Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 21, no. 3, pp. 281–312 (2017)
DOI: 10.7155/jgaa.00417

On Aligned Bar 1-Visibility Graphs

Franz J. Brandenburg [1] *Alexander Esch* [1] *Daniel Neuwirth* [1]

[1]University of Passau, 94030 Passau, Germany

Abstract

A graph is called a bar 1-visibility graph if its vertices can be represented as horizontal segments, called bars, and each edge corresponds to a vertical line of sight which can traverse another bar. If all bars are aligned at one side, then the graph is an *aligned bar 1-visibility graph*, $AB1V$ graph for short.

We consider $AB1V$ graphs from different angles. First, we study combinatorial properties and K_5 subgraphs. Then, we establish a difference between maximal and optimal $AB1V$ graphs, where optimal $AB1V$ graphs have the maximum of $4n-10$ edges. We show that optimal $AB1V$ graphs can be recognized in $\mathcal{O}(n^2)$ time and prove that an $AB1V$ representation is determined by an ordering of the bars either from left to right or by length. Finally, we introduce a new operation, called path-addition, that admits the addition of vertex-disjoint paths to a given graph and show that $AB1V$ graphs are closed under path-addition. This distinguishes $AB1V$ graphs from other classes of graphs. In particular, we explore the relationship to other classes of beyond-planar graphs and show that every outer 1-planar graph is an $AB1V$ graph, whereas $AB1V$ graphs are incomparable, e.g., to planar, k-planar, outer fan-planar, outer fan-crossing free, fan-crossing free, bar $(1, j)$-visibility, and RAC graphs.

Submitted:	Reviewed:	Revised:	Reviewed:	Revised:
May 2016	November 2016	December 2016	December 2016	December 2016
	Accepted:	Final:	Published:	
	January 2017	January 2017	February 2017	
	Article type:	Communicated by:		
	Regular paper	M. Kaykobad and R. Petreschi		

Supported by the Deutsche Forschungsgemeinschaft (DFG), grant Br835/18-2. An extended abstract of this paper has been presented at WALCOM 2016 [11].
E-mail addresses: brandenb@informatik.uni-passau.de (Franz J. Brandenburg)

1 Introduction

There is recent interest in beyond-planar graphs, which comprise classes of graphs that extend the planar graphs and are defined by restrictions on crossings. Particular examples are 1-planar graphs [33, 32], fan-planar [6, 5] and fan-crossing free graphs [16], quasi-planar graphs [2], right angle crossing (RAC) graphs [20], bar visibility graphs [18], bar $(1, j)$-visibility graphs [13], rectangle visibility graphs [27], and map graphs [14, 38]. Besides, there are specializations, such as outer 1-planar graphs [4, 26], outer fan-planar graphs [5], outer fan-crossing free graphs, and $AB1V$ graphs. The latter were introduced by Felsner and Massow [25] who called them semi bar 1-visibility graphs, since the bars are semi transparent. However, the alignment of the bars is the typical feature of $AB1V$ representations.

Visibility is a major topic in computational geometry [31] and graph drawing [19]. A *bar visibility representation* displays each vertex as a horizontal bar and each edge as a vertical line of sight between the bars of the endvertices, which, in case of k-visibility, traverses up to k other bars. Bars are non-intersecting segments. There are several versions of visibility including *distinct*, *strong* and *weak*. In the distinct case, the endpoints of the bars must have different x-coordinates [29, 25]. In the distinct and strong versions there is an edge if and only if there is a visibility, whereas in the weak version there is a visibility if there is an edge. Thus edges can be omitted. Clearly, graphs in the weak version are exactly the subgraphs of graphs in the other versions. This assumption is relevant for a comparison with other classes of graphs which are generally closed under taking subgraphs.

Every weak visibility graph is planar and vice versa. Hence, weak visibility graphs can be recognized in linear time. Some planar graphs do not have a strong visibility representation, including $K_{2,3}$ [18] and some 3-connected graphs [3]. The recognition problem for strong visibility graphs is $\mathcal{NP}$-complete [3] and there is a characterization of strong visibility graphs [34, 37, 39].

In the non-planar case with k-visibility, a line of sight for an edge can traverse up to k other bars. Simply speaking, an edge can cross up to k vertices. The planar case corresponds to $k = 0$. Bar 1-visibility graphs were introduced by Dean et al. [18] and further investigated by Sultana et al. [35] and by Evans et al. [22], who also compared them with other classes of beyond-planar graphs and considered the strong and weak versions of 1-visibility. Bar 1-visibility graphs are specialized to bar $(1, j)$-visibility [13] and bar $(1, 1)$-visibility graphs [9] by restricting the number of edges that may traverse a bar to $j > 0$ and $j = 1$, respectively. In particular, bar 1-visibility graphs of size n have up to $6n - 20$ edges and an $\mathcal{NP}$-hard recognition problem. However, 1-planar graphs are a proper subclass of bar $(1, 1)$-visibility [9] and bar 1-visibility graphs [21]. Hence, in contrast to the planar case, bar 1-visibility is stronger than 1-planarity.

An outer planar graph has a drawing with all vertices in the outer face. In particular, all vertices can be placed on a line with edges as circular arcs above this line. Outer planar graphs are an important subclass of the planar graphs. Each outer planar graph has at least one vertex of degree at most two. The graphs have book-thickness one and treewidth two, and K_4 and $K_{2,3}$ are the forbidden minors. The restriction from planar to outer planar transfers a general to an aligned bar visibility representation, where all bars start at a common line, e.g., at the bottom or at the left. In the planar case, outer planar and weak aligned bar visibility graphs coincide [17], whereas aligned bar 1-visibility is stronger than outer 1-planarity, as we shall show.

Aligned bar 1-visibility representations and graphs were introduced by Felsner and Massow [25] who used the distinct version of visibility and allow a line of sight to traverse up to k bars. A distinct $AB1V$ representation, $dAB1V$ for short, is characterized by two permutations, the t- and r-orders. The *t-order* is the left-to-right (or top-down) ordering of the bars of the vertices and the *r-order* is an ordering of the bars by length. Felsner and Massow established important properties of $dAB1V$ graphs. For example, they showed that every $dAB1V$ graph has a vertex of degree four and has at most $4n - 10$ edges. This bound is tight for all $n \geq 4$. Moreover, they established that $AB1V$ graphs are 5-colorable, have clique number five and geometric thickness two, and they showed that an r-order can be computed from a t-order of a $dAB1V$ graph.

In this work, we extend the research on $AB1V$ graphs. There are two main features in $AB1V$ graphs: clusters and paths. A *cluster* is a K_5 subgraph, i.e., the maximum complete $AB1V$ graph. First, we show that every cluster has a special $AB1V$ representation and can uniquely be associated with a vertex. Thereafter, we prove that maximal $AB1V$ graphs have at least $3.5n - 9$ edges and that there are sparse maximal $AB1V$ graphs with $3.5n - 1$ edges for every $n \geq 21$. Then we establish a quadratic-time recognition algorithm for optimal $AB1V$ graphs. Complementary to a result by Felsner and Massow, we show that the t-order can be computed from the r-order of a $dAB1V$ graph in linear time. Given an $AB1V$ representation, one can easily introduce a vertex-disjoint path between two vertices. This observation has led to the introduction of path-addition, which is a new operation on graphs and is further studied in [12]. Finally, we explore the relationship of weak $AB1V$ and other classes of beyond-planar graphs. First, there is a proper hierarchy for the versions of visibility and $AB1V$ graphs. Then we show that every outer 1-planar graph is an $AB1V$ graph but not conversely. An $AB1V$ representation of an outer 1-planar graph can be constructed in linear time. However, $AB1V$ graphs are incomparable, e.g., to planar, k-planar, outer fan-planar, outer fan-crossing free, fan-crossing free, bar $(1, j)$-visibility, and RAC graphs.

The paper is organized as follows. In Section 2 we introduce basic concepts. Structural properties and maximal graphs are studied in Section 3 and path-additions in Section 4. In Section 5 we investigate recognition problems. The relationship to other graph classes is discussed in Section 6 and we conclude with some open problems.

2 Preliminaries

We consider simple, undirected graphs $G = (V, E)$ with vertices listed in some arbitrary order. Let $N(v)$ denote the set of neighbors of a vertex v including v and $G[U]$ the subgraph induced by $U \subseteq V$.

Graphs are defined by *aligned bar 1-visibility representations*, $AB1V$ for short. As suggested by Felsner and Massow [25], we rotate the drawings, which, due to the alignment, are more intuitive and compact than ordinary visibility representations. Vertical bars were also used by Wismath [39]. In an $AB1V$ representation, each vertex is represented by a vertical bar, which is a closed interval with bottom at $y = 0$ and top at some $y > 0$. Each edge $e = (u, v)$ corresponds to a horizontal line of sight between the bars of u and v, which can traverse another bar. Thus, there is a *1-visibility* between the bars of u and v.

We distinguish between *distinct*, *strong* and *weak* visibility [19] and denote the respective classes of aligned bar 1-visibility graphs by $dAB1V$, $sAB1V$ and $wAB1V$, respectively. In addition, we use $mAB1V$ for *maximal* $AB1V$ graphs, which cannot be extended by adding a further edge, and $oAB1V$ for *optimal* $AB1V$ graphs with the maximum of $4n - 10$ edges for graphs of size n. Recall that there is an edge if and only if there is a visibility in the distinct and strong versions, where in the distinct case all bars have a different length. The weak version admits the omission of edges.

A *partial* $AB1V$ representation is the $AB1V$ representation of an induced subgraph $G[U]$. An *extension* of a partial $AB1V$ representation is an $AB1V$ representation of $G[U \cup W]$ for sets of vertices U and W so that the restriction to U is the $AB1V$ representation of $G[U]$.

From Felsner and Massow [25] we adopt the t- and r-orders for the description of $AB1V$ representations, which are the orderings of the bars from left to right and by length, respectively. Then a partial $AB1V$ representation is the restriction of the t- and r-orders to the vertices of U. For convenience, we say that vertex u is left of vertex v if the left to right ordering holds for their bars in an $AB1V$ representation. Accordingly, we say that a vertex is between other vertices. Note that each $AB1V$ representation has a reflection with an ordering from right to left representing the same graph.

We denote the t- and r-orders of two vertices by $u <_t v$ and $u <_r v$, respectively, and we extend these relations to sets of vertices and *bucket orders* [24]. For disjoint sets of vertices X and Y we write $X <_t Y$ if $x <_t y$ for every $x \in X$ and $y \in Y$, however, the ordering of the vertices of X and Y is still unclear or may be exchanged. The sets X and Y are called buckets. A bucket order is extended to an order by ordering the elements in each bucket [24]. The notation $X <_t Y$ is also used for induced subgraphs, and $X <_r Y$ is used accordingly. For convenience, we omit braces for singleton sets and inside brackets.

Aligned visibility representations were introduced by Cobos et al. [17]. They considered the planar case with non-transparent bars and implicitly use the t- and r-orders by assigning an n-tuple to an aligned bar visibility representation of a graph of size n, where the i^{th} entry is the length of the i^{th} bar from the left.

In general, we shall consider distinct $AB1V$ representations with all bars of

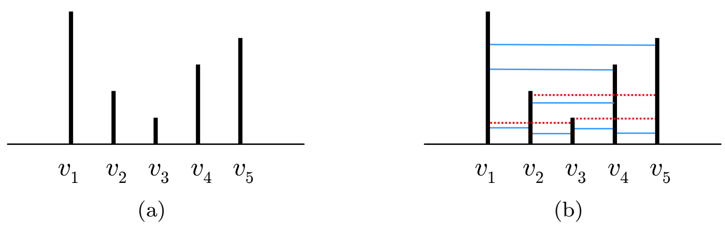

Figure 1: $AB1V$ representation of a K_5 with planar and 1-visibility lines of sight drawn as bold and dotted horizontal lines in (b)

different length, that are represented by the t- and r-orders. For convenience, we omit the prefix d and explicitly refer to the other versions of visibility if this is relevant. For a comparison with other classes of graphs we use $wAB1V$ graphs, since they are closed under taking subgraphs.

3 Combinatorial Properties

In their introductory paper on $AB1V$ graphs, Felsner and Massow [25] observed that each $AB1V$ graph has a vertex of degree at most four and that the clique number is five, i.e., K_5 is the largest complete $AB1V$ (sub-) graph. We call a K_5-subgraph a *cluster*. In fact, clusters and their $AB1V$ representation play a prominent role in $AB1V$ graphs.

3.1 Clusters

Forthcoming, we shall often consider K_5-subgraphs and subgraphs that are induced by vertices with long bars in an $AB1V$ representation, i.e., we consider the top-k vertices in r-order and a 1-visibility above a certain level. Two long bars prevent a 1-visibility between two short bars to their left and right.

Definition 1 *A subgraph $G[X]$ of an $AB1V$ graph G is called a* cluster *if $G[X] = K_5$. Let $\mathcal{C}(G)$ denote the set of clusters of G. The w-*clustered graph *$\mathcal{CG}_w(G) = (\mathcal{C}(G), F_w)$ of G has a vertex for each cluster of G and there is an edge between two clusters $G[X]$ and $G[Y]$ if and only if $|X \cap Y| \geq w$, i.e., $G[X]$ and $G[Y]$ have at least w vertices in common, where $1 \leq w \leq 4$.*

For $AB1V$ graphs the following is immediate:

Lemma 1 *A subgraph $G[v_1, \dots, v_5]$ of an $AB1V$ graph G is a cluster if and only if $v_1 <_t \dots <_t v_5$ in t-order implies $v_3 <_r \{v_1, v_2, v_4, v_5\}$, v_2 or v_4 has the second shortest bar, and there is no other vertex u of G with $v_1 <_t u <_t v_5$ and $v_3 <_r u$.*

An $AB1V$ representation of K_5 is depicted in Fig. 1.

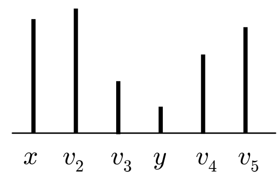

Figure 2: An $AB1V$ representation of two clusters with four vertices in common

The bars of the vertices of a cluster C are $\mathcal{U}$-shaped with the shortest bar in the middle and the second shortest bar next to it. The top-3 vertices of C in r-order can be permuted and the longest bar may be next to the bar in the middle. This is excluded in a V-shape. Hence, a K_5 has six $AB1V$ representations up to reflection, some of which may be excluded if more vertices are taken into account. The left to right order it tight in the sense that no other vertex u can be placed between the vertices of C if the bar of u is longer than the shortest bar of the vertices of C. In reverse, if u must be placed between the vertices of a cluster, then u has a short bar. In particular, if u is a neighbor of the vertex in the middle of C, then u is placed between the vertices of C and has a bar that is shorter than the bars of the vertices of C.

Lemma 2 *If $G[x, v_2, v_3, v_4, v_5]$ and $G[y, v_2, v_3, v_4, v_5]$ are two clusters of an $AB1V$ graph G which differ in one vertex, namely x and y, then in any $AB1V$ representation, either x is extreme in t-order and y is minimum in r-order, or conversely, see Fig. 2.*

Proof: Suppose that $v_2 < v_3 < v_4 < v_5$ in t-order. Then $v_2 <_t \{x, y\} <_t v_5$ implies that one of $G[x, v_2, v_3, v_4, v_5]$ and $G[y, v_2, v_3, v_4, v_5]$ is not a cluster by Lemma 1, since the longer bar of x and y is between the bars of the vertices of the other cluster. If x is to the left of v_2, v_3, v_4, v_5, then $v_3 <_r \{x, v_2, v_4, v_5\}$ by Lemma 1 and $v_3 <_r y$ implies that there is no 1-visibility between v_3 and an extreme vertex. The case with x at the right is similar. □

A cluster C may be associated with its vertex with the shortest bar, which may be taken as a representative of C.

Lemma 3 *For an $AB1V$ graph $G = (V, E)$, there is a one-to-one mapping $\kappa : \mathcal{C}(G) \to V$ assigning each cluster to a vertex.*

Proof: Given an $AB1V$ representation, let $\kappa(G[X]) = v_3$ if $G[X]$ is a cluster with vertices $v_1 <_t \ldots <_t v_5$ in t-order. By Lemma 1, the bar of v_3 is the shortest of the bars of the vertices of X and v_3 is placed in the middle of the bars of the vertices of X. If two (or more) clusters X and Y were assigned to v_3, then a long bar from a vertex of Y must be placed between the bars of X or vice versa, contradicting Lemma 1. □

The assignment of clusters to vertices implies an upper bound on the number of clusters and all clusters can be computed in linear time from a given $AB1V$

representation. Note that a subgraph induced by a vertex v and four neighbors of v with longer bars than v do not necessarily induce a cluster. The condition on the vertex with the second shortest bar from Lemma 1 must be satisfied.

Corollary 1 *An $AB1V$ graph of size n has at most $n-4$ clusters, which, given an $AB1V$ representation, can be listed in linear time.*

Proof: A cluster can be assigned to all but the two vertices at the left and right ends in an $AB1V$ representation, such that $n-4$ vertices remain. The bound is achieved if the $AB1V$ representation has a V-shape with a monotone t-order and a bitonic r-order.

For the computation, consider the vertices in decreasing r-order and check that $\kappa^{-1}(v_i)$ is a cluster. □

We can also consider clusters in decreasing r-order of their representatives.

Lemma 4 *Let $v_1 <_r \ldots <_r v_n$ be the r-order of the vertices of an $AB1V$ graph. For $i = 1, \ldots, n-4$, there is a cluster $\kappa^{-1}(v_i)$ if and only if v_i has four neighbors with longer bars w_i, x_i, y_i, z_i in $G[v_{i+1}, \ldots, v_n]$ which induce K_4, i.e., if $w_i <_r \{x_i, y_i, z_i\}$, then there is no vertex v_j with $i+1 \leq j \leq n$ and $w_i <_t v_j <_t v_i$ or $v_i <_t v_j <_t w_i$.*

Proof: The "only-if" direction follows from Lemma 1 and for the "if-direction" observe that a vertex v_j as specified hinders w_i or v_i to be 1-visible from x_i, y_i, and z_i. □

Once we know the clusters, we would like to know how they are interrelated. This is expressed by the clustered graphs $\mathcal{CG}_w(G)$ that are parameterized by the number of common vertices.

A *stripe* is an outer planar graph with vertices of degree at most four that consists of two parallel paths with spokes as depicted in Fig. 3(b).

Lemma 5 *The* 4-clustered graph $\mathcal{CG}_4(G)$ *of an $AB1V$ graph G consists of a set of paths and the* 3-clustered graph $\mathcal{CG}_3(G)$ *is a subgraph of a stripe.*

Proof: Consider a cluster $\kappa^{-1}(x_1) = G[X]$ with vertices $x_1 <_r \ldots <_r x_5$ in r-order. If there is a cluster $G[Y]$ with four vertices in common with X and a further vertex $x_6 \in Y - X$ with a longer bar than the bar of x_1, then Lemma 2 implies that x_6 must be placed to the left or to the right of the vertices of X, $x_2 <_r x_6$ in r-order, $Y = \{x_2, \ldots, x_6\}$, and $\kappa(G[Y]) = x_2$. Hence, there is an edge in $\mathcal{CG}_4(G)$ from $\kappa^{-1}(x_1)$ to $\kappa^{-1}(x_2)$, where x_2 is the neighbor of x_1 with the next longer bar. If $x_6 <_r x_1$, then $\kappa(G[Y]) = x_6$, and the roles of X and Y are exchanged. Thus, there may be an (incoming) edge to $G[X]$ from a cluster whose representative has a shorter bar. Hence, a vertex of $\mathcal{CG}_4(G)$ has degree at most two. Cycles are excluded by the increasing length of the bars of the assigned vertices. A path is interrupted at $G[X]$ if there is no cluster that is assigned to vertex y, where y has the second longest bar of the vertices of X.

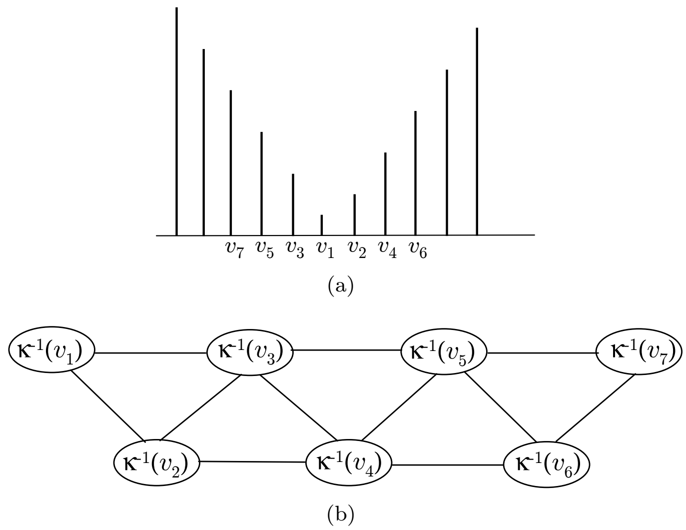

Figure 3: A V-shaped $AB1V$ representation of a graph G and the 3-clustered graph $\mathcal{CG}_3(G)$

Accordingly, if clusters $G[X]$ with $X = \{x_1, \ldots, x_5\}$ and $x_1 <_r \{x_2, \ldots, x_5\}$ and $G[Y]$ have at least three common vertices, and the new vertices of $Y - X$ have longer bars than those of X, then $Y = \{x_2, \ldots, x_6\}$ for some x_6 as stated in Lemma 2 or $Y = \{x_3, x_4, x_5, y_1, y_2\}$, $x_3 <_r \{y_1, y_2\}$, y_1 and y_2 are placed to the left or to the right of the vertices of X, and $X \cap Y = \{x_3, x_4, x_5\}$. Hence, there are at most two (outgoing) edges in $\mathcal{CG}_3(G)$ from a cluster $\kappa^{-1}(x_1)$ to the clusters of the two next neighbors of x_1 in r-order $\kappa^{-1}(x_2)$ and $\kappa^{-1}(x_3)$, provided these clusters exist. If they do, then there is an edge from $\kappa^{-1}(x_1)$ to $\kappa^{-1}(x_2)$ and to $\kappa^{-1}(x_3)$ in $\mathcal{CG}_3(G)$. Accordingly, there are edges in $\mathcal{CG}_3(G)$ from clusters with representatives with shorter bars.

If $x_1 <_r \ldots <_r x_n$ is the r-order of the vertices of G, then there are edges $(\kappa^{-1}(x_i), \kappa^{-1}(x_{i+j}))$ for $j = -2, -1, 1, 2$ in $\mathcal{CG}_3(G)$, provided there are clusters. Hence, $\mathcal{CG}_3(G)$ is a subgraph of a stripe, which is an outer planar graph as depicted in Fig. 3(b). □

If the $AB1V$ representation of G has a monotone t-order and a bitonic r-order, then $\mathcal{CG}_4(G)$ and $\mathcal{CG}_3(G)$ have $n-4$ vertices and are a path and a stripe, respectively, see Fig. 3. On the contrary, there are $AB1V$ graphs such that the clustered graphs $\mathcal{CG}_1(G)$ and $\mathcal{CG}_2(G)$ are complete graphs on $n-4$ vertices. If $v_1 <_t \ldots <_t v_n$ and $v_3 <_r \ldots <_r v_n <_r v_2 <_r v_1$ are the t- and r-orders, then there are clusters $\kappa^{-1}(v_i)$ for $i = 3, \ldots, n-2$ which each contain the two leftmost vertices with the longest bars v_1 and v_2, and therefore are mutually connected in the clustered graphs.

Finally, we consider disjoint clusters. Two clusters $G[X]$ and $G[Y]$ of an $AB1V$ graph G are *disjoint* if in any $AB1V$ representation of G the vertices of X

are to the left of the vertices of Y, i.e., $X <_t Y$, or vice versa. We say that $G[Y]$ *nests* in $G[X]$ if there are two vertices x_1 and x_2 of X with $x_1 <_t Y <_t x_2$. Thus (the bars of) the vertices of Y are placed between x_1 and x_2 and, by Lemma 1, x_1 and x_2 can be chosen so that no other vertex of X is placed between them.

Lemma 6 *If $G[X]$ and $G[Y]$ are two vertex disjoint clusters of an $AB1V$ graph G, then in any $AB1V$ representation, $G[X]$ and $G[Y]$ are disjoint or nest, and the bars of the nesting cluster are shorter than the bars of the other cluster.*

Proof: Otherwise, there are vertices $x_1 < y_1 < x_2 < y_2$ in t-order with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, which contradicts Lemma 1. □

3.2 Maximality

Next, we consider extremal graphs and Turán-type theorems on the maximum number of edges. An $AB1V$ graph G is *maximal* if no further edge can be added without violating aligned bar 1-visibility, and G is *optimal* if it has $4n-10$ edges. Hence, a graph G is maximal if there is no $AB1V$ supergraph with the same set of vertices and more edges, and G is optimal if there is no $AB1V$ graph of the same size and more edges. These notions coincide for planar graphs and maximal and optimal planar graphs of size n have exactly $3n-6$ edges and are triangulated. However, for outer 1-planar [4], 1-planar [8], and $(1, j)$-visibility graphs [13], there are maximal graphs that are not optimal. Surprisingly, there are maximal 1-planar graphs with less than $2.65\,n$ edges.

Optimal $AB1V$ graphs were characterized by Felsner and Massow [25], who proved that a distinct $AB1V$ representation results in an optimal graph if and only if the top-4 vertices in r-order are placed in pairs at the left and right ends and induce K_4. In other words, the four longest bars are placed at the left and right ends and the shortest of them is not the extreme left or right. Thereafter, any other vertex has at least four neighbors and there are exactly four new neighbors if the vertices are taken in decreasing r-order. In consequence, $AB1V$ graphs have a density of at most $4n-10$ [25], where the *density* is a function of the maximal number of edges of graphs of size n. However, there are sparser maximal $AB1V$ graphs, and there are maximal $AB1V$ representations with graphs with only $2n-3$ edges, e.g., if the bars are placed from left to right in increasing (or decreasing) r-order.

We aim at $AB1V$ graphs with a unique $AB1V$ representation. However, this is impossible, since the t-order can be reversed and the vertices with the three longest bars can be permuted in r-order.

Lemma 7 *There is an $AB1V$ graph B with a unique t-order of the vertices (up to reflection), and there are $AB1V$ graphs G_k for $k \geq 1$ where the t-order is a unique bucket order with buckets of size two (up to reflection).*

Proof: Graph B has 15 vertices from the sets $V = \{v_1, \ldots, v_{10}\}$ and $U = \{u_1, \ldots, u_5\}$. For $k \geq 1$, graph G_k extends B by adding vertices from $k+2$

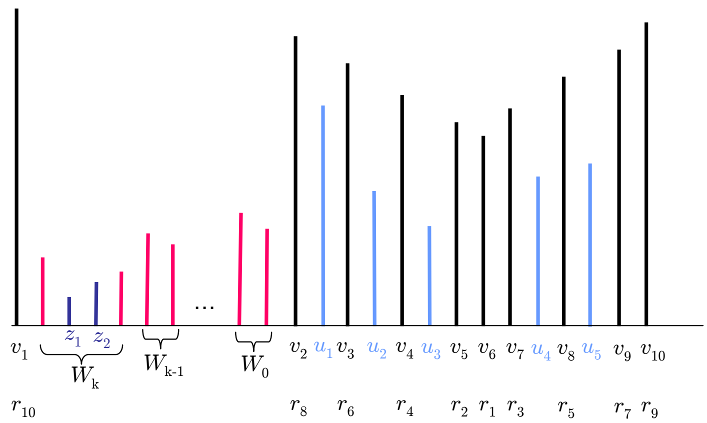

Figure 4: An $AB1V$ graph G_k consisting of six clusters with vertices $v_1, \ldots, v_{10}$, two paths with vertices u_1, u_2, u_3 and u_4, u_5 and supplementary edges, and a $k+1$ pair sets (buckets) $W_0, \ldots, W_k$. An r-order of the vertices of the clusters is $r_1 <_r \ldots <_r r_{10}$.

pair sets $Z = \{z_1, z_2\}$ and $W_i = \{w_i, w_i'\}$ for $i = 0, \ldots, k$. The sets V, U, Z and $W_0, \ldots, W_k$ are pairwise disjoint. For $i = 0, \ldots, k$, the vertices of W_i are added incrementally and are placed immediately to the right of v_1 if v_1 appears at the left. The vertices w_i and w_i' of W_i cannot be distinguished and can be swapped both with respect to their length and with respect to their position, except for $i = k-1$. Hence, there is a bucket order $W_k < \ldots < W_0$, both for the t-order (up to reflection) and the r-order. Both vertices of Z are placed one bar apart from the bar of v_1.

An $AB1V$ representation of G_k is given in Fig. 4.

The subgraph $B[V]$ consists of a sequence $C_1, \ldots, C_6$ of six clusters with the sets of vertices $\{v_1, v_2, v_3, v_9, v_{10}\}$, $\{v_2, v_3, v_8, v_9, v_{10}\}$, $\{v_2, v_3, v_4, v_8, v_9\}$, $\{v_3, v_4, v_7, v_8, v_9\}$, $\{v_3, v_4, v_5, v_7, v_8\}$, and $\{v_4, v_5, v_6, v_7, v_8\}$, respectively. The first and the last clusters are disjoint, and two adjacent clusters C_i and C_{i+1} with $1 \leq i < 6$ have four common vertices. Note that $C_1, \ldots, C_6$ forms a path-decomposition of width four of $B[V]$ [7]. This graph shall be used later (Lemma 11) as an example of a non-fan-crossing free graph.

The subgraph $B[U]$ adds two paths $(v_1, u_1, u_2, u_3, v_6)$ and (v_6, u_4, u_5, u_{10}) from the central vertex v_6 to the extreme ones. For a distinct $AB1V$ representation, we add the edges $(u_1, v_1), (u_1, v_2), (u_1, v_3)$, (u_1, u_4), (u_2, u_1), (u_2, v_3), (u_2, v_4), (u_2, v_5), (u_3, u_2), (u_3, v_4), (u_3, v_5), (u_3, v_6), (u_4, v_6), (u_4, v_7), (u_4, v_8), (u_4, u_5), (u_5, v_7), (u_5, v_8), (u_5, v_9), (u_5, v_{10}). The edges can be retrieved from the $AB1V$ representation in Fig. 4. In total, the base B has 15 vertices and 50 edges, and therefore is an optimal $AB1V$ graph.

There is an edge between the vertices w_0 and w_0' of W_0 and both have v_1, v_2 and u_1 as neighbors. For $i = 1, \ldots, k$ there is an edge between the vertices of W_i and both have v_1 and the vertices of W_{i-1} as neighbors. Finally, each vertex $z_1, z_2 \in Z$ adds four edges, namely $(z_1, v_1), (z_1, w_k), (z_1, w'_k), (z_1, z_2)$ and $(z_2, v_1), (z_2, w_k), (z_2, w'_k), (z_2, w_{k-1})$. The edge (z_2, w_{k-1}) distinguishes z_1 from z_2 and w_{k-1} from w'_{k-1}.

The r-order of the vertices of V in Fig. 4 is $v_1 > v_{10} > v_2 > v_9 > v_3 > v_8 > v_4 > v_7 > v_5 > v_6$ or $r_1 <_r \ldots <_r r_{10}$. Moreover, $u_3 <_r u_2 <_r u_1$, $u_4 <_r u_5$ and $\{u_3, u_4\} <_r v_6$, $u_1 <_r v_4, u_2 <_r v_5$ and $u_5 <_r v_7$. The r-order of B is not unique, since, e.g., the length of the top-4 bars can be exchanged and the length of the bars of $\{u_1, u_2, u_3\}$ and $\{u_4, u_5\}$ are not related. Apart from this, B has the r-order from above. In addition, for G_k we have $W_i <_r W_{i-1}$ for $i = 1, \ldots, k$ and $W_0 <_r u_1$, as we shall show. However, the vertices of W_i can be exchanged, except for $i = k - 1$.

Obviously, B and G_k with $k \geq 0$ are $dAB1V$ graphs. The given visibility representation of G_k is maximal, i.e., all edges are listed. The base $B = G_k[V \cup U]$ is optimal and each set W_i for $i = 0, \ldots, k$ adds two vertices and seven edges, since vertex w_i has only vertex v_1 as a left neighbor with a bar that is longer than the one of w_i for $0 = 1, \ldots, k$. Hence, one edge to a left neighbor is missing. Finally, Z adds two vertices and eight edges.

It remains to show that the t-order of B and of G_k given in Fig. 4 is unique up to reflection and an exchange of w_i and w'_i. Therefore, we use the properties of $AB1V$ representations of clusters and prove four claims.

Claim 1. *The vertices of V admit two t-orders with a bucket, namely,* $v_6 <_r v_5 <_r v_7 <_r v_4 <_r v_8 <_r \{v_1, v_2, v_3, v_9, v_{10}\}$ *or* $v_1 <_r v_{10} <_r v_2 <_r v_9 <_r v_3 <_r \{v_4, v_5, v_6, v_7, v_8\}$.

Proof: The clusters $C_1 = B[v_1, v_2, v_3, v_9, v_{10}]$ and $C_{10} = B[v_2, v_3, v_8, v_9, v_{10}]$ have four vertices in common. Then either v_1 or v_8 has the shortest bar among these vertices by Lemma 2. First, assume $v_8 <_r v_1$. Then v_8 is in the middle of v_2, v_3, v_9, v_{10} and $v_8 <_r \{v_1, v_2, v_3, v_9, v_{10}\}$, i.e., the bar of v_8 is short. Since v_4 is a neighbor of v_8, it must be placed close to v_8 and it is placed between the extreme vertices of v_2, v_3, v_9, v_{10}. Then the bar of v_4 is shorter than the bar of v_8 by Lemma 1. By the same reasoning, the bars of v_5, v_6 and v_7 are shorter than the bar of v_8 and are placed between the extreme bars of v_2, v_3, v_9, v_{10}. Next, consider cluster $C_3 = B[v_2, v_3, v_4, v_8, v_9]$. Since v_4 has the shortest bar among these vertices, it is placed in the middle of them by Lemma 1. As before, the bar of v_7 (and also the bars of v_5 an v_6) must be shorter than the bar of v_4. By the same reasoning, we obtain $v_5 <_r v_7$ and $v_6 <_r v_5$ from C_4, C_5 and C_6. Hence, assuming $v_8 <_r v_1$ implies the r-order $v_6 <_r v_5 <_r v_7 <_r v_4 <_r v_8 <_r \{v_1, v_2, v_3, v_9, v_{10}\}$.

Otherwise, $v_1 <_r v_8$ implies the r-order $v_1 <_r v_{10} <_r v_2 <_r v_9 <_r v_3 <_r \{v_4, v_5, v_6, v_7, v_8\}$. This is shown as follows: For the clusters C_1 and C_2 and $v_1 <_r v_8$, Lemma 2 implies that v_1 has the shortest bar and is placed in the middle of v_2, v_3, v_9, v_{10}. Now, the assumption $v_4 <_r v_{10}$ leads to a contradiction. Then v_4 has the shortest bar of the vertices of cluster $B[v_2, v_3, v_4, v_8, v_9]$. Otherwise, if a vertex $v \in \{v_2, v_3, v_8, v_9\}$ has the shortest bar, then the clusters $B[v_2, v_3, v_8, v_9, v_{10}]$ and $B[v_2, v_3, v4, v_8, v_9]$ are both assigned to v, contradicting

Lemma 3. Now, $v_4 <_r \{v_2, v_3, v_8, v_9\}$ implies $v_4 <_r v_1$ by Lemma 1, since v_4 must be placed between the extreme vertices of the cluster $B[v_1, v_2, v_3, v_9, v_{10}]$. In consequence, we have $v_1 <_r v_{10} <_r v_4$. We can now proceed as before, and from the sequence of clusters $C_1, \ldots, C_6$ we obtain the r-order as claimed. □

Next, we determine the t-order of B.

Claim 2. *The r-order $v_1 <_r v_{10} <_r v_2 <_r v_9 <_r v_3 <_r \{v_4, v_5, v_6, v_7, v_8\}$ is infeasible.*

Proof: The clusters C_1 and C_6 are disjoint. Since $v_3 \in C_1$ has a shorter bar than its four neighbors in C_6, v_3 must be placed between the vertices of C_6 and there are at least two vertices of C_6 on either side of v_3. Then C_1 and C_6 nest by Lemmas 2 and 6 and the vertices of C_1 are placed between two vertices v_x and v_y of C_6. Vertex v_{10} has the second longest bar of the vertices of C_1 and therefore it is placed next to the middle bar v_1 in the $\mathcal{U}$-shape of C_1 by Lemma 1. By the edge (v_8, v_{10}) it must be placed close to v_8, so that $v_x = v_8$ and v_8 and v_{10} are on the same side of v_1. Fig. 5 depicts the $AB1V$ representation with v_{10} left of v_1. Otherwise, the t-order is reversed.

Consider u_5 with neighbors v_7, v_8, v_9, v_{10}. Then u_5 must be placed immediately to the left of v_8 and it has a small bar that is shorter than the bar of v_{10}. To see this, first u_5 must be placed between its four neighbors, since v_{10} is not 1-visible from a bar to the left of the bars of C_6 that are long, or to the right of C_1, since C_1 is $\mathcal{U}$-shaped. Hence, u_5 must be placed immediately to the right of v_8. Its bar must be shorter than the bar of v_{10}, since the edge (v_8, v_{10}) intersects the bar of vertex x_1. Then $x_1 = v_9$ and $y_1 = v_7$. Vertex u_4 cannot be placed left of $y_1 = v_7$ because of the edge (u_4, u_5). Hence, u_4 must be placed left of v_8 and its bar is shorter than the bar of u_5 because of the edge (u_5, v_7). Then u_6 must be placed furthest to the left.

By the same reasoning, u_1 must be placed between x_2 and x_3 and its bar is shorter than the one of v_1. It determines $y_2 = x_4$. Similarly, u_2 must be placed right of $y_2 = v_4$ and determines $y_3 = v_5$, and u_3 must be placed right of $y_3 = v_5$ and determines $y_4 = v_6$. However, there is a contradiction, since v_6 must be placed furthest to the left and to the right. □

Claim 3. *The r-order $v_6 <_r v_5 <_r v_7 <_r v_4 <_r v_8 <_r \{v_1, v_2, v_3, v_9, v_{10}\}$ implies the t-order $v_1 <_t \ldots <_t v_{10}$ or its reversal, and the vertices of U are placed between vertices of V as shown in Fig. 4.*

Proof: By the same reasoning as in the proof of Claim 2, cluster C_6 must nest in C_1, since v_8 has four neighbors in C_1 with longer bars, and therefore must be placed between the vertices of C_1. Then all vertices of C_1 are placed between two vertices of C_1 by Lemmas 2 and 6. Vertex v_5 has the second longest bar of the vertices of C_6 and therefore must be placed next to the bar of v_6 and on the same side as v_3. Assume that v_3 and v_5 are to the left of v_6. Otherwise, the t-order is reversed.

As a neighbor of v_5, vertex u_2 must be placed between v_3 and v_5 and its bar must be shorter than the one of v_5 because of the edge (v_3, v_5), which intersects

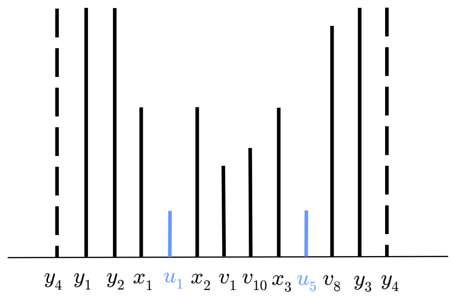

Figure 5: An $AB1V$ representation of $B[V \cup \{u_1, u_5\}]$ with $C_1 <_r C_6$, $\{x_1, x_2, x_3\} = \{v_2, v_2, v_9\}$, and $\{y_1, y_2, y_3, y_4\} = \{v_4, v_5, v_6, v_7\}$. Vertex y_4 can be placed left or right.

the leftmost bar y_1 of C_6. Then $y_1 = v_4$ and u_3 must be placed between v_4 and v_5. As a neighbor of v_4, vertex u_1 cannot be placed to the far left. It is placed left of v_3 and enforces that v_1 and v_2 are to the left. The left to right order $v_1 <_t v_2$ is determined by C_1 and C_2, whose placement by Lemma 2 implies that v_1 is extreme for C_2. Similarly, vertices u_4 and u_5 determine the t-order to the right. The vertices of U are inserted at fixed positions. Hence, the left to right order of the vertices of $V \cup U$ is as shown in Fig. 4, or its reversal. □

Finally, we consider the extension of B to G_k by the insertion of the pair sets of vertices $W_i = \{w_i, w'_i\}$ for $i = 1, \ldots, k$, which are buckets for the t- and r-orders. The ordering of the elements of a bucket is left open.

Claim 4. *If the left to right order of the vertices of $V \cup U$ is as depicted in Fig. 4 with $v_1 <_t v_2 <_t u_1 <_t u_3$ at the left and $u_1 <_r \{v_1, v_2, v_3\}$, then the left to right order is $v_1 <_t W_k <_t \ldots <_t W_0 <_t v_2 <_t u_1$ and $v_1 <_t w_k <_t z_1 <_t z_2 <_t w_{k-1} <_t w'_{k-1}$. The r-order is $W_i <_r W_{i-1}$ for $i = 0, \ldots, k$, $Z <_r W_k$, $z_1 <_r z_2$, $w'_{k-1} <_r w_{k-1}$, and $W_0 <_r u_1$.*

Proof: We proceed by induction and first consider $W_0 = \{w_0, w'_0\}$. Since both vertices are neighbors of u_1 and the bar of u_1 is shorter than the bars of v_1, v_2, w_0 and w'_0 must be placed between the extreme vertices of cluster C_1. They are placed either between v_1 and v_2 or between v_2 and u_1, and their bars must be shorter than the bar of u_1 because of the edge (v_1, u_1). Placing one of them at either side of v_2 leads to a contradiction, since an edge incident to v_1 or to u_1 cannot be realized in an $AB1V$ representation.

If the base B is extended only by W_0, then both placements are possible, where an edge (w_0, v_3) or (w'_0, v_3) must be added depending on the vertex with the longer bar.

The subgraphs $G_k[W_{i-1} \cup W_i]$ are a K_4 for $i = 1, \ldots, k$. First, consider $W_0 \cup W_1$. If w_0 and w'_0 are placed between v_2 and u_1, then w_1 and w'_1 cannot be placed at all. As the bars of w_0 and w'_0 are shorter than the ones of v_1 and v_2, w_1 and w'_1 must be placed to the right of v_1. Each vertex between v_1 and

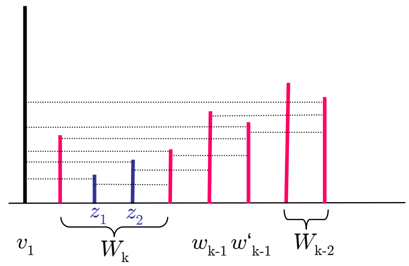

Figure 6: The $AB1V$ representation of G_k to the right of v_1 if v_1 is leftmost. The dotted lines represent visibility lines that traverse another bar.

v_2 must have a bar that is shorter than the shortest of the bars of w_0 and w'_0 because of the edges incident to v_1. If a vertex from W_1 were placed between v_1 and v_2, then an edge from the vertex of W_1 with the shorter bar to v_1 or to the rightmost of the vertices of W_1 cannot be realized in an $AB1V$ representation. If all of w_0, w_1, w'_0, w'_1 were placed to the right of v_2, then the edge from the one with the shortest bar to v_1 cannot be realized.

Hence, both vertices of W_0 must be placed between v_1 and v_2. Since u_2 is a neighbor of both, the leftmost of them must have the longer bar. Suppose that w_i is left of w'_i for $i = 1, \ldots, k$. The other case is symmetric and exchanges the roles. Then w_1 and w'_1 cannot be placed to the left of v_1, since the bar of w'_0 is not 1-visible from there. Then $\{w_1, w'_1\} <_r \{w_0, w'_0\}$ by the edges from w_0 and w'_0 to v_1 and u_1. By the reasoning as before, w_1 and w'_1 must be placed between v_1 and w_0 with bars that are shorter than the ones of W_0 and with the leftmost of the vertices of W_1 having the longer bar.

The same arguments are used in the inductive step with w_{i-1} and w'_{i-1} taking the roles of v_2 and u_1, where w_{i-1} is to the left of w'_{i-1} and has the longer bar.

Finally, consider the vertices on the left and z_1, z_2. As before, z_1 and z_2 must be placed to the right of v_1 and their bars must be shorter than the ones of w_k and w'_k. Then they must be placed between w_k and w'_k, as depicted in Fig. 6. Now, the leftmost vertex v_{k-1} of W_{k-1} is 1-visible from z_2 and the edge (z_2, w_{k-1}) determines the t- and r-orders of the vertices of Z and W_{k-1}. □

The proof of Lemma 7 now follows from Claims 1-4. □

Next we show that the graphs from Fig. 4 are almost best possible if we aim at sparse maximal $AB1V$ graphs.

Theorem 1 *Every maximal $AB1V$ graph of size n has at least $3.5n - 9$ edges.*

Proof: Consider an $AB1V$ representation of a maximal $AB1V$ graph G of size at least 6. For $n < 6$ the bound obviously holds. Let $X = \{v_1, v_2, v_p, v_q, v_{n-1}, v_n\}$ consist of the four extreme and the top-4 vertices such that $(v_1, v_2, \ldots, v_p, \ldots, v_q, \ldots, v_{n-1}, v_n)$ is the left to right order and v_1, v_p, v_q, v_n are the top-4 vertices in

r-order. Here $v_2 = v_p$ and $v_q = v_{n-1}$ are allowed. Clearly, the vertices at the left and right are among the top-4 if G is maximal.

We count edges in decreasing r-order of the vertices and assign an edge (u, v) to the vertex with the shorter bar.

Consider vertices $v \in V - X$. If v is placed between v_p and v_q, then it has two neighbors with longer bars to the left and right, and therefore, v adds four edges to G. Every $v \in V - X$ between v_1 and v_p has at least v_1 to the left and two neighbors with longer bars to the right, and therefore adds at least three edges to G. The right side is similar.

Suppose that $v \in V - X$ has only v_1 as a left neighbor with a longer bar. We call v a *leftish vertex* and denote the set of leftish vertices by L. The set R at the right is defined accordingly. By definition, all vertices u with $u \neq v_1$ to the left of a leftish vertex v in an $AB1V$ representation of a maximal $AB1V$ graph have a shorter bar than the bar of v. In addition, if we aim at a maximum number of leftish vertices, then the bar of u cannot be extended beyond the bar of v. Otherwise, u were a neighbor to the left of v with a longer bar and v were no longer a leftish vertex. Even more, if u is not a neighbor of v before the extension of the bar, then (u, v) is a new edge by the extension and G is not maximal. Hence, such extensions should be excluded.

A vertex u is called a *left blocker* of a leftish vertex $v \in L$ in a given $AB1V$ representation if $v_1 <_t v <_t u <_t v_p$ in t-order, $u <_r v$ in r-order, and v_1 is a neighbor of both u and v. Thus, the blocker is to the right of v, it has a shorter bar, and the bar of v is traversed by the line of sight of the edge (v_1, u). We denote the set of left blockers by B. Accordingly, consider right blockers and denote the set of right blockers by B'.

For the lower bound on the number of edges, we show that we can assign a blocker to each vertex $v \in L \cup R$. To see this, first, consider an $AB1V$ representation with $v_1 <_t v <_t v_p$ for a leftish vertex $v \in L$. Let $w \in V - X$ be the vertex between v_1 and v and with the longest bar, which is shorter than the bar of v. Then increase the length of the bar of w so that thereafter $v <_r w$ and there is no other vertex between v and w in r-order. The modified $AB1V$ representation also represents G if v has no blocker, and v is no more a leftish vertex. Second, a vertex w cannot be a blocker for more than one vertex, since two or more vertices of L prevent the 1-visibility of the leftmost vertex v_1 from w.

Hence, $|L| \leq |B|$ and $|R| \leq |B'|$, and these sets are pairwise disjoint. Since $|L|+|R|+|B|+|B'| \leq |V-X| = n-6$, we have $|L|+|R| \leq n/2-3$. In consequence, at least $n/2-3$ vertices each add at least four edges to G, each vertex of L and R adds three edges, the vertices in second and next to last position add three and the top-4 vertices add 6 edges to G. Hence, $m \geq 3(|L|+|R|)+4(n-6-|L|-|R|)+12 \geq 4n - 12 - |L| - |R| \geq 4n - 12 - n/2 + 3 = 3.5n - 9$, where m is the number m of edges of G. □

The graphs from Fig. 4 are maximal but not optimal, since edges are missing at the left, and they are close to sparsest maximal $AB1V$ graphs.

Theorem 2 *There are maximal $AB1V$ graphs of size $n \geq 21$ with $\lfloor 3.5\,n - 1 \rfloor$ edges.*

Proof: Consider the graphs G_k from Lemma 7. Since the left to right order given in Fig. 4 is a unique bucket order with buckets W_i of size two (up to reflection) it suffices to show that any t-order does not allow the addition of further edge. The base $B = G_k[V \cup U]$ has 15 vertices and 50 edges and is optimal. Each bucket W_i adds 2 vertices and 7 edges. One edge is missing towards optimality, since the leftmost vertex of W_i in the given t-order has only the leftmost vertex v_1 as a neighbor with a longer bar to the left. Finally, Z adds two vertices and eight edges. Then, every G_k with $k \geq 21$ and k odd has $3.5\,n - 1.5$ edges. For every even $k \geq 22$, we add a vertex with a short bar immediately to the left of v_8 in the $AB1V$ representation of G_{k-1} in Fig. 4. This adds four edges. □

Hence, there are tight bounds, both for densest and sparsest maximal $AB1V$ graphs.

Corollary 2 *The densest maximal $AB1V$ graphs have $4n - 10$ edges and the sparsest maximal $AB1V$ graphs have $3.5n - c$ edges, and both bounds are tight up to a constant c with $1 \leq c \leq 9$.*

In consequence, the various versions of visibility induce a strict hierarchy of classes of $AB1V$ graphs. Moreover, every $wAB1V$ graph is an induced subgraph of an optimal (or m, d, s)$AB1V$ graph. For the latter result, an $AB1V$ representation of a graph G is turned into an optimal $AB1V$ representation of a supergraph H by first varying the length of bars of the same length such that they are all different and then expanding the two bars at the left and right ends to the top-4 bars in r-order so that their vertices induce K_4. Then, H is an optimal $AB1V$ graph [25] and G is an induced subgraph.

Corollary 3 *$oAB1V \subset mAB1V \subset dAB1V \subset sAB1V \subset wAB1V$.*

Proof: The inclusions are obvious. The first proper inclusion is shown in Theorem 2. If $v_1 < \ldots < v_n$ for the t- and r-orders, then the resulting graphs have only $2n-3$ edges and are distinct but not maximal, which proves the second inclusion. Third, the guarded double chains with vertices $\{1, \ldots, n\}$ and edges $(i, i+1)$ and $(i, i+2)$ for $1 \leq i < n-1$ and $(1, n), (1, n-1), (2, n), (2, n-1), (n-1, n)$ and $n \geq 10$ have an aligned bar 1-visibility representation, where the four longest bars are placed in pairs at the left and right and induce K_4 and all other bars have the same length. These graphs cannot be represented with all bars of distinct length, since the fifth longest bar can see the bars of the four extreme vertices $1, 2, n-1, n$. Finally, strong $AB1V$ graphs of size at least five have degree at least four, which does not hold for weak $AB1V$ graphs. □

4 Path-Addition

In this section we show that $wAB1V$ graphs are closed under path-addition. *Path-addition* is a new operation on graphs which arises naturally from $AB1V$ representations. The operation is very powerful and is the opposite of taking minors. Some classes of graphs are closed under path-addition, including bar 1-visibility, quasi-planar, fan-crossing free, RAC, and non-planar graphs, whereas others are not, including planar, treewidth-k, k-planar, bar $(1, j)$-visibility, and fan-planar graphs, see [12].

Definition 2 *A* path-addition *takes a graph* $G = (V, E)$*, two vertices* $u, v \in V$*, and an internally vertex-disjoint path* $P = (u, w_1, \ldots, w_t, v)$ *with* $w_i \notin V$ *for* $1 \leq i \leq t$ *from* u *to* v *and results in* $G' = (V \cup W, E \cup Q)$ *where* $W = \{w_1, \ldots, w_t\}$ *and* Q *is the edges of* P*. We denote* G' *by* $G \oplus P$*.*

Definition 3 *A class of graphs* $\mathcal{G}$ *is* closed under path-addition *if the graph* $G \oplus P$ *is in* $\mathcal{G}$ *for every graph* G *in* $\mathcal{G}$ *of size* n *and for every internally vertex-disjoint path* P *of length at least* $n - 1$ *between two vertices* u *and* v *of* G*.*

For the closure of $wAB1V$ (and other classes) under path-addition, we add a path P of sufficient length so that P can be added independent of the position of the endvertices in an $AB1V$ representation. In consequence, the length of added paths increases at least exponentially. Path-additions allow the construction of any complete graph as a minor. First, add paths until there are k vertices, and then add a new path between any pair of vertices. Then we have a K_k minor.

Theorem 3 *The class of* $wAB1V$ *graphs is closed under path-addition.*

Proof: Consider an $AB1V$ representation of G_i after paths $P_1, \ldots, P_i$ have been added to a $wAB1V$ graph G. Increase the length of the bars of the vertices of G_i by $k_{i+1} + 1$, where k_{i+1} is the length of P_{i+1}, so that they are long, and let the bars of the internal vertices of P_{i+1} have length $1, \ldots, k_{i+1}$, respectively, so that they are short. Insert the bars of the internal vertices of P_{i+1} between the bars of G_i such that there is a 1-visibility between two consecutive vertices of P_{i+1}. This can be done in many ways if P_{i+1} is longer than the distance of its endvertices in the t-order of G_i. □

The addition of a path P to a strong $AB1V$ graph G does not preserve $AB1V$, since the inserted vertices of P induce 1-visibility to vertices of G. Then a set of supplementary edges F must be added to preserve aligned bar 1-visibility and F depends on the given $AB1V$ representation.

5 Recognition of Optimal $AB1V$ Graphs

The recognition problem of beyond-planar graphs is $\mathcal{NP}$-hard, in general. However, it is known that outer 1-planar graphs can be recognized in linear time [4, 26], map graphs in cubic time [15], maximal outer fan-planar graphs in polynomial time [5] and optimal 1-planar graphs in linear time [10]. Here, we solve the recognition problem for optimal $AB1V$ graphs. In addition, we show that the t-order can be computed in linear time from the r-order of a $dAB1V$ graph. This complements a result by Felsner and Massow [25] who computed the r-order from the t-order. Their algorithm runs in quadratic time and can be implemented to run in linear time, although this is not stated.

The recognition algorithms use the fact that the vertex with the i-th longest bar either has four neighbors with longer bars and then must be placed between them, or it must be placed just outside the extreme vertices with the longest bars. This procedure is stepwise generalized from a given r-order via optimal graphs to distinct graphs and at the expense of the running time.

In particular, we use the following fact, which was mentioned above:

Proposition 1

- *Every $AB1V$ graph has a vertex of degree at most four [23].*
- *The maximum complete subgraphs have size at most five [23].*
- *G is an optimal $AB1V$ graph if and only if the vertices with the four longest bars are in pairs at the left and right ends and build K_4 [23].*
- *If G is an optimal $AB1V$ graph, then all vertices have degree at least four.*
- *For $i = 1, \dots, n-4$, the i-th vertex in r-order has four neighbors with a higher r-order, i.e., with longer bars, and the subgraph induced by the vertex and its neighbors is K_5 or $K_5 - e$.*

Our recognition algorithm for optimal $AB1V$ graphs proceeds in two phases. First, it computes all clusters. Then it incrementally adds a vertex at a time and chooses a vertex v so that v induces a cluster with the already inserted vertices. Thereby, it takes the vertices in decreasing r-order.

Lemma 8 *There is a linear-time algorithm that checks whether a partial $AB1V$ representation of a cluster of an optimal $AB1V$ graph G can be extended to an $AB1V$ representation of G, and, if so, computes an $AB1V$ representation, i.e., a t- and an r-order of G.*

Proof: The algorithm implements Proposition 1. Let $V = \{v_1, \dots, v_n\}$ be the set of vertices of G and let $v_1, \dots, v_5$ be the vertices of the given cluster with $v_1 >_r v_2 >_r v_3 >_r v_4 >_r v_5 >_r v_i$ for $i > 5$ in r-order and v_5 in the middle of v_1, v_2, v_3, v_4 in the t-order of a partial $AB1V$ representation of $G[v_1, \dots, v_5]$ according to Lemma 1.

For $i = 5, \dots, n$ and $V_i = \{v_1, \dots, v_i\}$, if there is a partial $AB1V$ representation of $G[V_i]$ and the vertices of V_i are assigned the $n+1-i$ longest bars, then the algorithm gets a vertex $v_{i+1} \in V - V_i$ which has four neighbors in V_i, assigns the bar of length $n - i$ and places the bar in the middle (at the median) of the bars of its four neighbors from V_i. This is a partial $AB1V$ representation of $G[V_{i+1}]$ with $V_{i+1} = V_i \cup \{v_{i+1}\}$. If such a vertex does not exist or the placement is invalid, then the algorithm stops and rejects.

Processing a vertex takes $\mathcal{O}(1)$ time. If the algorithm succeeds, it has computed a t- and an r-order of G and if it fails, then either G is not $AB1V$ or the initial cluster did not represent the top-5 vertices in r-order. □

Theorem 4 *There is a quadratic-time algorithm that checks whether a graph G is an optimal $AB1V$ graph, and, if so, computes an $AB1V$ representation.*

Proof: The top-5 vertices in r-order induce a cluster C. There are at most $O(n)$ clusters by Corollary 1, which are computed in $O(n)$ time. Since the bars of the vertices of C are longer than the bars of all other vertices, we take any r-order that defines a cluster.

The algorithm takes each cluster of G and successively tests all 120 t-orders of the vertices as a partial $AB1V$ representation for an extension according to Lemma 8. The algorithm succeeds if it finds an extension and rejects otherwise. If G is an optimal $AB1V$ graph, the algorithm must succeed and it cannot succeed otherwise. □

Next, we show that the left-to-right order of the bars can be computed from an $AB1V$ graph together with the knowledge of the length of the bars. The converse was shown in [25].

Theorem 5 *There is a linear-time algorithm that computes a t-order given an r-order of a $dAB1V$ graph G such that the t- and r-orders describe a $dAB1V$ representation of G.*

Proof: Consider the vertices of G in decreasing r-order and for $i = 1, \ldots, n$ let $W_i = \{v_1, \ldots, v_i\}$ denote the set of i vertices with the longest bars. The algorithm successively extends a partial $AB1V$ representation of $G[W_i]$ to a partial $AB1V$ representation of $G[W_{i+1}]$. It starts with an $AB1V$ representation of $G[W_4]$ and tries all 12 t-orders of the vertices up to reflection.

For $i = 5, \ldots, n$, if v_i has four neighbors in W_{i-1}, then the partial $AB1V$ representation is extended by placing v_i in the middle of its four neighbors in W_{i-1}. If v_i has three neighbors in W_{i-1}, then one of its neighbors is the first (resp. last) vertex in the actual partial $AB1V$ representation, which is extended by placing v_i in second (last but one) place. Finally, if v_i has only two neighbors in W_{i-1}, then v_i is an extreme vertex and is placed to the left (right) of the vertices of W_{i-1} if the former leftmost (rightmost) vertex is its neighbor. The algorithm stops if the placement was invalid.

The correctness is due to the fact that the bars of $v_i, \ldots, v_n$ are shorter than the bar of v_{i-1}. Hence, the bar of v_i is 1-visible from the bars of its neighbors from W_{i-1}, and it is 1-visible from two, three, or four neighbors depending on its position in the partial $AB1V$ representation.

Each run with an extension of a partial $AB1V$ representation of $G[W_4]$ takes linear time, which implies $O(n)$ time in total. □

Together with a result from [25] we can conclude:

Corollary 4 *If G is a dAB1V graph, then an AB1V representation can be computed in linear time if (i) a t-order or (ii) an r-order is given.*

6 Relationship to other Classes of Graphs

For the comparison of $AB1V$ graphs with other classes of graphs we consider weak $AB1V$ graphs, since the other versions, e.g., $sAB1V$ graphs, are too restrictive, since the graphs have vertices of degree four. Moreover, the other classes of graphs are generally closed under taking subgraphs, in particular, all classes of graphs listed in Fig. 13.

6.1 Inclusion Relations

Common drawings and visibility representations are equivalent in the planar case with non-transparent bars, both in the general case [37, 39] and in the outer planar case with aligned visibility [17]. However, visibility representations are more powerful than common drawings if crossings are allowed. The 1-planar graphs are a proper subset of bar $(1,1)$-visibility graphs [9] and bar 1-visibility graphs [22]. Hence, single edge-edge crossings are weaker than single edge-vertex crossings. A similar result holds for $AB1V$ graphs.

Theorem 6 *There is a linear time algorithm that constructs a wAB1V representation of an outer 1-planar graph.*

Proof: Recall that a graph is outer 1-planar if it admits a drawing with all vertices in the outer face and at most one crossing per edge [4, 26].

The linear time algorithm of Auer et al. [4] tests whether a graph G is outer 1-planar and, if so, augments G to a graph H by adding edges and constructs a (planar) maximal outer 1-planar embedding of H. The embedding of H consists of planar triangles and kites. A kite is a K_4 that is embedded with a pair of crossing edges.

Let X be the set of crossing edges in the embedding of H and let $H - X$ be the graph after their removal. Then $H - X$ is an outer planar graph with a Hamiltonian cycle where all inner faces are triangles or quadrangles. The dual of $H - X$ is a tree T with t- and q-vertices corresponding to triangles and quadrangles, respectively. The outer face is ignored. We root T at a leaf r and thereby orient T from r.

First, we construct a (planar) bar visibility representation of $H-X$. Consider the root of T which corresponds to a triangle or a quadrangle $(v_1, \ldots, v_k)$ with $k = 3, 4$ and with the edge (v_1, v_2) in the outer face. Then the t-order is the ordering of the vertices of the Hamiltonian cycle of H with $v_1 = 1$ and $v_2 = n$. Now, two vertices u and v span an interval on the Hamiltonian cycle.

The length of each bar, i.e., the r-order of $H-X$, is computed by successively removing a leaf from T. This extends Mitchell's algorithm for the recognition of maximal outer planar graphs by successively removing a vertex of degree two [30]. A leaf b of T is in one-to-one correspondence to a vertex v of degree two if

b is a triangle, and to two vertices u and v of degree two with an edge (u, v) if b is a quadrangle. Suppose that $i-1$ vertices have been removed so far, where $i = 1, \ldots, n-i-j$ for $j = 0, 1$ and $j = 0$ if and only if the root is a triangle. Assign v a bar of length i if b is removed and v is a degree-two vertex in the triangle b. The edges (u, v) and (v, w) incident to v are lines of sight at level i, i.e., at the top end of the bar of v. Increase i by one. Accordingly, assign bars of length i and $i+1$ to u and v (in any order) if b is a quadrangle and u and v are the degree two vertices corresponding to b and draw the lines of sight at the top of the bars. Increase i by two. Finally, assign bars of length n and $n-1$ to the vertices v_1 and v_2 corresponding to the root and bars of length $n-2$ resp. $n-3$ to the other vertices of the root.

The algorithm preserves the invariant that all bars have a distinct length and that the bars of all vertices in the interval between u and v have bars that are shorter than the bars of u and v if there is an edge (u, v). The latter is due to the fact that the vertices between u and v have been processed before u and v. It guarantees that an edge at level i is unobstructed.

Finally, we reinsert the crossing edges of X. For each pair of crossing edges (a, c) and (b, d) there is a quadrangle (a, b, c, d) with $a <_t b <_t c <_t d$ in t-order and $\{b, c\} <_r \{a, d\}$ in r-order. By the previous assignment, the bars of b and c have length i and $i+1$ for some i, respectively. Then the bar of a vertex w in the interval from a to d has length at most $i-1$ if $w \neq b, c$. Suppose the bar of b has length i, the case where the bar of c has length i is similar. Then draw the edge (a, c) as a planar line of sight at level $i+1$ and draw the edge (b, d) as a line of sight at level $(i-0.5)$ which traverses the bar of c. No other bars are affected. Hereby, we adapt the technique of Brandenburg [9] for the bar $(1,1)$-visibility representation of 1-planar graphs to $AB1V$ representations. For integer coordinates, scale by two.

In total, we have obtained a weak distinct $AB1V$ representation of G where lines of sight are ignored if there is no edge. All stages of the algorithm take linear time. □

To establish proper inclusion relations we use the fact that outer 1-planar graphs are planar [4], that K_5 is an $AB1V$ graph, and that K_6 is a bar 1-visibility graph and not an $AB1V$ graph. Note that bar 1-visibility graphs are a proper subclass of quasi-planar graphs [22].

Corollary 5 *Every outer 1-planar graph is a $wAB1V$ graph, but not conversely, and every $wAB1V$ graph is a bar 1-visibility graph, but not conversely.*

Hence, $AB1V$ graphs are quasi-planar, which is a large class of beyond-planar graphs containing graphs with up to 6.5n-20 edges [1]. Quasi-planar graphs are defined via embeddings and exclude a mutual crossing of three edges. Therefore, they include all 1-planar, RAC, fan-planar, bar 1-visibility [22], and rectangle visibility graphs, and all graphs with geometric thickness two. Since $AB1V$ graphs have geometric thickness two [25] and are bar 1-visibility graphs, we can conclude:

Corollary 6 *A graph G is quasi-planar if G is a $wAB1V$ graph, but not conversely.*

6.2 Incomparability Results

In this section we prove that w$AB1V$ graphs are incomparable to some classes of beyond-planar graphs. Two graph classes are *incomparable* if there are mutual counterexamples, i.e., graphs that are in one class and not in the other. As aforesaid, we consider classes that are closed under taking subgraphs. The incomparability holds for any version of visibility, i.e., $\{o, m, d, s, w\}AB1V$ graphs.

First, consider planar graphs. Since there are planar graphs that are not $wAB1V$ graphs, a class of graphs $\mathcal{G}$ is not contained in $wAB1V$ if $\mathcal{G}$ includes the planar graphs.

Theorem 7 *The classes of planar and $wAB1V$ graphs are incomparable.*

Proof: Every $wAB1V$ graph has a vertex of degree at most four, namely the one with the shortest bar in an $AB1V$ representation. However, there are planar graphs of degree five, such as the dual of the dodecahedron graph or of the football graph C_{60}.

Conversely, K_5 is an $AB1V$ graph and not planar. Moreover, optimal $AB1V$ graphs are denser than planar graphs. □

For the incomparability of $wAB1V$ and k-planar respectively bar $(1, k)$-visibility graphs we use the fact that the closure of bipartite graphs $K_{2,m}$ under path-addition results in graphs that are neither k-planar nor $(1, k)$-visibility graphs [12], but are $wAB1V$ graphs.

Theorem 8 *For every $k \geq 0$, the classes of k-planar (bar $(1, k)$-visibility) graphs and $wAB1V$ graphs are incomparable.*

Proof: There are planar graphs that are not $wAB1V$ graphs, which proves one direction. Conversely, consider graphs that are obtained from $K_{2,n}$ by at least one path-addition between each pair of vertices from the set of n vertices. Each such graph is a $wAB1V$ graph by the closure of $wAB1V$ under path-addition, as established in Theorem 3. However, for $n \geq 4k + 9$ the graphs are not k-planar and not bar $(1, k)$-visibility graphs, as shown in [12]. □

Similarly, there are counterexamples for outer fan-planar and fan-planar graphs.

Theorem 9 *The classes of $wAB1V$ and outer fan-planar graphs are incomparable.*

Proof: Outer fan-planar graphs of size n have at most $3n - 5$ edges [6], but there are $AB1V$ graphs with $4n - 10$ edges.

Conversely, consider the outer fan-planar graph G from Fig. 7, which is built in three stages. First, G has a central K_5 with vertices $v_1, \ldots, v_5$, which is drawn as a pentagram with edges (v_i, v_{i+1}) for $i = 1, \ldots, 5$ with $v_6 = v_1$ in the outer face. There is a subgraph associated with each outer edge as depicted in Fig. 7(b). In the second stage, for $i = 1, \ldots, 5$ with $v_6 = v_1$, there are four new vertices such that the subgraph induced by $\{v_i, v_{i+1}\}$ and $\{u_1^i, \ldots, u_4^i\}$ is $K_{2,4}$. Third, let $v_i = u_0^i$ and $v_{i+1} = u_5^i$ and add three new vertices w_j^i, x_j^i, y_j^i for $i = 1, \ldots, 5$ and $j = 0, \ldots, 4$, which together with vertices u_j^i, u_{j+1}^i induce a cluster. In total, G has 100 vertices and 290 edges.

G is outer fan-planar as shown in Fig. 7.

Assume, for a contradiction, that G has an $AB1V$ representation (in which edges may be omitted). Lemma 1 applies to each cluster C. In particular, there is no other vertex with a long bar between the vertices of C which disturbs the $AB1V$ representation of C. We say that two vertices x and y *see* each other if both are members of a cluster. Then there is a planar line of sight between the bars of x and y at level l, where l is the length of the shorter bar. A vertex *disturbs* if it prevents a bar 1-visibility of two other vertices.

Suppose $v_{i_1} <_t \ldots <_t v_{i_5}$ is the t-order of the vertices of the central K_5. Since there is a triangle by the vertices $v_{i_2}, v_{i_3}, v_{i_4}$, at least one of the edges of the triangle is an outer edge and has an associated $K_{2,4}$. Let (v_s, v_t) be the outer edge. Suppose that $v_s <_r v_t$. Then v_s has at most the second shortest bar of the vertices of the central K_5 and therefore $v_s <_r \{v_{i_1}, v_{i_5}\}$ by Lemma 1.

Consider a partial $AB1V$ representation with vertices $V' = \{v_{i_1}, v_p, v_q, v_{i_5}\}$ and $U = \{u_1, \ldots, u_4\}$ of the associated $K_{2,4}$. Note that $\{v_{i_1}, u_1\}$, $\{v_{i_5}, u_4\}$, $\{u_j, u_{j+1}\}$ for $j = 1, 2, 3$, and $\{v_{i_1}, v_p\}, \{v_p, v_q\}$ and $\{v_q, v_{i_5}\}$ pairwise see each other if the fifth vertex $w \in \{v_{i_2}, v_{i_3}, v_{i_4}\} - \{v_p, v_q\}$ is ignored. Towards a contradiction, we show that there is no partial $AB1V$ representation of $G[V' \cup U]$ which respects the pairs of vertices that see each other.

First, observe that the vertices of U cannot be placed between v_p and v_q, i.e., $v_p <_t U <_t v_q$, since the least vertex of U in r-order is not 1-visible from v_p and v_q, and there is no $K_{2,4}$. Second, if some vertex $u \in U$ is placed to the left of v_p, then all vertices of U must be placed to the left of v_p, since u must see one or two vertices of U, and this relation is transitive on U. This holds accordingly for v_{i_1} and for v_q and v_{i_5} with a placement to the right. Finally, suppose that all $u \in U$ are placed to the left of v_p. If the vertices of U are placed between v_{i_1} and v_p, then their bars must be shorter than the bar of v_p. Now v_q cannot see u_4 since v_p disturbs. Otherwise, if the vertices of U are placed to the left of v_{i_1}, then v_p cannot see u_1, since v_{i_1} disturbs. The placement of the vertices of U to the right is similar and also leads to a contradiction. □

Next, we address fan-crossing free graphs. First, we show that there are graphs with a unique fan-crossing free embedding, which may be useful in further studies of fan-crossing free graphs.

Lemma 9 *A fan-crossing free embedding of K_5 has one pair of crossing edges and is unique up to graph automorphism (labeling of the vertices).*

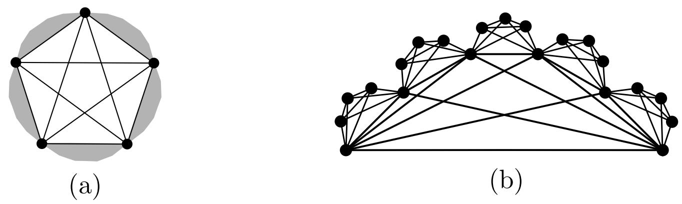

Figure 7: An outer fan-planar graph which is not an $AB1V$ graph. Gray areas in (a) are the subgraphs displayed in (b).

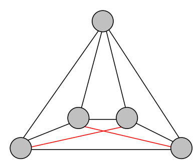

Figure 8: A fan-crossing-free embedding of K_5

Proof: There is a fan-crossing free embedding of K_5 as shown in Fig. 8. Clearly, any embedding of K_5 has a pair of crossing edges with a crossing point. Then the remaining fifth vertex cannot be placed in a triangle with the crossing point and two other vertices of K_5 in a fan-crossing free embedding. Hence, only the embedding as in Fig. 8 remains, which is unique up to a graph automorphism [36]. Note that we exclude crossings of edges incident to a common vertex. $\square$

Lemma 10 *There are fan-crossing free graphs with an almost fixed fan-crossing free embedding.*

Proof: Consider the graph $A = G_k[V_0]$ with vertices $v_1, \ldots, v_{10}$ in t-order and $r_1, \ldots, r_{10}$ in r-order from Fig. 4 and a fan-crossing free embedding in Fig. 12. We prove property (*) which implies that all but the placement of vertex r_{10} is fixed. The outer face is opposite to the face containing r_6 in the embedding of the initial K_5.

For convenience, we consider the vertices $\{r_1, \ldots, r_{10}\}$ of A in the r-order of the $AB1V$ representation from Fig. 4 such that there are six clusters $C_i = A[r_i, \ldots, r_{i+4}]$ for $i = 1, \ldots, 6$. We successively add the vertices $r_6, \ldots, r_{10}$ to an initial fan-crossing free embedding of the first cluster C_1 and establish the following property:

(*) In any fan-crossing free embedding $\mathcal{E}(A)$, vertex r_i for $i = 6, \ldots, 9$ must be placed in a triangle $\Delta_i = \Delta(r_{i-3}, r_{i-2}, r_{i-1})$ and the edges of Δ_i are planar. Finally, r_{10} is placed in Δ_8.

In consequence, all vertices r_j with $j \geq i$ are placed in Δ_i and there is a nesting of $\mathcal{E}(A)$ as shown in Fig. 12.

For the proof of (*), first observe that vertex r_i for $i \geq 6$ cannot be placed in a triangle $\Delta(u, v, x)$ of the planarization of $\mathcal{E}(A[r_1, \ldots, r_{i-1}])$ if x is a crossing point of two edges. A planarization of an embedding treats a crossing point as a special vertex of degree four which subdivides the crossed edges and belongs to four triangles. All faces are triangles by construction. A planarization converts an embedding with crossings into a planar embedding. The observation is due to the fact that r_i has four neighbors in $\{r_1, \ldots, r_{i-1}\}$ and the edges to at least two vertices not in $\Delta(u, v, x)$ inevitably introduce a fan-crossing if r_i is placed in $\Delta(u, v, x)$.

Hence, the triangles Δ of the planarization of $\mathcal{E}(A[r_1, \ldots, r_{i-1}])$ with a crossing point are excluded for a placement of any vertex r_j for $j \geq i$.

From now on we construct a fan-crossing free embedding of A. By Lemma 9 the embedding of the first cluster $A[r_1, \ldots, r_5]$ is as in Fig. 8. Since the vertices $r_1, \ldots, r_5$ are not yet determined, we rename them to $x_1, \ldots, x_5$.

We choose the outer face as given and insert vertex r_6 in the triangle $\Delta(x_1, x_2, x_3)$. Since r_6 has four neighbors in $\{x_1, \ldots, x_5\}$, at least one edge of (x_1, x_3) and (x_2, x_3) is crossed by an edge (r_6, x_4) or (r_6, x_5).

Towards a contradiction, suppose that $\Delta(x_1, x_2, x_3)$ is not the triangle of r_3, r_4, r_5, i.e., $\{x_1, x_2, x_3\} \neq \{r_3, r_4, r_5\}$. Then both edges (x_1, x_3) and (x_2, x_3) are crossed by edges incident to r_6, as illustrated in Fig. 9. Assume that r_6 has neighbors x_1 and x_3, which implies $x_2 = r_1$ and thereby determines the first vertex of $x_1, \ldots, x_5$. The two other cases are similar. For the placement of r_7, there are only two regions left, the outer face $\Delta(x_3, x_4, x_5)$ and an inner face with vertices x_1, r_6, x_4 and two crossing points, but both fail. Since $x_2 = r_1$, r_7 cannot be placed in the inner face, since two of its neighbors from $\{r_4, r_5, r_6\}$ are in the outer face and cannot be reached from r_7 without a fan-crossing. If r_7 is placed in the outer face, then edge (r_6, r_7) introduces a fan-crossing.

Assume that edge (x_1, x_3) is crossed by (r_6, x_4). This enforces $x_5 = r_1$. The case where (x_2, x_3) is crossed by (r_6, x_5) and $x_4 = r_1$ is similar. Then r_7 can be placed in one of the triangles $\Delta(x_1, x_2, r_6), \Delta(x_2, x_3, r_6)$ or $\Delta(x_2, x_3, r_1)$, see Fig. 10. The outer face is excluded, since (r_6, r_7) would introduce a fan-crossing.

If r_7 is placed in $\Delta(x_2, x_3, r_1)$ (the dark shaded area in Fig. 10), then $x_1 = r_2$ and the edges (r_3, x_5) and (x_2, x_3) are crossed by the edges (r_7, x_4) and (r_7, r_6), respectively. Thereafter, only $\Delta(x_1, x_2, r_6)$ is left for a placement of r_8; however, edge (r_7, r_8) introduces a fan-crossing.

Hence, r_7 must be placed in another triangle, which implies $x_4 = r_2$. Now we have $\{r_3, r_4, r_5\} = \{x_1, x_2, x_3\}$ such that r_6 is placed in the "good" triangle as stated in (*).

Next, suppose that r_7 is placed in $\Delta(x_2, x_3, r_6)$, see Fig. 11; the other case is symmetric and yields another labeling of x_1, x_2, x_3 by r_3, r_4, r_5. Then (x_1, r_7) crosses (x_2, r_6), which implies $x_1 = r_3$ and leaves $\Delta(x_2, x_3, r_7)$ and $\Delta(x_3, r_6, r_7)$ for a placement of r_8. The triangle $\Delta(x_2, x_3, r_1)$ is excluded, since edge (r_6, r_8) introduces a fan-crossing.

If r_8 is placed in $\Delta(x_2, x_3, r_7)$, then (r_6, r_8) crosses (r_7, x_3) and leaves no face for a placement of r_9. Hence, r_8 must be placed in $\Delta(x_3, r_6, r_7)$, which implies $x_2 = r_4$ and $x_3 = r_5$. Now all initial vertices $x_1, \ldots, x_5$ are determined. If the

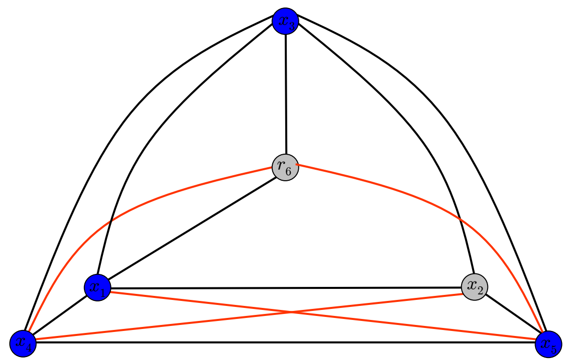

Figure 9: A fan-crossing-free embedding if vertex r_6 is placed in a bad face with only two neighbors in the face

next vertex r_9 is placed in the triangle of r_6, r_8, x_3, then there is no face for r_{10}. Hence, r_9 must be placed in the triangle of r_6, r_7, r_8 and (r_9, r_5) crosses (r_6, r_8). Thereafter, r_{10} can be placed in $\Delta(r_6, r_7, r_9)$ or in $\Delta(r_7, r_8, r_9)$. If the sequence is continued by some $r_{11}, r_{12}, \ldots$, then r_{10} must be placed in $\Delta(r_7, r_8, r_9)$.

In summary, the vertices $r_1, \ldots, r_5$ are determined and property (*) is preserved, which implies an almost fixed embedding. □

Lemma 11 *There are $wAB1V$ graphs that are not fan-crossing free.*

Proof: Our counterexample G consisting of the graph $A = G_k[V_0]$ from Fig. 4 together with a single vertex z which is connected to $v_2, \ldots, v_5$ or r_2, r_4, r_6, r_8 and is placed between v_3 and v_4 or r_4 and r_6 with a short bar. Thus $z = u_1$ and $u_2, \ldots, u_6$ are removed.

As shown in Lemma 10, there is a unique fan-crossing free embedding of $\mathcal{E}(A)$ up to the choice of two triangles for the placement of r_{10}. However, there is no face left for a placement of vertex z with the given neighbors without introducing a fan-crossing. A candidate triangle with vertices r_5, r_7, r_8 fails because of the edges (r_2, z) and (r_6, z) and similarly, the triangle with vertices r_5, r_6, r_8 fails because of the edges (r_4, z) and (r_2, z).

More counterexamples can be constructed by adding more vertices to G, as in Lemma 10. □

In consequence, we can establish a further incomparability.

Theorem 10 *The classes of $wAB1V$ graphs and (a) RAC graphs and (b) fan-crossing free graphs are incomparable.*

Proof: There are planar graphs of minimum degree five which are not $wAB1V$ graphs, and conversely the graph G from Lemma 11 is $wAB1V$ and not fan-crossing free and therefore not RAC, since every RAC graph is fan-crossing free. □

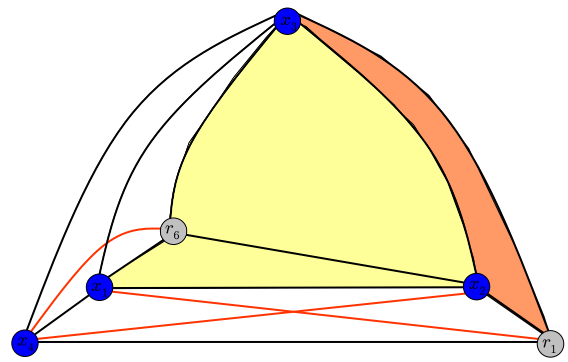

Figure 10: A fan-crossing-free embedding after vertex r_6 is placed. Vertex r_7 can be placed in one of the shaded triangles and a placement in the darker shared triangle fails.

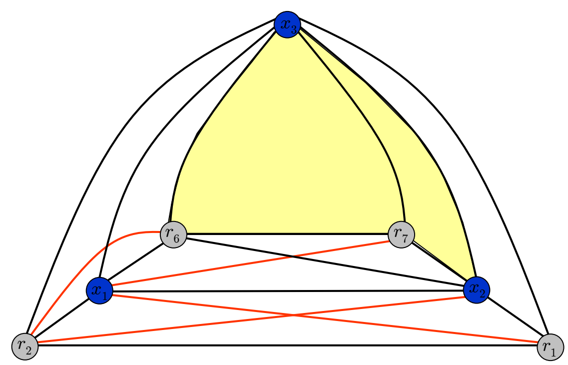

Figure 11: A fan-crossing-free embedding after placing r_6 and r_7

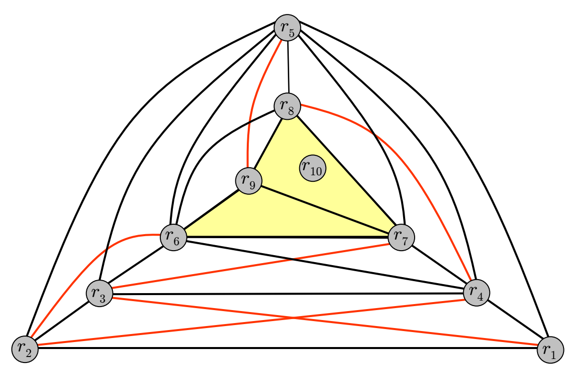

Figure 12: A fan-crossing-free embedding of the A graph from the proof of Lemma 10. Vertex r_{10} can be placed in the shaded area. There is no face left for vertex z with neighbors r_2, r_4, r_6, r_8 from the proof of Lemma 11.

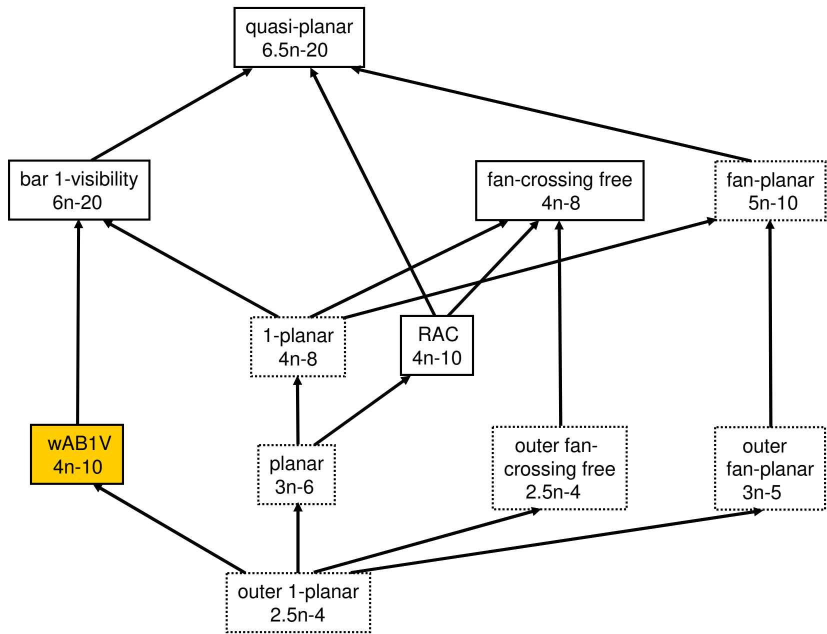

Figure 13: Relationships between $AB1V$ graphs and other classes of beyond-planar graphs. An arrow indicates proper inclusion and no path to/from $AB1V$ indicates incomparability. The formula below the class name gives the maximum number of edges. Classes with a closed boundary are closed under path-addition, whereas classes with a dotted boundary are not closed under path-addition [12].

The relationships of the class of $wAB1V$ graphs to other classes of beyond-planar graphs are illustrated in Fig. 13. An "arrow" indicates a proper inclusion, whereas "no path" indicates incomparability. The incomparability has been proved for $wAB1V$ and all shown classes except for fan-planar and outer fan-crossing free graphs. Outer fan-crossing free graphs have not yet been studied in detail. They may be included in $wAB1V$, but not conversely, since K_5 is not outer fan-crossing free by Lemma 9. Accordingly, there are fan-planar graphs that are not $wAB1V$ by Lemma 9 and fan-planar graphs are not closed under path-addition [12]. We conjecture that all illustrated incomparabilities hold.

The bar (i, j)-visibility graphs [13] and the bar $(1, 1)$-visibility graphs [9] are shown to be incomparable with $wAB1V$. These classes range between the classes of 1-planar and bar 1-visibility graphs.

Clearly, the subclasses IC-planar [28] and NIC-planar [40] of the 1-planar graphs are also incomparable with $wAB1V$, since they include the planar graphs and do not include graphs with $4n - 10$ edges.

Finally, $wAB1V$ graphs have geometric thickness two [25], and graphs with geometric thickness two are quasi-planar. The relationship of $wAB1V$ and RVG (rectangle visibility graphs) [27] is open and we conjecture incomparability.

7 Conclusion

In this work we extended the studies of Felsner and Massow [25] on $AB1V$ graphs and added new properties and relationships.

Our studies have revealed new problems, such as the closure of classes of graphs under path-addition and the relation of $AB1V$ graphs to other classes of beyond-planar graphs. Of interest are recognition problems, in particular for weak $AB1V$ graphs, and the relationship of $AB1V$ graphs to further classes of beyond-planar graphs. It might be worthwhile to study the relationship between classes of optimal or maximal graphs.

8 Acknowledgements

We wish to thank the reviewers for their valuable suggestions.

References

[1] E. Ackerman and G. Tardos. On the maximum number of edges in quasi-planar graphs. *J. Comb. Theory, Ser. A*, 114(3):563–571, 2007. doi:10.1016/j.jcta.2006.08.002.

[2] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997. doi:10.1007/BF01196127.

[3] T. Andreae. Some results on visibility graphs. *Discrete Applied Mathematics*, 40(1):5–17, 1992. doi:10.1016/0166-218X(92)90018-6.

[4] C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, and J. Reislhuber. Outer 1-planar graphs. *Algorithmica*, 74(4):1293–1320, 2016. doi:10.1007/s00453-015-0002-1.

[5] M. A. Bekos, S. Cornelsen, L. Grilli, S. Hong, and M. Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. In C. A. Duncan and A. Symvonis, editors, *GD 2014*, volume 8871 of *LNCS*, pages 198–209. Springer, 2014. doi:10.1007/978-3-662-45803-7_17.

[6] C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, A. Symvonis, and I. G. Tollis. Fan-planarity: Properties and complexity. *Theor. Comput. Sci.*, 589:76–86, 2015. doi:10.1016/j.tcs.2015.04.020.

[7] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theor. Comput. Sci.*, 209(1-2):1–45, 1998. doi:10.1016/S0304-3975(97)00228-4.

[8] F. Brandenburg, D. Eppstein, A. Gleißner, M. T. Goodrich, K. Hanauer, and J. Reislhuber. On the density of maximal 1-planar graphs. In W. Didimo and M. Patrignani, editors, *GD 2012*, volume 7704 of *LNCS*, pages 327–338. Springer, 2012. doi:10.1007/978-3-642-36763-2_29.

[9] F. J. Brandenburg. 1-visibility representation of 1-planar graphs. *J. Graph Algorithms Appl.*, 18(3):421–438, 2014. doi:10.7155/jgaa.00330.

[10] F. J. Brandenburg. Recognizing optimal 1-planar graphs in linear time. *Algorithmica*, pages 1–28, 2016. doi:10.1007/s00453-016-0226-8.

[11] F. J. Brandenburg, A. Esch, and D. Neuwirth. On aligned bar 1-visibility graphs. In M. Kaykobad and R. Petreschi, editors, *WALCOM 2016*, volume 9627 of *LNCS*, pages 95–106. Springer, 2016. doi:10.1007/978-3-319-30139-6_8.

[12] F. J. Brandenburg, A. Esch, and D. Neuwirth. Path-additions. *CoRR*, abs/1605.02891, 2016. URL: http://arxiv.org/abs/1605.02891.

[13] F. J. Brandenburg, N. Heinsohn, M. Kaufmann, and D. Neuwirth. On bar $(1, j)$-visibility graphs - (extended abstract). In M. S. Rahman and E. Tomita, editors, *WALCOM 2015*, volume 8973 of *LNCS*, pages 246–257. Springer, 2015. doi:10.1007/978-3-319-15612-5.

[14] Z. Chen, M. Grigni, and C. H. Papadimitriou. Map graphs. *J. ACM*, 49(2):127–138, 2002. doi:10.1145/506147.506148.

[15] Z. Chen, M. Grigni, and C. H. Papadimitriou. Recognizing hole-free 4-map graphs in cubic time. *Algorithmica*, 45(2):227–262, 2006. doi:10.1007/s00453-005-1184-8.

[16] O. Cheong, S. Har-Peled, H. Kim, and H. Kim. On the number of edges of fan-crossing free graphs. *Algorithmica*, 73(4):673–695, 2015. doi:10.1007/s00453-014-9935-z.

[17] F. Cobos, J. Dana, F. Hurtado, A. Márquez, and F. Mateos. On a visibility representation of graphs. In F. J. Brandenburg, editor, *Graph Drawing*, volume 1027 of *LNCS*, pages 152–161. Springer, 1995. doi:10.1007/BFb0021799.

[18] A. M. Dean, W. Evans, E. Gethner, J. D. Laison, M. A. Safari, and W. T. Trotter. Bar k-visibility graphs. *J. Graph Algorithms Appl.*, 11(1):45–59, 2007. doi:10.7155/jgaa.00136.

[19] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice Hall, 1999.

[20] W. Didimo, P. Eades, and G. Liotta. Drawing graphs with right angle crossings. *Theor. Comput. Sci.*, 412(39):5156–5166, 2011. doi:10.1016/j.tcs.2011.05.025.

[21] P. Eades, S.-H. Hong, N. Katoh, G. Liotta, P. Schweitzer, and Y. Suzuki. A linear time algorithm for testing maximal 1-planarity of graphs with a rotation system. *Theor. Comput. Sci.*, 513:65–76, 2013. doi:10.1016/j.tcs.2013.09.029.

[22] W. S. Evans, M. Kaufmann, W. Lenhart, T. Mchedlidze, and S. K. Wismath. Bar 1-visibility graphs vs. other nearly planar graphs. *J. Graph Algorithms Appl.*, 18(5):721–739, 2014. doi:10.7155/jgaa.00343.

[23] I. Fabrici and T. Madaras. The structure of 1-planar graphs. *Discrete Math.*, 307(7-8):854–865, 2007. doi:10.1016/j.disc.2005.11.056.

[24] R. Fagin, R. Kumar, M. Mahdian, D. Sivakumar, and E. Vee. Comparing partial rankings. 20(3):628–648, 2006. doi:10.1137/05063088X.

[25] S. Felsner and M. Massow. Parameters of bar k-visibility graphs. *J. Graph Algorithms Appl.*, 12(1):5–27, 2008. doi:10.7155/jgaa.00157.

[26] S. Hong, P. Eades, N. Katoh, G. Liotta, P. Schweitzer, and Y. Suzuki. A linear-time algorithm for testing outer-1-planarity. *Algorithmica*, 72(4):1033–1054, 2015. doi:10.1007/s00453-014-9890-8.

[27] J. P. Hutchinson, T. Shermer, and A. Vince. On representations of some thickness-two graphs. *Computational Geometry*, 13:161–171, 1999. doi:10.1016/S0925-7721(99)00018-8.

[28] D. Král and L. Stacho. Coloring plane graphs with independent crossings. *Journal of Graph Theory*, 64(3):184–205, 2010. doi:10.1002/jgt.20448.

[29] F. Luccio, S. Mazzone, and C. K. Wong. A note on visibility graphs. *Discrete Mathematics*, 64(2-3):209–219, 1987. doi:10.1016/0012-365X(87)90190-7.

[30] S. L. Mitchell. Linear algorithms to recognize outerplanar and maximal outerplanar graphs. *Inform. Process. Lett.*, 9(5):229–232, 1979.

[31] J. O'Rourke. *Art Gallery Theorem and Algorithms*. Oxford University Press, 1987.

[32] J. Pach and G. Tóth. Graphs drawn with a few crossings per edge. *Combinatorica*, 17:427–439, 1997. doi:10.1007/BF01215922.

[33] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. aus dem Math. Seminar der Univ. Hamburg*, 29:107–117, 1965. doi:10.1007/bf02996313.

[34] P. Rosenstiehl and R. E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete & Computational Geometry*, 1:343–353, 1986. doi:10.1007/BF02187706.

[35] S. Sultana, M. S. Rahman, and S. Roy, Arpita Tairin. Bar 1-visibility drawings of 1-planar graphs. In P. Gupta and C. D. Zaroliagis, editors, *ICAA 2014*, volume 8321, pages 62–76. Springer, 2014. doi:10.1007/978-3-319-04126-1_6.

[36] Y. Suzuki. Re-embeddings of maximum 1-planar graphs. *SIAM J. Discr. Math.*, 24(4):1527–1540, 2010. doi:10.1137/090746835.

[37] R. Tamassia and I. G. Tollis. A unified approach a visibility representation of planar graphs. *Discrete Comput. Geom.*, 1:321–341, 1986. doi:10.1007/BF02187705.

[38] M. Thorup. Map graphs in polynomial time. In *Proc. 39th FOCS*, pages 396–405. IEEE Computer Society, 1998. doi:10.1109/SFCS.1998.743490.

[39] S. Wismath. Characterizing bar line-of-sight graphs. In *Proc. SoCG 1985*, pages 147–152. ACM Press, 1985. doi:10.1145/323233.323253.

[40] X. Zhang. Drawing complete multipartite graphs on the plane with restrictions on crossings. *Acta Math. Sinica, English Series*, 30(12):2045–2053, 2014.