

Circumference of essentially 4-connected planar triangulations

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Submitted: February 2020 Reviewed: July 2020 Revised: August 2020

Accepted: January 2021 Final: January 2021 Published: January 2021

Article type: Regular paper

Communicated by: G. Liotta

Abstract. A 3-connected graph G is essentially 4-connected if, for any 3-cut $S \subseteq V(G)$ of G , at most one component of $G - S$ contains at least two vertices. We prove that every essentially 4-connected maximal planar graph G on n vertices contains a cycle of length at least $\frac{2}{3}(n + 4)$; moreover, this bound is sharp.

Keywords: circumference, long cycle, triangulation, essentially 4-connected, planar graph 2010 *MSC:* 05C38, 05C10

1 Introduction and Preliminaries

We consider finite, simple, and undirected graphs. The *circumference* $\text{circ}(G)$ of a graph G is the length of a longest cycle of G . A cycle C of G is an *outer independent cycle* of G if the set $V(G) \setminus V(C)$ is independent. (Note that an outer independent cycle is sometimes called a dominating cycle ([3]), although this is in contrast to the more commonly used definition of a dominating subgraph H of G , where $V(H)$ dominates $V(G)$ in the usual sense.) A set $S \subseteq V(G)$ ($S \subseteq E(G)$) is a k -cut (a k -edge-cut) of G if $|S| = k$ and $G - S$ is disconnected. A 3-cut (a 3-edge-cut) S of a 3-connected (3-edge-connected) graph G is *trivial* if at most one component of $G - S$ contains at least two vertices and the graph G is *essentially 4-connected* (*essentially 4-edge-connected*) if every 3-cut (3-edge-cut) of G is trivial. A 3-edge-connected graph G is *cyclically 4-edge-connected* if for every 3-edge-cut S of G , at most one component of $G - S$ contains a cycle.

Research partially supported by DAAD, Germany (as part of BMBF) and the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800. IF also partially supported by the Slovak Research and Development Agency under contract No. APVV-19-0153.

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It is well-known that for (3-connected) cubic graphs different from the triangular prism $K_3 \times K_2$ (which is essentially 4-connected only) these three notions coincide (see e.g. [6] and [16]). Obviously, the line graph $H = L(G)$ of a 3-connected graph G is 4-connected if and only if G is essentially 4-edge-connected. These two observations are reasons for the quite great interest in studying all these three concepts of connectedness of graphs intensively.

Zhan [17] proved that every 4-edge-connected graph has a Hamiltonian line graph. Broersma [3] conjectured that even every essentially 4-edge-connected graph has a Hamiltonian line graph and showed that this is equivalent to the conjecture of Thomassen [14] stating that every 4-connected line graph is Hamiltonian (which is known to be equivalent to the conjecture by Matthews and Sumner [12] stating that every 4-connected claw-free graph is Hamiltonian, as shown by Ryjáček [13]). Among others, the subclass of essentially 4-edge-connected cubic graphs is interesting due to a conjecture of Fleischner and Jackson [6] stating that every essentially 4-edge-connected cubic graph has an outer independent cycle which is equivalent to the previous three conjectures.

Regarding to the existence of long cycles in essentially 4-connected graphs we mention the following

Conjecture 1 (Bondy, see [8]) *There exists a constant c , $0 < c < 1$, such that for every essentially 4-connected cubic graph on n vertices, $\text{circ}(G) \geq cn$.*

Note that the conjecture of Fleischner and Jackson implies Conjecture 1 with $c = \frac{3}{4}$. Bondy's conjecture was later extended to all cyclically 4-edge-connected graphs (see [6]). Máčajová and Mazák [11] constructed essentially 4-connected cubic graphs on $n = 8m$ vertices with circumference $7m + 2$. We remark that the conjecture of Fleischner and Jackson and, therefore, also Bondy's Conjecture with $c = \frac{3}{4}$ (this is the result of Grünbaum and Malkevitch [7]) are true for planar graphs, which can be seen easily by the forthcoming Lemma 1. Many results concerning the circumference of essentially 4-connected planar graphs G can be found in the literature.

For the class of essentially 4-connected cubic planar graphs, Tutte [15] showed that it contains a non-Hamiltonian graph, Aldred, Bau, Holton, and McKay [1] found a smallest non-Hamiltonian graph on 42 vertices, and Van Cleemput and Zamfirescu [16] constructed a non-Hamiltonian graph on n vertices for all even $n \geq 42$. As already mentioned, Grünbaum and Malkevitch [7] proved that $\text{circ}(G) \geq \frac{3}{4}n$ for any essentially 4-connected cubic planar graph G on n vertices and Zhang [18] (using the theory of Tutte paths) improved this lower bound on the circumference by 1. Recently, in [10], an infinite family of essentially 4-connected cubic planar graphs on n vertices with circumference $\frac{359}{366}n$ was constructed.

In [9], Jackson and Wormald extended the problem to find lower bounds on the circumference to the class of arbitrary essentially 4-connected planar graphs. Their result $\text{circ}(G) \geq \frac{2n+4}{5}$ was improved in [5] to $\text{circ}(G) \geq \frac{5}{8}(n+2)$ for every essentially 4-connected planar graph G on n vertices. On the other side, there are infinitely many essentially 4-connected maximal planar graphs G with $\text{circ}(G) = \frac{2}{3}(n+4)$ ([9]). To see this, let G' be a 4-connected maximal planar graph on $n' \geq 6$ vertices and let G be obtained from G' by inserting a new vertex into each face of G' and connecting it with all three boundary vertices of that face. Then G is an essentially 4-connected maximal planar graph on $n = 3n' - 4$ vertices and, since G' is Hamiltonian, it is easy to see that $\text{circ}(G) = 2n' = \frac{2}{3}(n+4)$. It is still open whether there is an essentially 4-connected planar graph G that satisfies $\text{circ}(G) < \frac{2}{3}(n+4)$. Indeed, we pose the following (to our knowledge so far unstated) Conjecture 2, which has been the driving force in that area for over a decade.

Conjecture 2 *For every essentially 4-connected planar graph on n vertices, $\text{circ}(G) \geq \frac{2}{3}(n+4)$.*

By the forthcoming Theorem 1, Conjecture 2 is shown to be true for essentially 4-connected maximal planar graphs.

We remark that $G - S$ has exactly two components for every 3-connected planar graph G and every 3-cut S of G . Thus, in this case, G is essentially 4-connected if and only if S forms the neighborhood of a vertex of degree 3 of G for every 3-cut S of G . This property will be used frequently in the proof of Theorem 1.

A cycle C of G is a *good cycle* of G if C is outer independent and $\deg_G(x) = 3$ for all $x \in V(G) \setminus V(C)$. An edge xy of a good cycle C is *extendable* if x and y have a common neighbor $z \in V(G) \setminus V(C)$. In this case, the cycle C' of G , obtained from C by replacing the edge xy with the path (x, z, y) is again good (and longer than C). The forthcoming Lemma 1 is an essential tool in the proof of Theorem 1 (an implicit proof for cubic essentially 4-connected planar graphs can be found in [7], the general case is proved in [4]).

Lemma 1 *Every essentially 4-connected planar graph on $n \geq 11$ vertices contains a good cycle.*

Theorem 1 *For every essentially 4-connected maximal planar graph G on $n \geq 8$ vertices,*

$$\text{circ}(G) \geq \frac{2}{3}(n + 4).$$

2 Proof of Theorem 1

Suppose $n \geq 11$, as for $n \in \{8, 9, 10\}$, Theorem 1 follows from the fact that G is Hamiltonian ([2]). Using Lemma 1, let $C = [v_1, v_2, \dots, v_k]$ (indices of vertices of C are taken modulo k in the whole paper) be a longest good cycle of length k of G (i.e., $\text{circ}(G) \geq k$) and let $H = G[V(G) \setminus V(C)]$ be the graph obtained from G by removing all vertices of degree 3 which do not belong to C . Obviously, H is maximal planar and C is a Hamiltonian cycle of H . A face φ of H is an *empty face* of H if φ is also a face of G , otherwise φ is a *non-empty face* of H . Denote by $F_e(H)$ the set of empty faces of H and let $f_e(H) = |F_e(H)|$. Note that every face of G has at least two (of three) vertices on C . The three neighbors of a vertex of $V(G) \setminus V(C)$ induce a separating 3-cycle of G creating the boundary of a non-empty face of H , which has no edge in common with C because otherwise such an edge would be an extendable edge of C in G .

Let H_1 and H_2 be the spanning subgraphs of H consisting of the cycle C and of its chords lying in the interior and in the exterior of C , respectively. Note that $E(H_1) \cap E(H_2) = E(C)$ and H_1 and H_2 are maximal outerplanar graphs, both having k -gonal outer face and $k - 2$ triangular faces. Let T_i be the weak dual of H_i , $i \in \{1, 2\}$, which is the graph having all triangular faces of H_i as vertex set such that two vertices of T_i are adjacent if the triangular faces share an edge in H_i . Obviously, T_i is a tree of maximum degree at most three.

A face φ of H is a *j-face* if exactly j of its three incident edges belong to $E(C)$. Since $n \geq 11$, there is no 3-face in H and each face of H is a *j-face* with $j \in \{0, 1, 2\}$. Denote by $f_j(H_i)$ the number of empty *j-faces* of H_i . Since C does not contain any extendable edge, the following claim is obvious.

Claim 1 *Each face of H incident with an edge of any longest good cycle (in particular, each 1- or 2-face) is empty.*

An edge e of C incident with a *j-face* φ and an *l-face* ψ , where $j, l \in \{1, 2\}$, is a *(j, l)-edge*. Let φ be a 2-face of H_i . The sequence $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, is the *φ-branch* if $\varphi_2, \dots, \varphi_{r-1}$ are 1-faces of H_i , φ_r is a 0-face of H_i , and φ_j, φ_{j+1} ($1 \leq j \leq r - 1$) are adjacent (i.e. B_φ is a

minimal path in T_i with end vertices of degree 1 and 3). The *rim* $R(B_\varphi)$ of the φ -branch B_φ is the subgraph of C induced by all edges of C that are incident with an element of B_φ . Hence, it is easy to see:

Claim 2 *The rim of a φ -branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ is a path of length r .*

Claim 3 *Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i , let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ -branch of H_i , and let $v_0v_2 \in E(H_{3-i})$. If*

- (a) $R(B_\varphi) = (v_1, v_2, \dots, v_{r+1})$ *is the rim of B_φ or*
- (b) $R(B_\varphi) = (v_0, v_1, \dots, v_r)$ *is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$, or*
- (c) $R(B_\varphi) = (v_{3-r}, \dots, v_2, v_3)$ *is the rim of B_φ and $v_{-1}v_2 \in E(H_{3-i})$,*

then φ_r is empty.

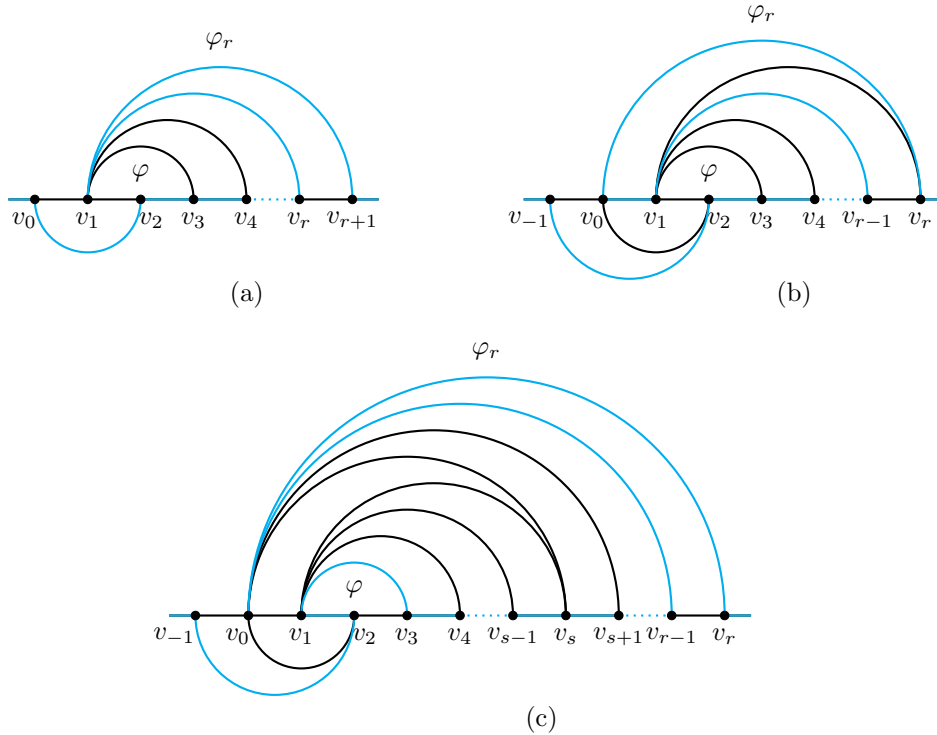


Fig. 1. A longest good cycle (cyan) sharing an edge with φ_r .

Proof.

(a) The cycle C' obtained from C by replacing the path $(v_0, v_1, \dots, v_{r+1})$ with the path $(v_0, v_2, \dots, v_r, v_1, v_{r+1})$ (Fig. 1(a)) is another longest good cycle of G and contains the edge v_1v_{r+1} incident with φ_r , thus φ_r is empty (by Claim 1).

(b) Let $\varphi_s = [v_0, v_1, v_s]$, for some s with $3 \leq s \leq r$, be a 1-face of H_i . The cycle C' obtained from C by replacing the path $(v_{-1}, v_0, \dots, v_r)$ by the path $(v_{-1}, v_2, \dots, v_{r-1}, v_1, v_0, v_r)$, for $s = r$

(Fig. 1(b)), or by the path $(v_{-1}, v_2, v_1, v_3, \dots, v_{r-1}, v_0, v_r)$, for $s \leq r - 1$ (Fig. 1(c)), is a longest good cycle of G and contains the edge v_0v_r incident with φ_r , thus φ_r is empty (by Claim 1).

(c) If $r \leq 3$, then φ_r is empty by (a) or (b). If $r \geq 4$, then $v_0v_3, v_{-1}v_3 \in E(H_i)$, thus $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction. \square

These tools will be used continuously in the following; we continue with the proof of Theorem 1. Hereby, we consider two cases. In the first case, both subgraphs H_1 and H_2 have some 0-faces. By using a customized discharging method, we distribute some weights from edges to faces to prove that sufficiently many faces are empty (each empty face will finally contain weight at most $\frac{2}{3}$). In the second case, there are only empty faces on one side of C , so that all vertices not in C are located on the other side of C . We have to prove that there are some additional empty faces on this side.

CASE 1. Let H_1 and H_2 both contain at least two 0-faces or one non-empty 0-face.

For every edge e of C we define the weight $w_0(e) = 1$. Obviously, $\sum_{e \in E(C)} w_0(e) = |E(C)| = k$.

First redistribution of weights.

Each edge of C sends weight to both incident faces as follows

Rule R1. A (1,1)-edge sends $\frac{1}{2}$ to both incident 1-faces.

Rule R2. A (1,2)-edge sends $\frac{2}{3}$ to the incident 1-face and $\frac{1}{3}$ to the incident 2-face.

Rule R3. A (2,2)-edge sends $\frac{1}{2}$ to both incident 2-faces.

The edges of C completely redistribute their weights to incident 1- and 2-faces. For an empty face φ , let $w_1(\varphi)$ be the total weight obtained by φ (in first redistribution). Obviously, for an empty face φ , it is

$$w_1(\varphi) = \begin{cases} 1, & \text{if } \varphi \text{ is a 2-face incident with two (2,2)-edges,} \\ \frac{5}{6}, & \text{if } \varphi \text{ is a 2-face incident with a (1,2)-edge and a (2,2)-edge,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 2-face incident with two (1,2)-edges,} \\ \frac{2}{3}, & \text{if } \varphi \text{ is a 1-face incident with a (1,2)-edge,} \\ \frac{1}{2}, & \text{if } \varphi \text{ is a 1-face incident with a (1,1)-edge,} \\ 0, & \text{if } \varphi \text{ is a 0-face.} \end{cases}$$

Moreover, $\sum_{\varphi \in F_e(H)} w_1(\varphi) = |E(C)| = k$.

Second redistribution of weights.

The weight of 2-faces of H exceeding $\frac{2}{3}$ will be redistributed to 1-faces and empty 0-faces of H by the following rules. Let φ be a 2-face of H_i with $w_1(\varphi) > \frac{2}{3}$ (i.e. incident with at least one (2,2)-edge) and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ -branch. Moreover, let α be a 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α .

- Rule R4.** φ sends $w_1(\varphi) - \frac{2}{3}$ to φ_r if φ_r is empty and $r \leq 3$.
- Rule R5.** φ sends $\frac{1}{6}$ to φ_j if φ_j ($2 \leq j \leq r - 1$) is a 1-face incident with a (1,1)-edge.
- Rule R6.** φ sends $\frac{1}{6}$ to φ_r if φ_r is empty and $r \geq 4$.
- Rule R7.** φ sends $\frac{1}{6}$ to α_2 if α is incident with a (1,2)-edge and α_2 is an empty 0-face.
- Rule R8.** φ sends $\frac{1}{6}$ to β_2 , where β is a 2-face of H_{3-i} having exactly one common vertex with φ and incident with two (1,2)-edges and β_2 is an empty 0-face of H_{3-i} adjacent to β .

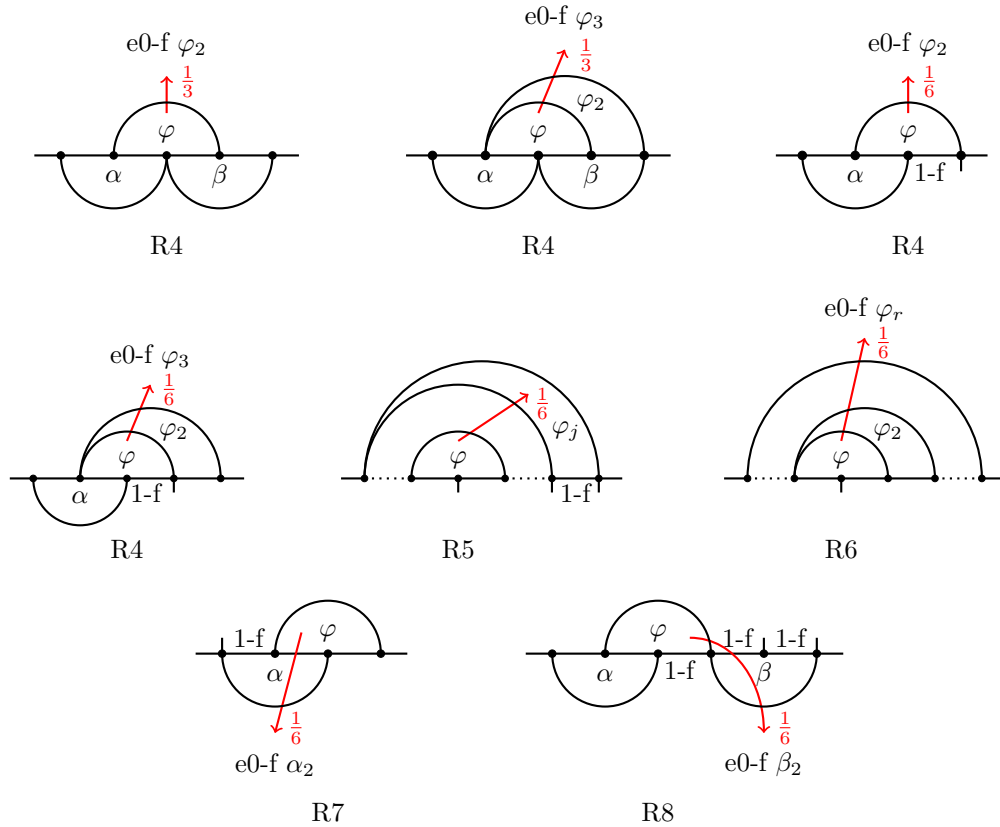


Fig. 2. Redistribution rules R4–8 (1-f is a 1-face and e0-f is an empty 0-face).

For an empty face φ , let $w_2(\varphi)$ be the total weight obtained by φ (after second redistribution). Obviously, $\sum_{\varphi \in F_e(H)} w_2(\varphi) = |E(C)| = k$ (as non-empty faces do not obtain any weight). In the following, we will show that the weight $w_2(\varphi)$ of each (empty) face φ does not exceed $\frac{2}{3}$ which will mean $k = \sum_{\varphi \in F_e(H)} w_2(\varphi) \leq \frac{2}{3} f_e(H)$. The maximal planar graph G has exactly $2n - 4$ faces. Each of $f_e(H) \geq \frac{3}{2}k$ empty faces of H is a face of G as well, and each of $n - k$ (pairwise non-adjacent) vertices of G not belonging to C (whose removal has created a non-empty face of H) is incident with three (“private”) faces of G . Hence $2n - 4 = |F(G)| = f_e(H) + 3(n - k) \geq \frac{3}{2}k + 3n - 3k$ and finally $k \geq \frac{2}{3}(n + 4)$ will follow.

Weight of a 2-face.

Let $\varphi = [v_1, v_2, v_3]$ be a 2-face of H_i and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$, $r \geq 2$, be the φ -branch. As already mentioned, $\frac{2}{3} \leq w_1(\varphi) \leq 1$. We check that the weight of φ exceeding $\frac{2}{3}$ will be shifted in the second redistribution.

1. Let φ be incident with two (2,2)-edges (note that $w_1(\varphi) = 1$). Denote $\alpha = [v_0, v_1, v_2]$ and $\beta = [v_2, v_3, v_4]$ the 2-faces of H_{3-i} adjacent to φ . Let α_2 and β_2 be the face of H_{3-i} adjacent to α and β , respectively. Each of the faces φ_2 , α_2 , and β_2 is either a 1-face or empty 0-face (by Claim 3a).

1.1. Let α_2 and β_2 be 0-faces (possibly $\alpha_2 = \beta_2$).

1.1.1. If edges v_0v_1 and v_3v_4 of C do not belong to the rim $R(B_\varphi)$ of B_φ , then $r = 2$, thus φ sends $\frac{1}{3}$ to empty 0-face φ_2 (by R4).

1.1.2. If v_0v_1 belongs to the rim $R(B_\varphi)$ and v_3v_4 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_0, v_1, v_3]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to α_2 (by R7). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$ and v_3v_4 belongs to $R(B_\varphi)$.)

1.1.3. If edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. Thus φ sends $\frac{1}{6}$ to α_2 and $\frac{1}{6}$ to β_2 (by R7).

1.2. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ be a 1-face and β_2 be a 0-face. (Similarly if α_2 is a 0-face and β_2 is a 1-face.)

1.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{3}$ to φ_r (by R4).

1.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends weight $\geq \frac{1}{6}$ to φ_r (by R4 or R6) and $\frac{1}{6}$ to β_2 (by R7).

1.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3 , then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. Thus $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.

1.2.3.1. If $v_{-1}v_0$ does not belong to the rim $R(B_\varphi)$, then φ_r is empty (by Claim 3b). Thus φ sends $\frac{1}{6}$ to φ_r (by R6) and $\frac{1}{6}$ to β_2 (by R7).

1.2.3.2. If $v_{-1}v_0$ belongs to the rim $R(B_\varphi)$, then $v_{-1}v_0$ is a (1,1)-edge. Thus φ sends $\frac{1}{6}$ to φ_j , a 1-face of B_φ incident with $v_{-1}v_0$ (by R5) and $\frac{1}{6}$ to β_2 (by R7).

1.3. Let $\alpha_2 = [v_{-1}, v_0, v_2]$ and $\beta_2 = [v_2, v_4, v_5]$ be 1-faces.

1.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{3}$ to φ_r (by R4). (Similarly if v_0v_1 does not belong to $R(B_\varphi)$.)

1.3.2. Let edges v_0v_1 and v_3v_4 belong to the rim $R(B_\varphi)$, then both are (1,2)-edges. If v_0v_1 and v_3v_4 are incident with φ_2 and φ_3 , then $\{v_0, v_2, v_4\}$ is a non-trivial 3-cut, a contradiction. If $\varphi_2 = [v_0, v_1, v_3]$ and $\varphi_3 = [v_{-1}, v_0, v_3]$, then $\{v_{-1}, v_2, v_3\}$ is a non-trivial 3-cut, a contradiction as well. (Similarly if $\varphi_2 = [v_1, v_3, v_4]$ and $\varphi_3 = [v_1, v_4, v_5]$.)

2. Let φ be incident with (2,2)-edge v_1v_2 and (1,2)-edge v_2v_3 (note that $w_1(\varphi) = \frac{5}{6}$). Denote $\alpha = [v_0, v_1, v_2]$ the 2-face of H_{3-i} adjacent to φ and let α_2 be the face of H_{3-i} adjacent to α . Each of the faces φ_2 and α_2 is either a 1-face or empty 0-face (by Claim 3a).

2.1. Let α_2 be 0-face.

2.1.1. If v_0v_1 does not belong to the rim $R(B_\varphi)$, then φ_r is empty (by Claim 3a). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).

2.1.2. If v_0v_1 belongs to the rim $R(B_\varphi)$, then v_0v_1 is a (1,2)-edge. Thus φ sends $\frac{1}{6}$ to α_2 (by R7).

2.2. Let α_2 be a 1-face incident with $v_{-1}v_0$ (i.e. $\alpha_2 = [v_{-1}, v_0, v_2]$).

2.2.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and φ_r is empty (by proof of Claim 3c). Thus φ sends $\frac{1}{6}$ to φ_r (by R4).

2.2.2. If v_3v_4 belongs to the rim $R(B_\varphi)$ and v_0v_1 does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and φ_r is empty (by Claim 3a). Thus φ sends $\frac{1}{6}$ to φ_r (by R4 or R6).

2.2.3. Let edges v_3v_4 and v_0v_1 belong to the rim $R(B_\varphi)$. If $v_{-1}v_0$ does not belong to $R(B_\varphi)$, then φ_r is empty (by Claim 3b). Thus φ sends $\frac{1}{6}$ to φ_r (by R6). Otherwise $v_{-1}v_0$ belongs to $R(B_\varphi)$, thus it is a (1,1)-edge incident with a 1-face φ_j of B_φ . Hence φ sends $\frac{1}{6}$ to φ_j (by R5).

2.3. Let α_2 be a 1-face incident with v_2v_3 (i.e. $\alpha_2 = [v_0, v_2, v_3]$). Since $v_0v_3 \in E(H_{3-i})$, φ_2 cannot be the 1-face $[v_0, v_1, v_3]$ in H_i .

2.3.1. If v_3v_4 does not belong to the rim $R(B_\varphi)$, then $r = 2$, thus φ sends $\frac{1}{6}$ to φ_2 (by R4).

2.3.2. If v_3v_4 belongs to the rim $R(B_\varphi)$, then $r \geq 3$ and $\varphi_2 = [v_1, v_3, v_4]$.

2.3.2.1. If v_3v_4 is incident with a 1-face of H_{3-i} (i.e., v_3v_4 is a (1,1)-edge), then φ sends $\frac{1}{6}$ to φ_2 (by R5).

2.3.2.2. Let v_3v_4 be incident with a 2-face β of H_{3-i} (necessarily, $\beta = [v_3, v_4, v_5]$). If $r = 3$, then φ_3 is empty (by Claim 3a), thus φ sends $\frac{1}{6}$ to φ_3 (by R4). If $r = 4$, then $\varphi_3 = [v_1, v_4, v_5]$ (as $\{v_0, v_3, v_4\}$ is a non-trivial 3-cut if $\varphi_3 = [v_0, v_1, v_4]$) and φ_4 is empty (by Claim 3a), thus φ sends $\frac{1}{6}$ to φ_4 (by R6). Finally, let $r \geq 5$. Necessarily $\varphi_3 = [v_1, v_4, v_5]$ (as for $r = 4$) and $\varphi_4 = [v_1, v_5, v_6]$ (as $\{v_0, v_3, v_5\}$ is a non-trivial 3-cut if $\varphi_4 = [v_0, v_1, v_5]$) are 1-faces of B_φ . If v_5v_6 is a (1,1)-edge, then φ sends $\frac{1}{6}$ to φ_4 (by R5). Otherwise v_5v_6 is a (1,2)-edge, thus it does not belong to β -branch (in H_{3-i}) and therefore β_2 is a 0-face, which is, moreover, empty (as the cycle obtained from C by replacing the path (v_0, \dots, v_5) by the path $(v_0, v_2, v_1, v_4, v_3, v_5)$ is a longest good cycle of G and contains the edge v_3v_5 incident with β_2 (Claim 1)). Hence φ sends $\frac{1}{6}$ to β_2 (by R8).

Weight of a 1-face.

To estimate the weight of a 1-face, we use the following simple observation:

Claim 4 *Each 1-face of H belongs to at most one branch.*

Let ψ be a 1-face incident with an edge e of C . If e is a (1,2)-edge, then ψ obtains weight $\frac{2}{3}$ from e (by R2) only. Otherwise e is a (1,1)-edge, thus ψ obtains $\frac{1}{2}$ from e (by R1). Furthermore, in this case, ψ can get $\frac{1}{6}$ from a 2-face φ (by R5) if ψ belongs to the φ -branch. Hence $w_2(\psi) \leq \frac{2}{3}$.

Weight of an empty 0-face.

Each empty 0-face ω belongs to at most two branches (in Case 1). Let φ be a 2-face of H_i with the φ -branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ such that $\varphi_r = \omega$, and let e be the edge incident with φ_r and φ_{r-1} (where $\varphi_{r-1} = \varphi$ for $r = 2$).

If φ is adjacent to two 2-faces, then ω gets through e the weight $\frac{1}{3}$ (by R4) for $r \leq 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. If φ is adjacent to one 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R4) and additionally $\frac{1}{6}$ (by R7) for $r = 2$ or the weight at most $\frac{1}{6}$ (by R4) for $r = 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. Finally, if φ is adjacent to no 2-face, then ω gets through e the weight $\frac{1}{6}$ (by R6) for $r \geq 4$ or the weight at most $2 \times \frac{1}{6}$ (by R8) for $r \leq 3$.

We showed that $w_2(\varphi) \leq \frac{2}{3}$ for each empty face φ and completed the Case 1. Thus, we can assume that in H_i are only empty faces and among them, at most one face is a 0-face. To complete the proof, we have to show that there are some empty faces in H_{3-i} as well.

CASE 2. Let H_i contain no 0-face or exactly one 0-face which is additionally empty.

Obviously, if H_i contains no 0-face, then it contains two 2-faces α_1 and α_2 (since T_i is a path and 2-faces of H_i are leaves of T_i). Note that, (only) in this case, the branches in H_i are not defined.

Remember that $H = G[V(C)]$ has $k \geq 7$ vertices (as otherwise G with at most $k + 2 \leq 8$ vertices is Hamiltonian). If H_i contains exactly one 0-face, then it contains three 2-faces α_1, α_2 and α_3 (since T_i is a subdivision of $K_{1,3}$ and 2-faces of H_i are leaves of T_i). We assume that H_{3-i} contains at least two 0-faces as otherwise all but at most one faces of H_{3-i} are empty and G has $n \leq |V(H)| + 1 = k + 1$ vertices and Theorem 1 follows immediately (with $n \geq 11$).

Distribution of points.

To estimate the number of empty 0- and 1-faces in H_{3-i} , each 2-face α_j of H_i ($j \in \{1, 2\}$ if H_i contains no 0-face and $j \in \{1, 2, 3\}$ if H_i contains one 0-face, respectively) will distribute 1 or 2 points to faces of H_{3-i} . Let α_j be adjacent to the faces φ and ψ of H_{3-i} .

Rule P1. If φ and ψ are 2-faces of H_{3-i} with branches $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ and $B_\psi = (\psi, \psi_2, \dots, \psi_t)$, then φ_r and ψ_t will each receive 1 point (or 2 points if $\varphi_r = \psi_t$) from α_j .

Rule P2. If φ and ψ are 1-faces of H_{3-i} , then φ and ψ will each receive 1 point from α_j .

Rule P3. If φ is a 2-faces of H_{3-i} with φ -branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ and ψ is a 1-face of H_{3-i} not belonging to B_φ , then φ_r and ψ will each receive 1 point from α_j .

Rule P4. If φ is a 2-faces of H_{3-i} with φ -branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ and ψ is a 1-face of H_{3-i} belonging to B_φ , then only ψ will receive 1 point from α_j .

For a face φ of H_{3-i} , let $p(\varphi)$ be the total number of points carried by φ (in the distribution of points).

Claim 5 $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq \sum_{\varphi \in F_e(H_{3-i})} p(\varphi)$.

Proof. We have to prove that each 1-face of H_{3-i} gets at most 1 point and that each 0-face of H_{3-i} gets points only if it is empty and it gets at most 2 points. Consequently, Claim 5 follows by simple counting.

Let β be a 1-face of H_{3-i} . Since β can only get points if it is adjacent to some α_j and there can only be one such face then $p(\beta) \leq 1$.

Let β be a 0-face of H_{3-i} . Since β can only get points if it belongs to a branch and it belongs to at most two branches (as there are at least two 0-faces in H_{3-i}), then $p(\beta) \leq 2$. Assume first that β gets a point by P1. Then there is α_j incident with two (2,2)-edges and adjacent 2-faces φ and ψ of H_{3-i} . Let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ with $\varphi_r = \beta$ be the branch which ends in β . By Claim 3a, $\varphi_r = \beta$ is an empty 0-face.

Thus, assume that β gets a point by P3. Then there is α_j incident with a (1,2)-edge with adjacent 1-face ψ in H_{3-i} and a (2,2)-edge with adjacent 2-face φ such that ψ does not belong to the branch $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ with $\varphi_r = \beta$. Since the common edge of α_j and ψ does not belong to the rim $R(B_\varphi)$, again by Claim 3a, $\varphi_r = \beta$ is an empty 0-face. □

Claim 6 $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.

Proof. If $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \geq 4$, then $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$ (by Claim 5). Assume $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \leq 3$.

1. Let H_i contains exactly one 0-face. As there are three 2-faces $\alpha_1, \alpha_2, \alpha_3$ in H_i (note, that T_i is a subdivided 3-star in this case), then $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) = 3$. Furthermore, only P4 was applied to each α_j ($j \in \{1, 2, 3\}$) hence there are three 1-faces with 1 point and they belong to three different branches.

Since $|V(H)| = k \geq 7$, there is $j \in \{1, 2, 3\}$ such that α_j is adjacent to a 1-face δ of H_i . Let φ be the adjacent 2-face of α_j in H_{3-i} and $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ be its branch.

1.1. If $r \geq 4$, then φ_2 and φ_3 are 1-faces of the same branch. Thus, at most one among φ_2 and φ_3 has a point and $f_1(H_{3-i}) \geq 4$.

1.2. If $r = 3$, then δ and φ are not adjacent (i.e. $\delta \neq \varphi_2$, since H has no multiple edges) and φ_3 is an empty 0-face (by Claim 3b), hence $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2. Let H_i contains no 0-face. Since $\sum_{\varphi \in F_e(H_{3-i})} p(\varphi) \leq 3$, there is $j \in \{1, 2\}$ such that P4 was applied to α_j . Let δ be the 1-face of H_i adjacent to α_j (since $|V(H)| = k \geq 7$), let φ and ψ be the 2-face and 1-face of H_{3-i} adjacent with α_j , respectively, and let $B_\varphi = (\varphi, \varphi_2, \dots, \varphi_r)$ be the branch of φ . We may assume $\alpha_j = [v_1, v_2, v_3]$ and $\varphi = [v_2, v_3, v_4]$.

2.1. Let $r \leq 4$.

2.1.1 If $\delta = [v_0, v_1, v_3]$, then v_0v_1 does not belong to the rim $R(B_\varphi)$ (otherwise $\varphi_2 = [v_1, v_2, v_4]$, $\varphi = [v_0, v_1, v_4]$ and v_0, v_3, v_4 is a non-trivial 3-cut, a contradiction) and φ_r is an empty 0-face (by Claim 3b). By P1–4, there is a face in H_{3-i} other than ψ and φ_r with a point, thus $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.

2.1.2 If $\delta = [v_1, v_3, v_4]$, then $\varphi_2 = [v_2, v_4, v_5]$ (since $v_1v_4 \in E(H_i)$), $\psi = \varphi_3 = [v_1, v_2, v_5]$, and $\{v_1, v_4, v_5\}$ is a non-trivial 3-cut, a contradiction.

2.2. Let $r = 5$. There are three 1-faces (in fact φ_2, φ_3 , and φ_4) all belonging to the same branch B_φ . We may assume that P4 was applied to α_j and P2 was applied to α_{3-j} , and all three 1-faces are adjacent to α_1 or α_2 (since otherwise there is another 1-face or empty 0-face and Claim 6 follows).

2.2.1. If $\alpha_{3-j} = [v_{-1}, v_0, v_1]$, then $\text{rim } R(B_\varphi) = (v_{-1}, \dots, v_4)$, thus $\varphi_2 = [v_1, v_2, v_4]$ and $\delta = [v_1, v_3, v_4]$, a contradiction to the simplicity of H .

2.2.2. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_0, v_1, v_3]$, then $\text{rim } R(B_\varphi) = (v_1, \dots, v_6)$ and φ_5 is an empty 0-face (by Claim 3b), thus $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2.2.3. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_1, v_3, v_4]$, then $\text{rim } R(B_\varphi) = (v_1, \dots, v_6)$. Hence $v_1v_6 \in E(H_{3-i})$ and consequently $\{v_1, v_4, v_6\}$ is a non-trivial 3-cut, a contradiction.

2.3. If $r \geq 6$, then there are at least four 1-faces in B_φ , thus $f_1(H_{3-i}) \geq 4$. □

Remember that each j -face of H_{3-i} is incident with j (“private”) edges of C , hence $2f_2(H_{3-i}) + f_1(H_{3-i}) = k$. As each of the $k-2$ triangular faces of H_i is empty, all non-empty faces of H belong to H_{3-i} and their number is $(k-2) - f_2(H_{3-i}) - f_1(H_{3-i}) - f_0(H_{3-i}) = (k-2) - \frac{1}{2}(k - f_1(H_{3-i})) - f_1(H_{3-i}) - f_0(H_{3-i}) = \frac{k}{2} - 2 - \frac{1}{2}(f_1(H_{3-i}) + 2f_0(H_{3-i})) \leq \frac{k}{2} - 4$ (by Claim 6). Finally, at most $\frac{k}{2} - 4$ vertices of G lie outside the cycle C (and exactly k vertices on C), hence $n \leq k + (\frac{k}{2} - 4)$ and $k \geq \frac{2}{3}(n + 4)$ follows, which completes the proof of Theorem 1.

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