on t) times the clique-width [15]. Consequently, the clique-width of any planarization is lower-bounded by a constant times its treewidth, which by Theorem 1 is $\Omega(n)$.

The treewidth and pathwidth of $K_{3,n}$ are at most three, by the path-decomposition and tree-decomposition shown in Figure 2. Its branch-width is also $O(1)$, as the branch-width and treewidth of all graphs are within constant factors of each other. As with any complete bipartite graph, the clique-width of $K_{3,n}$ is two: it can be constructed from a disjoint union of single vertices of two colors, by adding edges between all bichromatic pairs of vertices (Figure 5). The tree-depth of $K_{3,n}$ is at most three, obtained from a height-three tree that consists of a rooted path of three vertices together with n leaves attached to the bottom vertex of the path (Figure 3). □

We remark that this bound is optimal. No stronger bound than $\Omega(n)$ on these width parameters is possible. For $K_{3,n}$, and more strongly for any n-vertex graph of bounded width, the number of edges is $O(n)$, and therefore any simple drawing has $O(n^2)$ crossings. As a planar graph, the planarization of such a drawing necessarily has width at most the square root of this number of crossings. So for all of these width parameters, all simple drawings of n-vertex graphs of bounded width have planarizations of width $O(n)$.

5 Cutwidth and bounded-degree pathwidth

We have seen that planarization can blow up many width parameters. However, as we show in this section, cutwidth behaves particularly well under planarization.[3]

Theorem 2 *Let G be a graph with n vertices and m edges, of cutwidth w. Then G has a planarization with $O(n + wm)$ vertices, of cutwidth at most w.*

Proof: Consider a linear arrangement of G with edge separation number w, and use the positions in this arrangement as x-coordinates for the vertices. Assign the vertices y-coordinates that place them into convex and general position, draw the edges of G as straight line segments between the resulting points, and planarize the drawing by replacing each crossing by a vertex. Here, by "general position" we mean that no two points have the same x-coordinate, no five points form a pentagon in which two crossing points and a vertex have the same x-coordinate, no six points form a hexagon with three coincident diagonals, and no eight points form an octagon in which the crossing points of two pairs of diagonals have the same x-coordinate. This will all be true after a rotation by a sufficiently small but nonzero angle of any convex placement. In the resulting drawing, there can be no intersections of vertices or edges other than incidences and simple crossings, and no two vertices or crossing points can have the same x-coordinate. An example is shown in Figure 6.

We use the ordering by x-coordinates of the planarization as a linear arrangement of the planarization. The edge intersection number is the maximum number of edges in the drawing that can be cut by any vertical line, unchanged between G and its planarization.

Because of the convex position of the vertices of G, each edge (u, v) of G can only be crossed by other edges that cross exactly one of the two vertical lines

[3]After the appearance of the preprint version of this paper [10], we learned that this result has been obtained independently by van Geffen et al. [30].

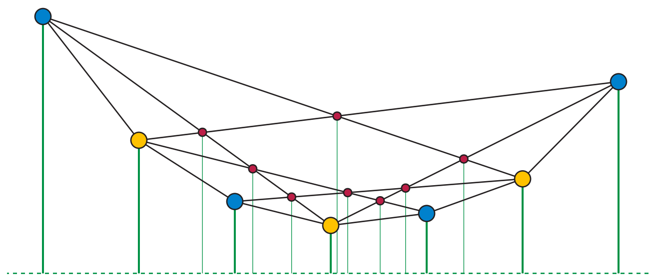

Figure 6: Planarizing a graph of low cutwidth (here $K_{3,4}$, drawn with edge separation number six) by lifting its linear arrangement to a convex curve.

through u and v; there are $O(w)$ such edges, so the number of crossings per edge is $O(w)$ and the total number of crossings is $O(wm)$. □

The lower bound of Corollary 1 implies that planarizations of $K_{3,n}$ have linear cutwidth. However, this does not contradict Theorem 2 because $K_{3,n}$ itself does not have bounded cutwidth. Its cutwidth is at least $3\lceil n/2 \rceil$, obtained in any linear arrangement at the cut between the first $\lceil n/2 \rceil$ vertices on the n-vertex side of the bipartition (together with any vertices from the other side that are mixed among them) and the remaining vertices of the graph. For instance, the drawing of $K_{3,4}$ in Figure 6 achieves the optimal cutwidth of six for this graph. An example showing that the bound on the number of vertices in Theorem 2 is tight is given by a disjoint union of $O(n/w)$ bounded-degree expander graphs, each having $O(w)$ vertices and crossing number $\Theta(w^2)$.

The graphs of bounded cutwidth must also have bounded degree, because every graph of maximum degree d has cutwidth at least $d/2$. If we explicitly bound the degree, then cutwidth becomes equivalent to pathwidth, as detailed in the next result:

Corollary 2 *Let G be a graph with bounded pathwidth and bounded maximum degree. Then G has a planarization with linear size and bounded pathwidth.*

Proof: If a graph has pathwidth w and maximum degree d, it has cutwidth at most dw [3], and so does its planarization (Theorem 2). Because the planarization has cutwidth at most dw, it also has pathwidth at most dw, because the vertex separation number of any linear arrangement is at most equal to the edge separation number (with equality when the separation number is achieved by a matching). □

A planarization of linear size follows from results of Dujmović et al. on the crossing number of bounded-degree graphs in minor-closed graph families [8], but their results do not control the pathwidth of the resulting planarization.

6 Bandwidth

The same construction used for planarizing graphs with low cutwidth also works for graphs of low bandwidth, and shows that the bandwidth of their planarizations remains low.

Theorem 3 *Let G be a graph with n vertices and m edges, of bandwidth w. Then G has a planarization with $O(n+w^2m)$ vertices, whose bandwidth is $O(w^4)$.*

Proof: We lift a linear arrangement of G with low span to a convex curve in the plane, as in the proof of Theorem 2. Within the span of any edge e of G, there are $O(w^2)$ other edges and $O(w^4)$ crossings of those edges, so the span of e in the planarization is $O(w^4)$. This bound applies also to the span of any segment of e created by crossings with other vertices. Each edge may be crossed by $O(w^2)$ other edges, so the total number of dummy vertices added is $O(w^2m)$. □

As with cutwidth, the graphs of bounded bandwidth must also have bounded degree, because every graph of maximum degree d has bandwidth at least $d/2$.

It is unclear whether the quartic dependence on w in Theorem 3 is optimal. It may be possible to reduce the bandwidth of the planarization by introducing artificial crossings to break up edges with long spans. However, we have not pursued this approach as we do not believe it will lead to better graph drawings.

7 Carving width and bounded-degree treewidth

If a graph has low carving width, we can use its carving decomposition (a tree with the vertices at its leaves, internal degree three, and with few edges spanning the cut determined by each tree edge) to guide a drawing of the graph that leads to a planarization with low carving width.

It is helpful, for our construction, to relate carving width to cutwidth.

Lemma 6 *If a graph G has cutwidth w and maximum degree d, then G has carving width at most* $\max(w, d)$*.*

Proof: We form a carving decomposition of G in the form of a caterpillar: a path with each path vertex having a single leaf connected to it (except for the ends of the path which have two connected leaves). The ordering of the leaves is given by a linear arrangement minimizing the edge separation number. Then the cuts of the carving decomposition that are determined by edges of the path are exactly the ones determining the edge separation number, w. The remaining cuts, determined by leaf edges of the tree, are crossed by the neighboring edges of each vertex, of which there are at most d. An example of this construction can be seen in Figure 6: the dashed horizontal green line represents the path from which the carving decomposition is formed, the heavy vertical green lines correspond to the leaf edges of the carving decomposition of $K_{3,4}$, and the thin vertical green edges correspond to the leaf edges of the carving decomposition of a planarization of $K_{3,4}$. □

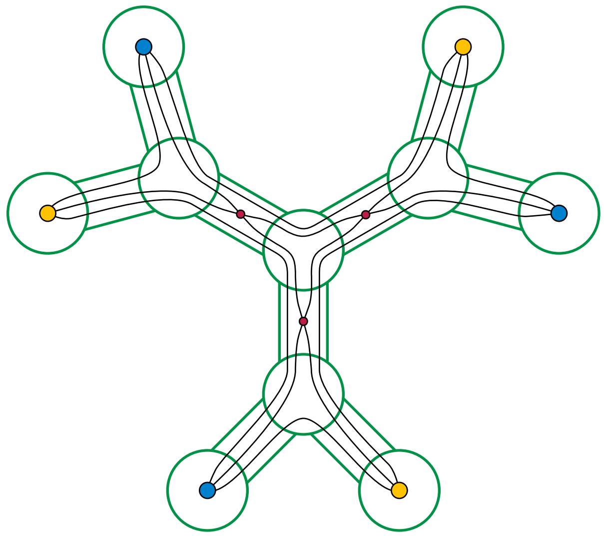

Figure 7: Using a carving decomposition of $K_{3,3}$ to guide a planarization.

Theorem 4 *If an n-vertex graph G has carving width w, then G has a planarization with $O(w^2 n)$ additional vertices that still has carving width at most w.*

Proof: Let T be the tree of a carving decomposition of G with width w. Draw T without crossings in the plane, with straight-line edges, and thicken the vertices of T to disks and the edges of T to rectangles without introducing any additional self-intersections of the drawing. Place each vertex of G in the disk of the corresponding leaf vertex of T. Route each edge of G as a curve through the rectangles and disks connecting its endpoints, so that within each rectangle it forms a monotone curve (with respect to the orientation of the rectangle) crossing at most once each other edge routed within the same rectangle, and so that, at each end of each rectangle, the curves are sorted by the ordering of their destination leaves in the planar embedding of T. With this sorted ordering, there need not be any crossings within the disks representing internal vertices of T, nor in the rectangles representing leaf edges of T (Figure 7). The $n-3$ remaining edges of T each contain at most $\binom{w}{2}$ crossings. So the total number of crossings is at most $(n-3)\binom{w}{2} = O(w^2 n)$.

This drawing cannot yet be recognized as a carving decomposition of a planarization of G, because some of its vertices (the dummy vertices introduced at crossings) are now placed along the edges of T rather than at leaves. However, by topologically sweeping the arrangement of monotone curves [9] within each rectangle corresponding to an edge of T, we can arrange the crossing points within that rectangle into a linear sequence, such that the portion of the drawing within that rectangle has edge separation number at most w for that sequence. Applying Lemma 6 (replacing the edge of T by a carving decomposition in the form of a caterpillar, with a leaf of the decomposition for each vertex added in the planarization to replace a crossing of G, and with the ordering of these leaves given by a topological sweep of the arrangement) produces a carving decomposition of the planarization with width w, as required. □

We note that this planarization technique resembles the "simple planarization" method of Di Battista et al. [7] for clustered graphs. In this respect, we may view the carving decomposition of G as a clustering to be respected by the planarization. A very similar drawing technique was also applied by Wood and Telle to a much more general class of decompositions with a planar graph underlying the decomposition rather than a tree, to show that bounded-degree graphs with these decompositions have linear crossing number [31, Lemma 4.1].

With a constant-factor loss in width we can obtain a tighter bound on the planarization size. To prove this, we apply a tree clustering technique of Frederickson [13] to the carving decomposition of G. Following Frederickson, we define a *restricted partition of order* z of an unrooted binary tree T (such as the tree of a carving decomposition) to be a partition of the vertices of T into connected subtrees with the following properties:

- Each subtree of the partition contains at most z vertices.
- If a subtree of the partition has more than two edges connecting it to other subtrees, then it contains exactly one vertex.
- If two subtrees of the partition are connected by an edge, then they cannot be merged into a single subtree while preserving the previous two properties.

Such a partition can be found easily by a greedy algorithm that repeatedly merges subtrees until no more merges are possible.

Lemma 7 (Frederickson [13]) *For every unrooted binary tree T with n vertices, every z, and every restricted partition of T of order z, there are at most $O(n/z)$ subtrees in the partition.*

Proof: If each subtree in a restricted partition is contracted into a single vertex, the result is again an unrooted tree with maximum degree three. Every leaf vertex of this contracted tree together with its parent must together have more than z vertices, or else they could be merged to form a larger tree with at most z vertices and at most two connecting edges to other subtrees. For the same reason, every pair of adjacent degree-two vertices in this contracted tree must together have more than z vertices. Therefore, the contracted tree can only have $O(n/z)$ leaf vertices and $O(n/z)$ adjacent pairs of degree-two vertices, from which it follows that it has $O(n/z)$ vertices altogether. □

Theorem 5 *If an n-vertex graph G has carving width w, then G has a planarization with $O(w^{3/2}n)$ additional vertices that still has carving width $O(w)$.*

Proof: To planarize G with $O(w^{3/2}n)$ additional vertices and carving width $O(w)$, proving Theorem 5, we first find a restricted partition of the carving decomposition T of G, of order $O(\sqrt{w})$.

Each subtree T_i of the restricted partition represents a subset of $O(\sqrt{w})$ vertices of G, possibly having up to $2w$ edges connecting it to the rest of G along the two edges of T connecting this subtree to the rest of T. Let V_i denote the subset of vertices of G within subtree T_i, together with up to two dummy vertices representing the two edges of T connecting T_i to the rest of T. We planarize the subgraph of edges that enter or pass through T_i by placing the vertices of V_i onto a circle, but otherwise in general position, and by drawing each edge as a straight line segment between two points of this circle. Each of the $O(n/\sqrt{w})$ subtrees contributes $O(w^2)$ crossings from this drawing (the maximum number of crossings for a graph on $O(\sqrt{w})$ vertices drawn with straight-line edges on a circle). Projecting the circle onto a line gives a linear arrangement of the subgraph, and of its planarization, with edge separation number $O(w)$, so by Lemma 6 the carving width of this subgraph and its planarization is also $O(w)$.

Let T' be the binary tree resulting from T by contracting each T_i into a point. As in Theorem 4 we draw T' in the plane, replacing each of its vertices by a disk and replacing each of its edges by a rectangle. We place the drawing of the subgraph associated with each subtree T_i into the corresponding disk of T'. We replace the (up to two) two dummy vertices representing connections from T_i to other subtrees with a bundle of edges passing from the disk to an adjacent rectangle. As in Theorem 4 we route edges across each rectangle by monotone curves that cross each other at most once. There are $O(n/\sqrt{w})$ rectangles, each having $O(w^2)$ crossings and carving width at most w, so the contributions to the total number of crossings and the carving width from this part of the construction are also $O(w^{3/2}n)$ and $O(w)$ respectively. □

An example showing that Theorem 5 is tight is given by a cluster graph (that is, a disjoint union of cliques) consisting of $O(n/\sqrt{w})$ disjoint cliques of size $O(\sqrt{w})$, each requiring $\Theta(w^2)$ crossings in any drawing.

Corollary 3 *Let G be a graph with treewidth or branchwidth w and maximum degree d. Then G has a planarization with $O((dw)^{3/2})$ additional vertices and treewidth and branchwidth $O(dw)$.*

Proof: Treewidth and branchwidth are always within a constant factor of each other [27] so we may concentrate on the results for branchwidth, and the corresponding results for treewidth will follow automatically.

A carving decomposition may be converted into a branch decomposition by replacing each leaf of the carving decomposition (representing a vertex of the given graph) with a subtree (representing edges adjacent to the given vertex), in such a way that each edge is represented at exactly one of its endpoints. This increases the width of the decomposition by at most a factor equal to the degree. In the other direction, a branch decomposition may be converted into a carving decomposition by replacing each leaf of the branch decomposition (representing an edge of the given graph) by a subtree of zero, one, or two leaves (representing endpoints of the edge) in such a way that each vertex is represented at exactly one of its incident edges.This increases the width of the decomposition by at most a factor of two. So, the carving width is at most the degree times the branchwidth, and at least half the branchwidth [23].

Therefore, if G has treewidth or branchwidth w and maximum degree d, it has carving width $O(dw)$. Plugging this bound into Theorem 5 (and translating the carving with of the planarization back into treewidth or branchwidth) gives the claimed results. □

As for pathwidth, a planarization of linear size for graphs of bounded carving width, or bounded degree and bounded treewidth or branchwidth, follows from results of Dujmović et al. on the crossing number of bounded-degree graphs in minor-closed graph families [8], but their results do not control the width of the resulting planarization.

8 Conclusions

We have shown that planarizing a graph may blow up its treewidth, pathwidth, branch-width, tree-depth, or clique-width, but that the cutwidth, bandwidth, and carving width remain bounded as a function of their original values. There are many additional properties of graphs that could be affected by planarization (for instance, connectivity and toughness, metric dimension, or the various types of centrality); it would be of interest to characterize which ones change only in a predictable and controlled way and which can change dramatically on planarization.

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