Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 21, no. 3, pp. 341–352 (2017)
DOI: 10.7155/jgaa.00419

A Necessary Condition and a Sufficient Condition for Pairwise Compatibility Graphs

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Abstract

In this paper we give a necessary condition and a sufficient condition for a graph to be a pairwise compatibility graph (PCG). Let G be a graph and let G^c be the complement of G. We show that if G^c has two disjoint chordless cycles then G is not a PCG. On the other hand, if G^c has no cycle then G is a PCG. Our conditions are the first necessary condition and the first sufficient condition for pairwise compatibility graphs in general. We also show that there exist some graphs in the gap of the two conditions which are not PCGs.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
May 2016	November 2016	December 2016	December 2016	January 2017

Published:
February 2017

Article type:
Regular paper

Communicated by:
M. Kaykobad and R. Petreschi

Work of Debajyoti Mondal is partially supported by NSERC.

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1 Introduction

Let T be an edge-weighted tree and let d_{min} and d_{max} be two non-negative real numbers such that $d_{min} \leq d_{max}$. A *pairwise compatibility graph (PCG) of* T *for* d_{min} *and* d_{max} is a graph $G = (V, E)$, where each vertex $u' \in V$ represents a leaf u of T and there is an edge $(u', v') \in E$ if and only if the distance between u and v in T, denoted by $d_T(u, v)$, lies within the range from d_{min} to d_{max}. We denote a pairwise compatibility graph of T for d_{min} and d_{max} by $PCG(T, d_{min}, d_{max})$. A graph G is a *pairwise compatibility graph* (PCG) if there exists an edge-weighted tree T and two non-negative real numbers d_{min} and d_{max} such that $G = PCG(T, d_{min}, d_{max})$. An edge-weighted tree T is called a *pairwise compatibility tree (PCT)* of a graph G if $G = PCG(T, d_{min}, d_{max})$ for some d_{min} and d_{max}. Figure 1(a) depicts a pairwise compatibility graph G and Fig. 1(b) depicts a pairwise compatibility tree T of G for $d_{min} = 4$ and $d_{max} = 5$. Evolutionary relationships among a set of organisms can be modeled as pairwise compatibility graphs [9]. Moreover, the problem of finding a maximal clique can be solved in polynomial time for pairwise compatibility graphs if one can find their pairwise compatibility trees in polynomial time [9].

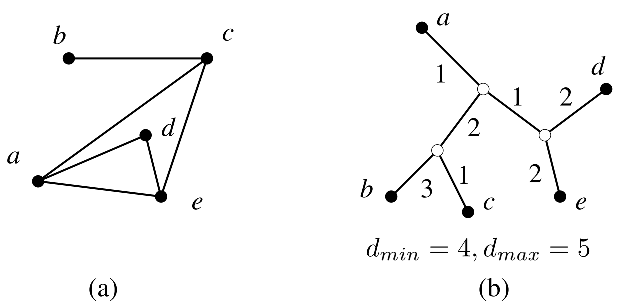

Figure 1: (a) A pairwise compatibility graph G and (b) a pairwise compatibility tree T of G.

Constructing a PCT of a given graph is a challenging problem. It is interesting that there are some classes of graphs with very restricted structural properties whose PCT are unknown. For example, it is unknown whether sufficiently large wheel graphs and grid graphs are PCGs or not. It is known that some specific graphs of 8 vertices, 9 vertices, 15 vertices, and 20 vertices are not PCGs [8, 13]. On the other hand, some restricted subclasses of graphs like, cycles, paths, trees, interval graphs, triangle free outerplanar 3-graphs, chordless cycles and single chord cycles, ladder graphs and some particular subclasses of bipartite graphs are known as PCGs [12, 13, 14, 11, 5]. It is also known that any graph of at most seven vertices [4] and any bipartite graph with at most eight vertices [10] are PCGs. Furthermore a lot of work has been done concerning some particular subclasses of PCG as leaf power graphs [1], exact leaf power graphs [3] and lately a new subclass has been introduced, namely the min-leaf power graphs [2]. However the complete characterization of a PCG is not known. We refer the reader to [7] for more details on PCG.

In this paper we give a necessary condition and a sufficient condition for a graph to be a pairwise compatibility graph based on the complement of the given graph. Let G be a graph and let G^c be the complement of G. We prove that if G^c has two disjoint chordless cycles then G is not a PCG. On the other hand, if G^c has no cycle then G is a PCG. We also show that there exist some graphs in the gap of the two conditions which are not PCGs.

The rest of the paper is organized as follows. In Section 2 we give some definitions that are used in this paper. In Section 3 we prove a necessary condition for PCG. We give a sufficient condition for PCG in Section 4. In Section 5 we show that there exist some graphs in the gap of the two conditions which are not PCGs. Finally, Section 6 presents some interesting open problems.

2 Preliminaries

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. Let V' and E' be subsets of V and E, respectively. The graph $G' = (V', E')$ is called a *subgraph* of G, and G' is an *induced* subgraph of G if E' is the set of all edges of G whose end vertices are in V'. The *complement* of G is the graph G^c with the vertex set V but whose edge set consists of the edges not present in G. A *chord* of a cycle C is an edge not in C whose endpoints lie in C. A *chordless cycle* of G is a cycle of length at least four in G that has no chord. For a vertex v of a graph G, $N(v) = \{u|(u, v) \in E\}$ denotes the *open neighborhood*. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two induced subgraphs of G. We call the subgraphs G_1 and G_2 *disjoint* if they do not share a vertex and there is no edge $(u, v) \in E$ such that $u \in V_1$ and $v \in V_2$. The cycles C_1 and C_2 drawn by thick lines in the graph in Fig. 2 are disjoint cycles. On the other hand, the cycle C_1 drawn by thick line and the cycle C_3 indicated by dotted lines are not disjoint since the edge (c, g) has one end vertex c in C_1 and the other end vertex g in C_3.

Let $G = PCG(T, d_{min}, d_{max})$, and let v be a leaf of T. Then we denote the corresponding vertex of v in G by v' and vice versa. The following lemma is known on a forbidden structure of PCG.

Lemma 1 *[8] Let C be the cycle a', b', c', d' of four vertices. If $C = PCG(T, d_{min}, d_{max})$ for some tree T and values d_{min} and d_{max}, then $d_T(a, c)$ and $d_T(b, d)$ cannot be both greater than d_{max}.*

A graph $G(V, E)$ is an *LPG (leaf power graph)* if there exists an edge-weighted tree T and a nonnegative number d_{max} such that there is an edge (u, v) in E if and only if for their corresponding leaves u', v' in T, we have $d_T(u', v') \leq d_{max}$. We write $G = LPG(T, d_{max})$ if G is an LPG for a tree T with a specified d_{max}. Again G is an *mLPG (minimum leaf power graph)* if there exists an edge-weighted tree T and a nonnegative number d_{min} such that there is an edge (u, v) in E if and only if for their corresponding leaves u', v'

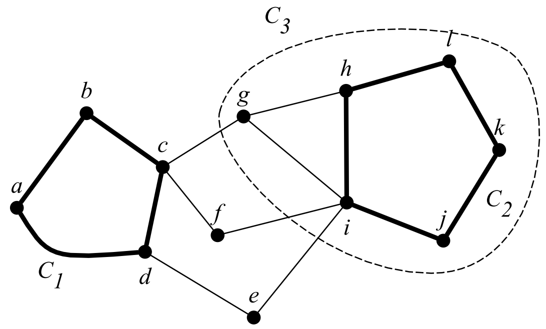

Figure 2: A graph G with two disjoint chordless cycles whose complement is not a PCG.

in T, we have $d_T(u', v') \geq d_{min}$. We write $G = mLPG(T, d_{min})$ if G is an mLPG for a tree T with a specified d_{min}. Both LPG and mLPG are subclasses of PCG [5]. The following lemmas are known on LPG and mLPG.

Lemma 2 *[6] Let C_n be a cycle of length $n \geq 5$, then $C_n \notin mLPG$.*

Lemma 3 *[5] The complement of every graph in LPG is in mLPG and conversely, the complement of every graph in mLPG is in LPG.*

3 Necessary Condition

In this section our aim is to prove that for a given graph G, if G^c has two disjoint chordless cycles of length four or more then G is not a PCG. We first prove the following lemma.

Lemma 4 *Let $G = (V, E)$ be a graph. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two induced subgraphs of G with no common vertices in G_1 and G_2. Assume that $V_2 \subset N(u')$ and $V_1 \subset N(v')$ in G for every $u' \in V_1$ and every $v' \in V_2$. If neither G_1 nor G_2 is an mLPG, then G is not a PCG.*

Proof: Let T_1 be a PCT of G_1 such that $G_1 = PCG(T_1, d_{min1}, d_{max1})$ and T_2 be a PCT of G_2 such that $G_2 = PCG(T_2, d_{min2}, d_{max2})$. Since G_1 is not an mLPG, there exist two leaves a, c in T_1 such that $d_{T_1}(a, c) > d_{max1}$, and two leaves b, d in T_2 such that $d_{T_2}(b, d) > d_{max2}$.

Assume for a contradiction that $G = PCG(T, d_{min}, d_{max})$. Then T has two subtrees T_1 and T_2 such that $G_1 = PCG(\ T_1, d_{min}, d_{max})$ and $G_2 = PCG(T_2, d_{min},\ d_{max})$. Since there exists a pair of leaves a, c in T_1 such that $d_{T_1}(a, c) > d_{max}$ and exists a pair of leaves b, d in T_2 such that $d_{T_2}(b, d) > d_{max}$, both $d_T(a, c)$ and $d_T(b, d)$ are greater than d_{max} in T. Then G does not have the edges (a', c') and (b', d'). Hence the induced subgraph of vertices a', b', c', d'

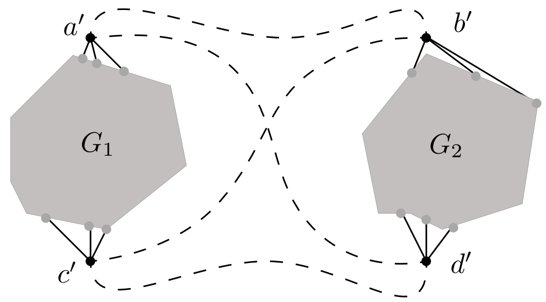

Figure 3: Illustration for Lemma 4.

is a cycle in G because $V_2 \subset N(u')$ and $V_1 \subset N(v')$ in G for each $u' \in V_1$ and each $v' \in V_2$ (see Fig. 3), a contradiction to Lemma 1. □

Since every vertex of G_1 is a neighbor of every vertex of G_2 in Lemma 4, there is no edge (u, v) in G^c where $u \in V_1$ and $v \in V_2$. That means G_1^c and G_2^c are disjoint for the graphs G_1 and G_2 in Lemma 4. Thus the following lemma is immediate.

Lemma 5 *Let G be a graph. Let H_1 and H_2 be two disjoint subgraphs of G^c. If neither H_1^c nor H_2^c is an mLPG, then G is not a PCG.*

It would be interesting to investigate what could be the smallest subgraph H_1^c or H_2^c, where there always exists a pair of leaves with weighted distance greater than d_{max} in its PCT. Our goal is to show that H_1^c or H_2^c could be a chordless cycle or the complement of a cycle.

Lemma 6 *Let C_n be a cycle of length n vertices. If $n \geq 5$, then C_n is not an mLPG. If $n \geq 4$, then C_n^c is not an mLPG.*

Proof: By Lemma 2 and Lemma 3, the claim is true for $n \geq 5$. The proof for the case when $n = 4$ can be obtained using a known result that every LPG is a chordal graph [7]. However, here we give an independent proof for completeness.

Let C_4 be the cycle a', b', c', d' of four vertices. Let $C_4^c = PCG(T_1, d_{min1}, d_{max1})$. We prove that there exist a pair of leaves in T_1 whose weighted distance is greater than d_{max1}. Note that C_4^c contains only two edges (a', c') and (b', d'). For contradiction assume that $d_{T_1}(a, b)$, $d_{T_1}(b, c)$, $d_{T_1}(c, d)$, $d_{T_1}(d, a)$ are smaller than d_{min1}. Then $C_4^c = mLPG(T_1, d_{min1})$. By Lemma 3, $C_4 = LPG(T_1, d_{max})$ or $PCG(T_1, 0, d_{max})$ for some d_{max}. Since (a', c') and (b', d') are the non-adjacent pair in C_4, both $d_{T_1}(a, c)$ and $d_{T_1}(b, d)$ are greater than d_{max}. But both $d_{T_1}(a, c)$ and $d_{T_1}(b, d)$ can not be greater than d_{max} by Lemma 1, a contradiction. □

We now have the following theorem, which is the main result of this section. The proof of the theorem is immediate from Lemma 5 and Lemma 6.

Theorem 1 *Let G be a graph. Let H_1 and H_2 be two disjoint induced subgraphs of G^c. If each of H_1 and H_2 is either a chordless cycle of at least four vertices or C_n^c for $n \geq 5$, then G is not a PCG.*

4 Sufficient Condition

In this section we show that if the complement of a graph has no cycle then the graph is a PCG.

Salma *et. al.* [12] showed that every tree $\mathcal{T}$ is a PCG where $\mathcal{T} = PCG(T, 3, 3)$. They compute the edge-weighted tree T easily by taking a copy of $\mathcal{T}$ and attaching each vertex as a pendant vertex with its original one. Then set weight 1 to each edge. It can be easily verified that T is a PCT of $\mathcal{T}$ for $d_{max} = 3$ and $d_{min} = 3$. For the same settings it is also true that $\mathcal{T} = LPG(T, 3)$. We extend this technique and show that every forest is a LPG as in the following lemma.

Lemma 7 *Let $\mathcal{T}$ be a forest. Then $\mathcal{T} = LPG(T, 3)$. Furthermore the edge-weighted tree T can be found in linear time.*

Proof:

Let $\mathcal{T}$ be a forest of trees $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_k$. We find $\mathcal{T}_i = LPG(T_i, 3)$ for each tree as described above. Then for $2 \leq i \leq k$ we join T_i with $\sum T_{i-1}$, where $\sum T_{i-1}$ is the resultant merged trees up to $i - 1$, as follows. Take two internal vertices u and v from T_i and $\sum T_{i-1}$, respectively. If no internal vertex exists in T_i or $\sum T_{i-1}$ then u or v could be a leaf. Then add an edge between u and v in $\sum T_i$ with weight 3. In this way we get $T = \sum T_k$. Figure 4(a) and Figure 4(b) illustrate a forest $\mathcal{T}$ and an edge-weighted tree T for $\mathcal{T} = LPG(T, 3)$, respectively. It is easy to see that T for $\mathcal{T} = LPG(T, 3)$ can be constructed in linear time.

□

We now show that if the complement of a graph has no cycle then the graph is a PCG as in the following theorem.

Theorem 2 *Let G be a graph. If G^c has no cycle then G is a PCG.*

Proof: Let G be a graph. If G^c has no cycle then G^c is a forest. By Lemma 7, $G^c = LPG(T, 3)$. It can be easily verified that $G = mLPG(T, 4)$. Moreover, by Lemma 3 G is a PCG.

□

5 Characterizing the Intermediate Graphs

In Section 3 we have shown that if a graph G is a PCG, then G^c cannot contain two disjoint induced chordless cycles. Recall that two chordless cycles C_1 and

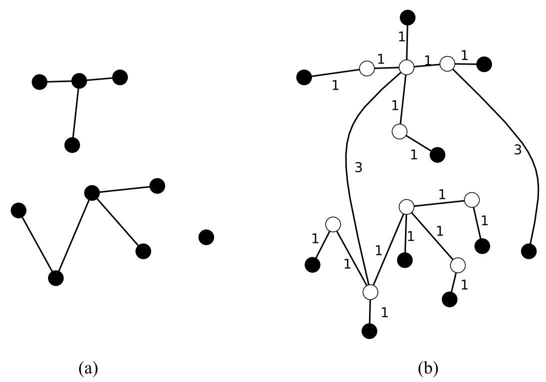

Figure 4: (a) A forest $\mathcal{T}$ and (b) the edge-weighted tree T for $\mathcal{T} = LPG(T, 3)$.

C_2 in G^c are disjoint if they are vertex disjoint, and there does not exist any edge (u, v) in G^c where $u \in C_1$ and $v \in C_2$. On the other hand, in Section 4 we have proved that if G^c does not contain any cycle, then G is a PCG. Some interesting classes of graphs that remain in the gap are as follows.

$\mathcal{G}^1$: A graph G belongs to $\mathcal{G}^1$ if G^c does not contain any chordless cycle. Fig. 5(a) shows such an example of G^c.

$\mathcal{G}^2$: A graph G belongs to $\mathcal{G}^2$ if G^c consists of two induced chordless cycles, where the cycles share some common vertices, e.g., see Fig. 5(b).

$\mathcal{G}^3$: A graph G belongs to $\mathcal{G}^3$ if G^c consists of two induced chordless cycles and some edges that are incident to both cycles, e.g., see Fig. 5(c).

$\mathcal{G}^4$: A graph G belongs to $\mathcal{G}^4$ if G^c contains only one induced chordless cycle, e.g., see Fig. 5(d).

We now show that there exist graphs in $\mathcal{G}^1$ and $\mathcal{G}^3$ that are not PCGs, which leaves the characterization for the remaining two graph classes $\mathcal{G}^2$ and $\mathcal{G}^4$ open.

Theorem 3 *Each of $\mathcal{G}^1$ and $\mathcal{G}^3$ contains a graph that is not a PCG.*

Proof: We first briefly review a result of Yanhaona *et al.* [13] that proves the existence of a bipartite graph, which is not a PCG. We then use this bipartite graph to construct instances of $\mathcal{G}^1$ and $\mathcal{G}^3$, which are not PCGs.

Review of Yanhaona *et al.*'s [13] Result: Let $H = (V, E)$ be a bipartite graph with vertex partition $V = (A \cup B)$, where $|A| = 5$, $|B| = 10$, and each set of three vertices in A is adjacent to a distinct vertex in B. Yanhaona *et al.* [13] proved that H is not a PCG. Specifically, they showed that for every pairwise compatibility tree T and a set L of five leaves in T, the following property holds in the corresponding pairwise compatibility graph $G = PCG(T, d_{min}, d_{max})$.

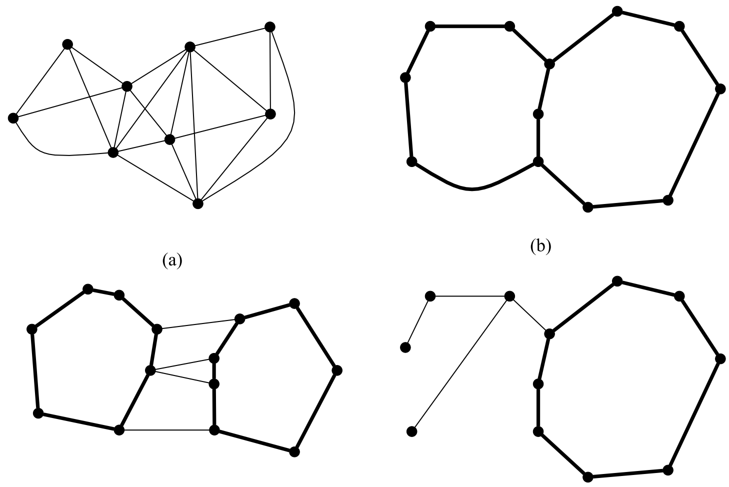

Figure 5: Illustration for open problems.

> *Neighborhood Property:* There exists a set $Q \subset L$ of three vertices in G such that any vertex $u \notin L$, which is adjacent to all the vertices of Q, must be adjacent to at least one of the vertices in $L \setminus Q$.

Now consider the graph H. Observe that for every set Q of three vertices in $A(= L)$, there exists a vertex $u \in B$, which is adjacent to all the vertices in Q, but not to the vertices of $A \setminus Q$. Therefore, the Neighborhood Property is violated for some choice of Q and u. Consequently, the graph H is not a PCG.

An interesting consequence of the above proof is the following. Any graph that contains H as a subgraph, but does not introduce any new edge joining a vertex in A to a vertex in B, is not a PCG. For example, one can insert some edges with both end vertices in A, and similarly, some edges with both end vertices in B, but this also yields a graph which is not a PCG. Specifically, let $\mathcal{H}$ be the class of graphs that includes all the graphs obtained from H by inserting edges in the same set of H. Then none of the graphs in $\mathcal{H}$ is a PCG.

A Negative Example for $\mathcal{G}^1$: Construct a graph H_1 from H by inserting edges in the set A such that the vertices in A form a clique of five vertices. Figure 6 shows an example of H_1. It is straightforward to observe that $H_1 \in \mathcal{H}$, and hence it is not a PCG. We now claim that H_1^c does not contain any induced chordless cycle, i.e., $H_1 \in \mathcal{G}^1$.

Suppose for a contradiction that H_1^c contains an induced chordless cycle $C = (v_1, v_2, v_3, \ldots, v_k)$. Since the set B forms a complete graph in H_1^c, the vertices of C cannot all belong to the set B. Without loss of generality assume that v_2 belongs to the set A. Since the vertices in A form an independent set in H_1^c, both v_1 and v_3 must belong to B. Since the set B forms a complete graph in H_1^c, the edge (v_1, v_3) must be a chord in C, which contradicts that C is an

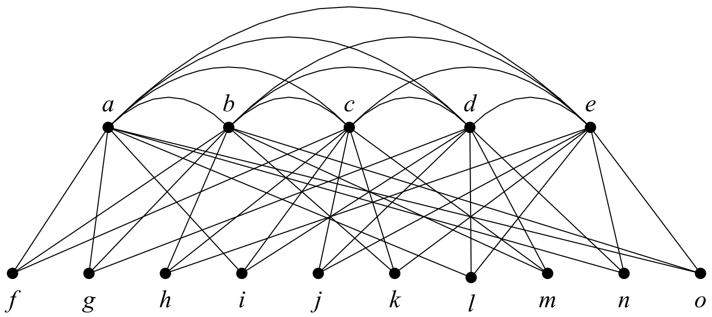

Figure 6: An example of H_1 whose complement is in $\mathcal{G}^1$.

induced chordless cycle in H_1^c.

A Negative Example for $\mathcal{G}^3$: Construct now a graph H_3 from H by inserting edges in set A (respectively, B) such that the vertices in A (respectively, B) form a complement of a C_5 (respectively, C_{10}). Figure 6 shows an example of H_3. It is straightforward to observe that $H_3 \in \mathcal{H}$, and hence it is not a PCG. In the following we verify that H_3^c consists of two induced chordless cycles and some edges that are incident to both cycles, i.e., $H_3 \in \mathcal{G}^3$.

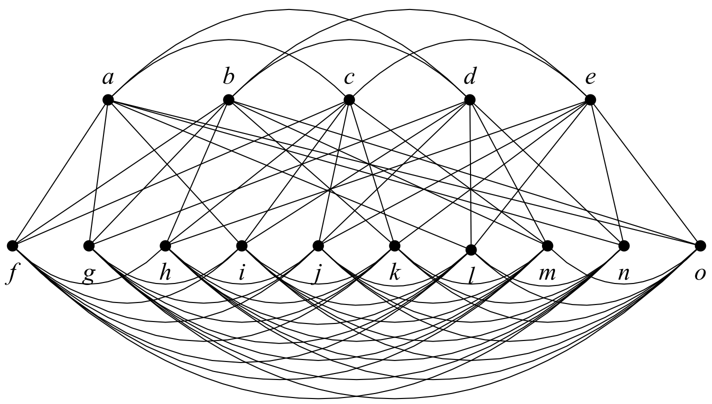

Figure 7: An example of H_3 whose complement is in $\mathcal{G}^3$.

Since A forms a complement of C_5 in H_3, the vertices in A form an induced cycle of five vertices in H_3^c. Similarly, since B forms a complement of C_{10} in H_3, the vertices in B form an induced cycle of ten vertices in H_3^c. Finally, since H is not a complete bipartite graph, and since H_3 does not introduce any new edge joining a vertex in A to a vertex in B, H_3^c must have some edge (u, v), where $u \in A$ and $v \in B$. □

6 Conclusion

In this paper we have given a necessary condition and a sufficient condition for a pairwise compatibility graph. We have shown that if the complement of a given graph G contains two disjoint chordless cycles or two disjoint complements of cycles then G is not a PCG. On the other hand, if the complement of G do not have any cycle then G is a PCG.

We have proved that some graphs lying in the gap between our two conditions are not PCGs. Below are the two interesting graph classes that lie in the gap, but for which no negative example is known.

$\mathcal{G}^2$: A graph G belongs to $\mathcal{G}^2$ if G^c consists of two induced chordless cycles, where the cycles share some common vertices, e.g., see Fig. 5(b).

$\mathcal{G}^4$: A graph G belongs to $\mathcal{G}^4$ if G^c contains only one induced chordless cycle, e.g., see Fig. 5(d).

It will be interesting to examine whether every graph that belongs to $\mathcal{G}^2$ or $\mathcal{G}^4$ is a PCG. Another interesting direction for future research would be to examine the computational complexity of PCG recognition in general.

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