

## Bar 1-Visibility Graphs and their relation to other Nearly Planar Graphs

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### Abstract

A graph is called a *strong* (resp. *weak*) *bar 1-visibility graph* if its vertices can be represented as horizontal segments (bars) in the plane so that its edges are all (resp. a subset of) the pairs of vertices whose bars have a  $\varepsilon$ -thick vertical line connecting them that intersects at most one other bar.

We explore the relation among weak (resp. strong) bar 1-visibility graphs and other nearly planar graph classes. In particular, we study their relation to *1-planar graphs*, which have a drawing with at most one crossing per edge; *quasi-planar graphs*, which have a drawing with no three mutually crossing edges; and the squares of *planar 1-flow networks*, which are upward digraphs with in- or out-degree at most one. Our main results are that 1-planar graphs and the (undirected) squares of planar 1-flow networks are weak bar 1-visibility graphs and that these are quasi-planar graphs.

Submitted: November 2013	Reviewed: September 2014	Revised: November 2014	Accepted: November 2014	Final: December 2014
Published: December 2014				
		Article type: Regular paper	Communicated by: J. S. B. Mitchell	

We thank Beppe Liotta for initiating our study of the relationships between Bar 1-visibility graphs and other classes of nearly planar graphs, and for his valuable and conscientious contributions to this paper. The research reported in this paper started at the 2013 McGill/INRIA/UVictoria Bellairs workshop. We gratefully acknowledge discussions with the other participants. Research supported by NSERC.

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## 1 Introduction

Developing a theory of graph drawing beyond planarity has received increasing interest in recent years. This is partly motivated by applications of network visualization, where it is important to compute readable drawings of non-planar graphs. Within this research framework, a rich body of papers has in particular been devoted to the study of the combinatorial properties of different types of drawings that are nearly planar, i.e., do not allow a specific restricted set of crossing configurations, such as the crossings cannot form too sharp angles (see, e.g., [12] for a survey). Another study of visualizations of non-planar graphs that are “close to planar” was conducted by Dean et al. [9], by introducing so-called bar  $k$ -visibility graphs and representations. Dean et al. were particularly interested in measurements of closeness to planarity of bar  $k$ -visibility graphs. In this work we shed some light on this question by investigating the relation of bar 1-visibility graphs with graphs that are known to be “close to planar”. Thus, we study the relation of bar 1-visibility graphs with nearly planar graphs, particularly 1-planar and quasi-planar graphs. Moreover, we investigate the relation of bar 1-visibility graphs with squares of planar graphs.

A *bar layout* consists of  $n$  horizontal non-intersecting line segments (bars). A pair of bars  $u$  and  $v$  are  *$k$ -visible* if and only if there is an axis-aligned rectangle of positive width touching  $u$  and  $v$  which intersects at most  $k$  bars in the layout. For a given bar layout, its (unique) *strong bar  $k$ -visibility graph* has a vertex for every bar and an edge  $uv$  if and only if the corresponding bars  $u$  and  $v$  are  $k$ -visible<sup>1</sup>. A *weak bar  $k$ -visibility graph* of a bar layout is any (spanning) subgraph of its strong bar  $k$ -visibility graph. Note that there are  $2^m$  weak bar  $k$ -visibility graphs if there are  $m$  edges in the strong bar  $k$ -visibility graph. A graph is a strong (weak) bar  $k$ -visibility graph if it is the strong (weak) bar  $k$ -visibility graph of some bar layout. Independently, Wismath [30] and Tamassia and Tollis [27] characterized strong bar 0-visibility graphs as exactly those that have a planar embedding with all cut vertices on the exterior face. Weak bar 0-visibility graphs are exactly the planar graphs [10]. Dean et al. [9] showed that  $K_n$  ( $n \leq 8$ ) is a strong bar 1-visibility graph, that  $K_9$  is not a strong bar 1-visibility graph, and that all  $n$ -vertex strong (and thus weak) bar 1-visibility graphs have fewer than  $6n - 20$  edges. Felsner and Massow [18] showed that there exists a strong bar 1-visibility graph that has thickness three, disproving an earlier conjecture [9] that all such graphs have thickness two or less.

While bar layouts represent the vertices of a graph as horizontal segments, a *topological drawing* of a graph  $G$  maps each vertex  $u$  of  $G$  to a distinct point  $p_u$  in the plane and each edge  $uv$  of  $G$  to a Jordan arc connecting  $p_u$  and  $p_v$ . In a topological drawing it is required that an edge does not pass through any other vertex except for its end-vertices, and any intersection point of two edges is either their common end-vertex or a crossing point. In this paper we consider *simple* topological drawings, where it is required that any two edges have at most one common point. A  *$k$ -planar graph* is one which admits a topological

<sup>1</sup>We denote by  $uv$  the undirected edge between  $u$  and  $v$ , and by  $(u, v)$  the edge directed from  $u$  to  $v$ .

drawing in which each edge is crossed by at most  $k$  other edges. Pach and Tóth proved that 1-planar graphs with  $n$  vertices have at most  $4n - 8$  edges, which is a tight upper bound [24] and that, in general,  $k$ -planar graphs are sparse. Korzhik and Mohar proved that recognizing 1-planar graphs is NP-hard [21]. A limited list of additional papers on  $k$ -planar graphs includes [3, 5, 7, 8, 16, 14, 17, 15, 20, 26, 29]. Sultana et al. [25] recently investigated the relation between 1-planar and bar 1-visibility graphs and showed that several restricted subclasses of 1-planar graphs are weak bar 1-visibility graphs.

A  $k$ -quasi-planar graph admits a topological drawing such that no  $k$  edges mutually cross; 3-quasi-planar graphs are commonly called *quasi-planar*, for short. Ackerman and Tardos showed that quasi-planar graphs with  $n$  vertices have at most  $6.5n - O(1)$  edges [2]. Di Giacomo et al. [11] described how to construct linear area  $k$ -quasi-planar drawings of graphs with bounded treewidth. Recently, Geneson et al. [19] showed that all semi-bar  $k$ -visibility graphs<sup>2</sup> are  $(k + 2)$ -quasi-planar. See also [1, 23] for additional references about  $k$ -quasi-planar graphs.

Another family of non-planar graphs, which are in some sense “close to planar” are the squares of directed planar graphs with bounded in- or out-degree. The *square*  $G^2$  of a graph  $G = (V, E)$  has vertex set  $V$  and all edges  $(u, v)$  (or  $uv$  in the case that  $G$  is undirected) where there is a path of length at most two from  $u$  to  $v$  in  $G$ . Observe that if for each vertex of a directed planar graph  $G$ , either in- or out-degree is bounded by a constant, then the number of edges in  $G^2$  is linear. This fact is captured by the notion of  $k$ -flow networks. A (*planar*)  $k$ -flow network is a (upward planar) directed graph in which every vertex  $v$  has  $\min\{\text{indeg}(v), \text{outdeg}(v)\} \leq k$ . The name of the class stems from the fact that at most  $k$  units of flow can pass through each vertex. Tarjan [28] studied 1-flow networks under the name of *unit flow networks*. Bessy et al. [4] studied the arc-chromatic number of  $k$ -flow networks under the name of  $(k \vee k)$ -digraphs. We let  $k$ -flow<sup>2</sup> denote the class of graphs that are the undirected squares of planar  $k$ -flow networks. Squares of graphs arise naturally in understanding bar 1-visibility graphs since a bar layout that represents a bar 0-visibility graph  $G$  also represents a family of weak bar 1-visibility graphs each of which is a spanning subgraph of  $G^2$ . That is, every weak bar 1-visibility graph is a spanning subgraph of the square of a bar 0-visibility graph. Thus, it is natural to consider which bar 0-visibility graphs have squares that are weak bar 1-visible.

While several properties of bar 1-visibility graphs have been investigated, it remains an open problem to provide their complete characterization. Recall that bar 1-visibility graphs are generally non-planar and contain at most  $6n - 20$  edges. Observe that this number is greater than the maximum number of edges in 1-planar graphs (at most  $4n - 8$ ) and smaller than the maximum number of edges in quasi-planar graphs (at most  $6.5n - O(1)$ ). Recall also that, every weak bar 1-visibility graph is a spanning subgraph of the square of a

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<sup>2</sup>Semi-bar visibility graphs require all horizontal bars to have minimum  $x$ -coordinate equal to zero [18].

bar 0-visibility graph. Motivated by these facts we study the relation of bar 1-visibility graphs with families of 1-planar, quasi-planar and squares of planar graphs. Our contribution is threefold: (i) We show that the class of weak bar 1-visibility graphs contains the class of 1-planar graphs, which proves a conjecture of Sultana et al. [25], (ii) we show that the class of bar 1-visibility graphs is contained in the class of quasi-planar graphs, and (iii) we show that 1-flow<sup>2</sup> graphs are weak bar 1-visibility graphs, and that this is not always true for 2-flow<sup>2</sup> graphs. An overview of our results is illustrated in Figure 1 and thoroughly described in Section 2. Proof details about the inclusion relationships of Figure 1 are given in Sections 3, 4, and 5.

We note that, recently, Brandenburg [6] independently showed (i).

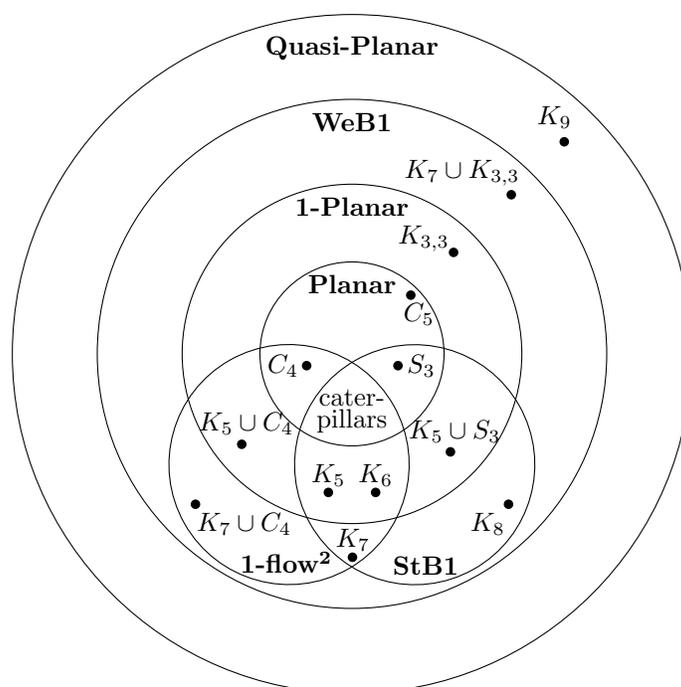


Figure 1: Relationships among graph classes proved in this paper.

## 2 Graph classes and their relationships

In this section we describe Figure 1. We abbreviate strong and weak bar 1-visibility graphs as *StB1* and *WeB1* graphs. Since a strong bar 1-visibility graph is a weak bar 1-visibility graph of the same bar layout, it follows that  $StB1 \subseteq WeB1$ . The observation that every planar graph is *WeB1* (it is in fact

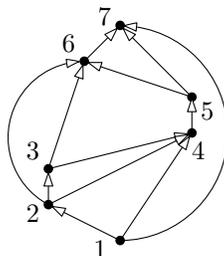


Figure 2: 1-flow graph  $G$  such that the square of the subgraph of  $G$  induced by vertices  $1, \dots, n$  is  $K_n$  ( $n \leq 7$ ).

a weak bar 0-visibility graph [10]); the fact that  $K_{3,3}$  is WeB1 ( $\equiv \equiv \equiv$ ); and the following simple lemma prove that  $\text{StB1} \subset \text{WeB1}$ .

**Lemma 1** *Any graph that is StB1 is either a forest or contains a triangle.*

**Proof:** Let  $G$  be StB1 and suppose  $G$  does not contain a triangle. We will show that  $G$  does not contain any cycle, which implies that it is a forest. For the sake of contradiction, assume that  $G$  contains a cycle  $C$ . Consider a strong bar 1-visibility layout of  $G$ . Let  $v$  be a vertex of  $C$  whose bar has right endpoint with minimum  $x$ -coordinate,  $x$ . Since  $v$  has at least two neighbors in  $C$ , their bars must share some  $x$ -coordinate with bar  $v$  and all must span  $x$ . Thus at least three bars span  $x$  implying a triangle in the graph, which is a contradiction.  $\square$

The number of edges in any 1-planar graph is known to be at most  $4n - 8$  [24]. Thus,  $K_7$  and  $K_8$  are not 1-planar (too many edges) but are StB1 as proved by Dean et al. [9]. The disjoint union  $K_7 \cup K_{3,3}$  is WeB1 but it is not 1-planar (because of  $K_7$ ) and it is not StB1 (because of  $K_{3,3}$  by Lemma 1). We show that all 1-planar graphs are WeB1 (see Section 3) and that all WeB1 graphs are quasi-planar (see Section 4).

In Section 5, we show that 1-flow<sup>2</sup> graphs are WeB1. We also show that 2-flow<sup>2</sup> graphs are not always WeB1. It is easy to see that if  $G^2 \neq G$  then  $G^2$  contains a triangle. Thus, since  $K_{3,3}$  is not planar and does not contain a triangle, it is not a 1-flow<sup>2</sup> graph. However, every planar bipartite graph  $G$  can be directed (from one bipartition to the other) so that  $G$  is a 1-flow network with  $G^2 = G$  and is thus a 1-flow<sup>2</sup> graph. Therefore, caterpillars and  $C_4$  are 1-flow<sup>2</sup> graphs. It is also easy to see that caterpillars are StB1. If  $G$  is the 1-flow graph of Figure 2, then the square of the subgraph of  $G$  induced by vertices  $1, \dots, n$  is  $K_n$  ( $n \leq 7$ ). In Section 5 we show that  $K_8$  is not the square of a 1-flow network, and that there exists a planar StB1 graph ( $S_3$ ) that is not the square of a 1-flow network.

### 3 1-planar graphs are WeB1

**Theorem 1** *If a graph  $G$  is 1-planar then  $G$  is WeB1.*

**Proof:** It suffices to prove the theorem for a maximal 1-planar graph  $G = (V, E)$  since a WeB1-representation of  $G$  is a WeB1-representation of every graph  $(V, E')$  with  $E' \subseteq E$ . Let  $\Gamma$  be a 1-planar drawing of  $G$ . Let  $ab$  and  $cd$  be a pair of edges that cross in  $\Gamma$ . By Proposition 1 [7], vertices  $a, b, c$  and  $d$  form a  $K_4$ . Thus, edges  $ac, cb, bd$ , and  $da$  exist in  $G$ . In case some of these edges are crossed, we can redraw them so that they become uncrossed. If, for example, edge  $ac$  was crossed in  $\Gamma$ , we could re-route it without introducing crossings by following edge  $ab$  from  $a$  to its intersection with  $cd$  and then following  $cd$  to  $c$ ; always following slightly to the  $c$ -side of  $ab$  and the  $a$ -side of  $cd$ .

Since  $G$  is a maximal 1-planar graph, the planar graph  $G_0$  obtained by removing all crossing edges from  $G$  is biconnected [14] and thus has an  $st$ -orientation [22], which is a partial order,  $\preceq$ , on the vertices  $V$  with a single source (minimal vertex) and a single sink (maximal vertex). We direct the edges of  $G_0$  to be consistent with this partial order; so  $uv$  is directed as  $(u, v)$  if  $u \preceq v$ . Let  $\vec{G}_0$  be the directed version of  $G_0$ , and let  $\Gamma_0$  be the drawing  $\Gamma$  restricted to  $G_0$ .

For every crossing pair of edges  $ab$  and  $cd$  in  $G$ , the (undirected) cycle  $C = acbda$  exists in  $G_0$  since none of its edges are crossed in  $\Gamma$ . We claim that the oriented version,  $\vec{C}$ , of  $C$  consists of two directed paths with common origin and common destination. This claim is a slight generalization of:

**Lemma 2 (Lemma 4.1 [10])** *Each face  $f$  of  $\vec{G}_0$  consists of two directed paths with common origin and common destination.*

In our case,  $\vec{C}$  may not be a face of  $\vec{G}_0$ ; it may contain vertices and edges in its interior. However, if our claim is violated, we can re-route the edges of the cycle  $C$  (as above) so that  $\vec{C}$  is a face of  $\vec{G}_0$  and contradict the previous lemma. Thus the claim holds and there must be two consecutive edges in  $C$  that are oriented in the same direction, say  $(a, c)$  and  $(c, b)$ . See for example Fig. 3(a).

We return the edge  $cd$  to the drawing  $\Gamma_0$  and direct it to be consistent with the partial order,  $\preceq$ , defined by the  $st$ -orientation. In place of the edge  $ab$ , we insert the directed path  $aucvb$  that contains two dummy vertices,  $u$  and  $v$  (specifically for this crossing). Note that, by the above discussion, this path is also consistent with the partial order. The dummy vertices  $u$  and  $v$  are placed as follows. Let  $x$  be the point where  $ab$  and  $cd$  intersect. We place  $u$  on the segment  $ax$  at  $\varepsilon$  distance from  $x$ . Similarly, we place  $v$  on the segment  $xb$  at  $\varepsilon$  distance from  $x$  (see Fig. 3(b)). By choosing  $\varepsilon$  small enough we have that none of the edges  $(a, u)$ ,  $(u, c)$ ,  $(c, v)$ ,  $(v, b)$  creates a crossing and the result, after every pair of crossing edges is replaced in this fashion, is an  $st$ -oriented plane graph  $G'$  with drawing  $\Gamma'$ . Since  $G'$  is planar and has an  $st$ -orientation,  $G'$  has a bar 0-visibility representation [30, 27].

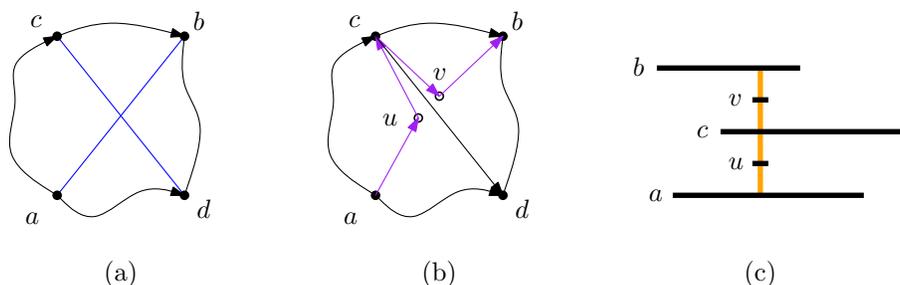


Figure 3: (a) At least two edges ( $ac$  and  $cb$ ) are oriented in the same direction around the cycle  $C$ . (b) One edge ( $ab$ ) in a pair of crossing edges is replaced with the path  $aucvb$  by adding dummy vertices  $u$  and  $v$ . (c) The visibility edges of the path  $aucvb$  are vertically aligned. (Only these bars are shown.)

The set of inserted paths are *nonintersecting*, meaning they are edge disjoint and do not *cross* at common vertices<sup>3</sup> in the drawing  $\Gamma'$ . Thus, we may construct a bar 0-visibility representation so that for each inserted path,  $aucvb$ , the visibility lines realizing the edges of the path are vertically aligned (Theorem 4.4 [10]). If we remove the bars representing dummy vertices  $u$  and  $v$ , the visibility lines become a line of sight between  $a$  and  $b$  that is crossed only by the bar representing vertex  $c$ . It follows that the bar 0-visibility representation, after removing all dummy bars, is a weak bar 1-visibility representation of  $G$ . See Figure 3(c). □

## 4 WeB1 graphs are Quasi-planar

**Theorem 2** *If a graph  $G$  is WeB1, then  $G$  is quasi-planar.*

**Proof:** Let  $R$  be a weak bar 1-visibility representation of  $G = (V, E)$ . We show that the set of *all* edges  $E'$  realized by the representation  $R$  (i.e., the strong bar 1-visibility graph of  $R$ ) forms a quasi-planar graph. Since  $E$  is a subset of  $E'$ ,  $G$  is quasi-planar.

We construct a quasi-planar drawing,  $Q$ , from the bar representation  $R$  as follows. In  $Q$ , place vertex  $v$  at the left endpoint,  $\ell(v)$ , of the bar representing  $v$  in  $R$ . The edges of  $E'$  are in one of two classes. Let  $E'_0 \subseteq E'$  be the edges, called *blue edges*, realized in  $R$  by a direct visibility between bars. Let  $E'_1 = E' - E'_0$  be the remaining edges of  $G'$ , called *red edges*, that is, those that are only realized by a visibility through another bar. For a blue edge  $(u, v)$ , with bar  $u$  below bar  $v$ , draw a polygonal curve in  $Q$  consisting of three segments: the middle segment is nearly identical to the *rightmost* vertical visibility segment that connects bar  $u$  with bar  $v$ , but it starts  $\gamma$  (a small, positive value) above bar  $u$ , ends  $\gamma$  below

<sup>3</sup>Two paths *cross* at a vertex  $v$  in a drawing  $\Gamma$  if  $v$  has four incident edges  $e_1, e_2, e_3$ , and  $e_4$  in clockwise order such that one path contains  $e_1$  and  $e_3$  while the other contains  $e_2$  and  $e_4$ .

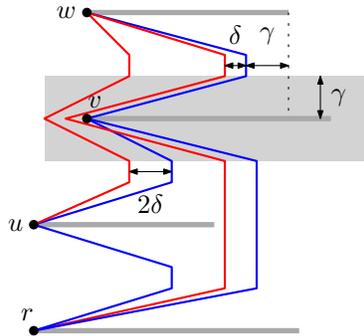


Figure 4: Construction of the quasi-planar drawing  $Q$ . The shaded region around bar  $v$  contains no vertical edge segments. The values of  $\gamma$  and  $\delta$  in the figure are larger than what they would be in a true construction.

bar  $v$ , and is shifted  $\gamma$  to the left. The first and third segments connect  $\ell(u)$  to the bottom of the middle segment and the top of the middle segment to  $\ell(v)$ , respectively.

We first show that the polygonal curves from two distinct blue edges do not cross. Let  $R(v)$  be the rectangular region composed of points less than  $L_\infty$ -distance  $\gamma$  from bar  $v$ . We choose  $\gamma > 0$  to be at most half the minimum positive difference between bar  $x$ -coordinates and bar  $y$ -coordinates. With this choice of  $\gamma$ , for any pair of vertices  $u \neq v$ , the regions  $R(u)$  and  $R(v)$  do not intersect. The vertical segment,  $\nu(u, v)$ , of an edge  $(u, v)$  connects a point on the boundary of  $R(u)$  with a point on the boundary of  $R(v)$ , since the left-shift by  $\gamma$  from a rightmost vertical visibility segment keeps the  $x$ -coordinate of  $\nu(u, v)$  within the  $x$ -range of bar  $u$  and bar  $v$ . In addition, the left-shift insures that  $\nu(u, v)$  is at least distance  $\gamma$  from the left end of any bar, and no closer than  $\gamma$  to the right end of any bar. So no vertical edge segment intersects  $R(u)$  for any  $u$ . All (nearly) horizontal segments that connect to  $\ell(v)$  lie within  $R(v)$  by construction, and do not intersect (except at  $\ell(v)$ ). Thus, no vertical edge segment intersects a (nearly) horizontal segment from another edge, and no (nearly) horizontal edge segment intersects a (nearly) horizontal segment from another edge. Thus the curves representing blue edges do not cross. See Figure 4.

For a red edge  $(u, w)$ , let  $v$  be the bar that is crossed by the *rightmost* 1-visibility segment,  $\sigma$ , that connects bar  $u$  with bar  $w$ . We call  $v$  the *bypass vertex* for the red edge  $(u, w)$ . Draw edge  $(u, w)$  as a polygonal curve in  $Q$  consisting of six segments: the first three connect  $\ell(u)$  to  $\ell(v)$  (as above) where the middle segment lies  $\gamma$  to the left of  $\sigma$ , and the last three connect  $\ell(v)$  to  $\ell(w)$  (as above) where, again, the middle segment lies  $\gamma$  to the left of  $\sigma$ . The edges  $(u, v)$  and  $(v, w)$  are in  $E'_0$  and therefore have polygonal curves in  $Q$  that lie on or to the right of the curve for  $(u, w)$ . In order to prevent the curve for  $(u, w)$  from intersecting the curves for  $(u, v)$  and  $(v, w)$  (except at  $\ell(u)$  and  $\ell(w)$ ), we shift all the points of the curve for  $(u, w)$ , except  $\ell(u)$  and  $\ell(w)$ , slightly to the

left. The amount of this shift depends on the red edges that have  $v$  as a bypass vertex. If  $k$  red edges with bypass vertex  $v$  have 1-visibility segments to the right of  $\sigma$  then the shift is by  $(k+1)\delta$ , where  $\delta$  is a positive value that is smaller than  $\gamma/|E'|^2$ . In this way, no two red edge curves with the same bypass vertex intersect.

Note that no (red or blue) vertical edge segments intersect region  $R(v)$  for any vertex  $v$ , and all (nearly) horizontal edge segments lie in such a region for some bar. Thus all edge curve intersections occur within such regions. See Figure 4.

In fact, two red edge curves intersect if and only if the bypass vertex of the edge whose curve has vertical segments further to the right is an endpoint of the other. One consequence of this is that if two red edge curves, say for edges  $(u, v)$  and  $(u, w)$ , share an endpoint then they do not intersect. If they did then the bypass vertex of one, say  $(u, w)$ , must be the unshared endpoint of the other, in this case  $v$ . This implies that there is a direct visibility between  $u$  and  $v$  and the edge  $(u, v)$  is blue not red.

Suppose, for the sake of contradiction, that the drawing  $Q$  is not quasi-planar. Consider a triple of edges (edge curves),  $(u_1, w_1)$ ,  $(u_2, w_2)$ , and  $(u_3, w_3)$ , that mutually intersect in  $Q$ . Since no two blue edges intersect, at most one edge in the triple is blue. In the first case, suppose all edges in the triple are red. Two red edges do not intersect if they share an endpoint, so all the endpoints  $u_1, u_2, u_3, w_1, w_2$ , and  $w_3$  are distinct. Let  $(u_1, w_1)$  be the edge whose curve has vertical segments furthest to the right of the three. The bypass vertex of  $(u_1, w_1)$  can be the endpoint of only one of the edges  $(u_2, w_2)$  and  $(u_3, w_3)$ . Thus  $(u_1, w_1)$  does not intersect the other; a contradiction.

In the second case, suppose  $(u_1, w_1)$  is the only blue edge in the triple of mutually intersecting edges. The intersection of a blue edge and a red edge must occur near the bypass vertex of the red edge, i.e. inside  $R(v)$  if  $v$  is the bypass vertex. Since both red edges,  $(u_2, w_2)$  and  $(u_3, w_3)$ , in the triple intersect edge  $(u_1, w_1)$ , they must have bypass vertices  $u_1$  or  $w_1$ . Because  $(u_2, w_2)$  and  $(u_3, w_3)$  intersect, they cannot share the same bypass vertex. Assume by renaming if necessary that  $(u_2, w_2)$  has bypass vertex  $u_1$  and  $(u_3, w_3)$  has bypass vertex  $w_1$ . Also because  $(u_2, w_2)$  and  $(u_3, w_3)$  intersect, the bypass vertex of one, say  $(u_2, w_2)$ , is an endpoint of the other. Thus one of the endpoints of  $(u_3, w_3)$  is  $u_1$  and its bypass vertex is  $w_1$ . This implies that three segments of the curve representing  $(u_3, w_3)$  are shifted versions of the curve representing the blue edge  $(u_1, w_1)$ . Thus  $(u_3, w_3)$  doesn't intersect the blue edge; a contradiction.  $\square$

## 5 Squares of planar 1-flow networks are WeB1

An acyclic digraph is called *upward planar* if it admits a planar drawing where all edges are represented by curves monotonically increasing in a common direction. An upward planar digraph with one source  $s$  and one sink  $t$ , embedded so that  $s$  and  $t$  are on the outer face, is called a *planar  $st$ -digraph*.

For a planar  $st$ -digraph  $G = (V, E)$ , let  $\text{left}(v)$  (resp.  $\text{right}(v)$ ) denote the

face of  $G$  separating the incoming from the outgoing edges in clockwise (resp. counterclockwise) order. A *topological numbering* of  $G$  is an assignment of numbers to the vertices of  $G$ , such that for every edge  $(u, v)$ , the number assigned to  $v$  is greater than the number assigned to  $u$ . The numbering is *optimal* if the range of the numbers assigned to the vertices is minimized.

Recall that a *planar  $k$ -flow network* is an upward planar digraph in which every vertex  $v$  has  $\min\{\text{indeg}(v), \text{outdeg}(v)\} \leq k$ . Recall also that  $k$ -flow<sup>2</sup> denote the class of graphs that are the undirected squares of planar  $k$ -flow networks.

As we already mentioned, a bar layout that represents a bar 0-visibility graph  $G$  also represents a family of weak bar 1-visibility graphs each of which is a spanning subgraph of  $G^2$ . In other words, every weak bar 1-visibility graph is a spanning subgraph of the square of a bar 0-visibility graph. In the following we investigate the reverse question, thus, we investigate which bar 0-visibility graphs have squares that are weak bar 1-visible.

**Theorem 3** *The square of a planar 1-flow network is WeB1.*

**Proof:** Let  $G'$  be a planar 1-flow network and  $G$  be a planar *st*-digraph for which  $G'$  is a spanning subgraph. We will prove in Lemma 3 that such  $G$  exists. The argument is a slight modification of the method used to prove Theorem 6.1 [10].

**Lemma 3** *Any planar 1-flow network is a spanning subgraph of an *st*-digraph that is also a 1-flow network.*

**Proof:** Let  $G'$  be a planar 1-flow network, i.e., an upward planar digraph with  $\min\{\text{indeg}(v), \text{outdeg}(v)\} \leq 1$ , for each vertex  $v$ . We add edges to  $G'$  to make it a planar 1-flow network  $G$ , with a unique source and a unique sink. For an upward planar drawing  $\Gamma'$  of  $G'$ , let  $t_1, \dots, t_k$  (resp.  $s_1, \dots, s_f$ ) be the sinks (resp. sources) of  $G'$  that are on the outer face, where  $t_1$  (resp.  $s_1$ ) has the largest (resp. smallest)  $y$ -coordinate (see Figure 5(a)). Add an edge from each of  $t_2, \dots, t_k$  to  $t_1$  and from  $s_1$  to each of  $s_2, \dots, s_f$  so that the resulting drawing  $\Gamma''$  is planar. Call the new planar 1-flow network  $G''$ .

Let  $t$  be a sink of  $G''$ . Consider a vertical half-line  $\ell$ , originating at  $t$  to  $+\infty$ . If  $t \neq t_1$ , half-line  $\ell$  crosses a boundary of an interior face  $f$  of  $\Gamma''$  that contains  $t$ , since otherwise  $t$  would have been on the outer face of  $\Gamma'$  and would not be a sink in  $G''$  (the edge  $(t, t_1)$  would be in  $G''$ ). We follow half-line  $\ell$  and the boundary of face  $f$  upward until we reach a sink  $t'$  of the face and add an edge  $(t, t')$  to  $G''$ . Vertex  $t'$  either has no outgoing edge, i.e., is a sink of  $G''$ , or already has two incoming edges. Thus, the addition of  $(t, t')$  keeps  $G''$  a 1-flow network. Moreover, edge  $(t, t')$  does not create any crossing and keeps the graph upward, therefore after this step  $G''$  is still a planar 1-flow network. The step cancels a sink of  $G''$ . We repeat this step until no other sink except for  $t_1$  remains. We perform a symmetric procedure for the remaining sources (see Figure 5(b)). The resulting graph  $G$  is a planar 1-flow network. Since only edges have been added,  $G'$  is a spanning subgraph of  $G$ .  $\square$

We come back to the proof of the theorem. In the following we show that the bar 0-visibility representation  $\Gamma$  of  $G$  produced by the algorithm of Tamassia

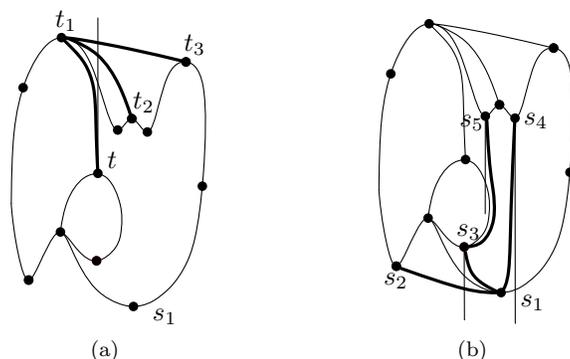


Figure 5: Illustration for the proof of Lemma 3. (a) Added thick edges cancel all sinks except  $t_1$ . (b) Added thick edges cancel all sources except  $s_1$ .

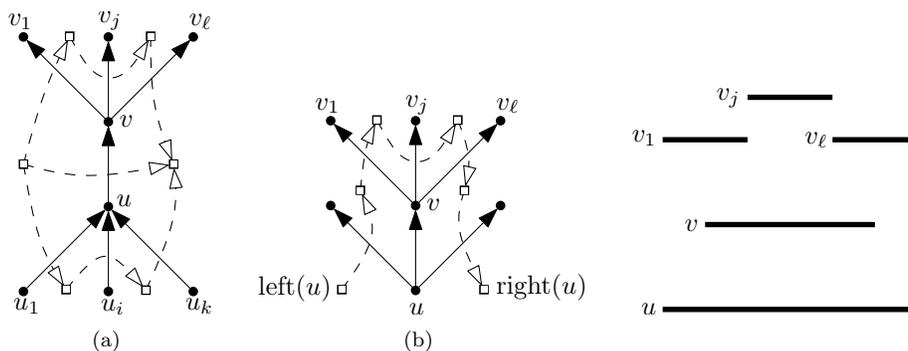


Figure 6: Illustration for the proof of Theorem 3

and Tollis [27] is a WeB1 visibility representation of  $G^2$ . Since  $G'$  is a spanning subgraph of  $G$ ,  $G'^2$  is a spanning subgraph of  $G^2$ , and therefore  $\Gamma$  is a WeB1 visibility representation of  $G'^2$ .

We first review the construction of  $\Gamma$ . Let  $G^*$  be the dual of  $G$ , where each of  $G^*$  is directed so that it crosses the corresponding edge of  $G$  from its left to its right. It is easy to see that  $G^*$  is a planar  $st$ -digraph [10]. Let  $\psi$  and  $\chi$  be the functions that assign an optimal topological numbering to the vertices of  $G$  and  $G^*$ , respectively. In  $\Gamma$ , vertex  $v$  is represented as a horizontal bar at  $y$ -coordinate  $\psi(v)$  and with end-points at  $x$ -coordinates  $\chi(\text{left}(v))$  and  $\chi(\text{right}(v)) - 1$ . We show that each edge of  $G^2$  of the form  $(u, w)$ , such that  $(u, v), (v, w) \in G$ , exists in  $\Gamma$  and is represented by a vertical line crossing only one vertex  $v$ . Assume that  $v$  has one incoming and several outgoing edges. The case when  $v$  has one outgoing and several incoming edges can be proven symmetrically.

Let  $(u, v)$  be the only incoming edge of  $v$ . If edge  $(u, v)$  is the only outgoing edge of  $u$  (Figure 6(a)),  $\chi(\text{left}(u)) = \chi(\text{left}(v))$  and  $\chi(\text{right}(u)) = \chi(\text{right}(v))$ . Therefore  $u$  and  $v$  are represented in  $\Gamma$  as two bars with the same left and

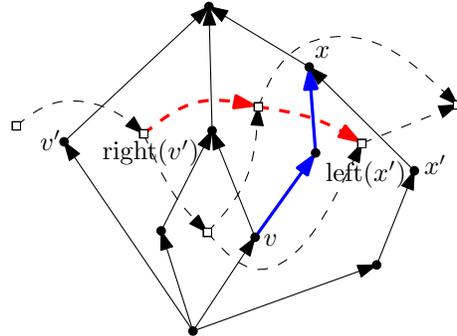


Figure 7: Illustration of the Lemma 4.3 [10]. The edges of the primal graph  $G$  are solid, the edges of the dual  $G^*$  are dashed. Duplicate edges in the dual are not shown.

right ends. If vertex  $u$  has more outgoing edges (Figure 6(b)),  $\chi(\text{left}(u)) < \chi(\text{left}(v))$  and  $\chi(\text{right}(v)) < \chi(\text{right}(u))$ . Thus generally it holds that  $\chi(\text{left}(u)) \leq \chi(\text{left}(v))$  and  $\chi(\text{right}(v)) \leq \chi(\text{right}(u))$  (see Figure 6(c)) and any vertical line that intersects bar  $v$  also intersects bar  $u$ . Thus, any vertical line that represents the edge from  $v$  to  $w$ , also crosses  $u$ .

It remains to show that there is no bar in  $\Gamma$  between  $u$  and  $v$  crossed by such a vertical line. Let  $x$  be a vertex different from  $u$  and  $v$ . By Lemma 4.3 [10], exactly one of the following directed paths exists (see Figure 7): (1) from  $v$  to  $x$  in  $G$  (blue bold path in the figure), (2) from  $x$  to  $v$  in  $G$ , (3) from  $\text{right}(v)$  to  $\text{left}(x)$  in  $G^*$  (red dashed bold path in figure), or (4) from  $\text{right}(x)$  to  $\text{left}(v)$  in  $G^*$ . The first case implies that  $\psi(v) < \psi(x)$  and therefore  $x$  is above  $v$  in  $\Gamma$ . The second case implies that the path from  $x$  to  $v$  passes through  $u$ , since  $(u, v)$  is the only incoming edge to  $v$ . Therefore  $\psi(x) < \psi(u)$  and  $x$  lies below  $u$ . In the third case,  $\chi(\text{right}(v)) < \chi(\text{left}(x))$  and in the fourth case,  $\chi(\text{right}(x)) < \chi(\text{left}(v))$ . Thus, there is no vertex  $x$  that prevents the edge  $(u, w)$  from existing in  $\Gamma$ .  $\square$

### 5.1 Limitations on the squares of planar 2-flow networks

We show that while the squares of planar 1-flow networks are WeB1, the squares of some planar 2-flow networks are not.

**Theorem 4** *There exists a planar 2-flow network whose square is not WeB1.*

**Proof:** Consider the graph  $G$  of Figure 8 oriented upward. It consists of a  $\sqrt{n} \times \sqrt{n}$  grid, rotated by  $45^\circ$ . The diagonals are present only in odd rows. Thus,  $G$  is a 2-flow network. Each vertex has out-degree in  $G^2$  indicated by its label in Figure 8. Consider the  $(\sqrt{n} - 2)^2$  vertices that are distance at least two from the upper boundary vertices in  $G$ . At least half of these vertices have out-degree 7 and the others have out-degree 6. Thus  $G^2$  has more than  $\frac{13}{2}(\sqrt{n} - 2)^2$

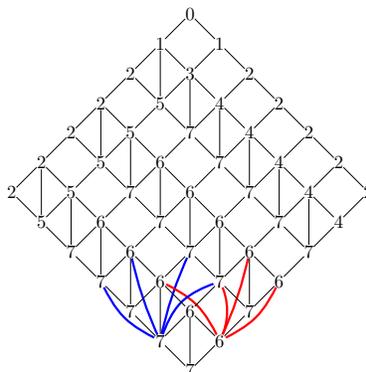


Figure 8: Illustration for the proof of Theorem 4.

edges, which exceeds the upper bound of  $6n - 20$  on the number of edges in a WeB1 graph [9], for sufficiently large  $n$ .  $\square$

## 5.2 Examples for different graph classes related to squares of planar 1-flow networks

The following two lemmata introduce examples of graphs that distinguish certain graph classes in Figure 1.

**Lemma 4**  $K_8$  is not the square of a planar 1-flow network.

**Proof:** Suppose  $G = (V, E)$  is a 1-flow network such that  $G^2 = K_8$ . First, if we view  $G$  as a partial order,  $\preceq$ , it must be a total order otherwise two vertices  $u \not\preceq v$  would not be connected in  $G^2$ . We number the vertices  $v_1 \preceq v_2 \preceq \dots \preceq v_8$  according to the total order so that  $(v_i, v_{i+1}) \in E$ , for all  $1 \leq i \leq 8$ . Observe that, if there exists an index  $i$ ,  $3 \leq i \leq 6$ , such that  $\text{indeg}(v_i) = \text{indeg}(v_{i+1}) = 1$  then  $G^2$  cannot contain the edge  $(v_1, v_{i+1})$ . Similarly, if there exists  $3 \leq i \leq 6$  such that  $\text{outdeg}(v_i) = \text{outdeg}(v_{i+1}) = 1$  then  $G^2$  cannot contain the edge  $(v_i, v_8)$ . These facts imply the following observation which will be repeatedly used in the proof of the lemma.

**Observation 1** For  $3 \leq i \leq 6$  it holds in  $G$  that (1) either  $\text{indeg}(v_i) > 1$  or  $\text{indeg}(v_{i+1}) > 1$ ; and (2) either  $\text{outdeg}(v_i) > 1$  or  $\text{outdeg}(v_{i+1}) > 1$ .

**Statement 1** For  $3 \leq i \leq 6$  it holds in  $G$  that either  $\text{indeg}(v_i) = 1$  and  $\text{outdeg}(v_i) > 1$ , or  $\text{indeg}(v_i) > 1$  and  $\text{outdeg}(v_i) = 1$ .

**Proof:** For the sake of contradiction assume that  $\text{indeg}(v_i) = \text{outdeg}(v_i) = 1$  then, by Observation 1,  $\text{indeg}(v_{i+1}) > 1$  and  $\text{outdeg}(v_{i+1}) > 1$ , which contradicts the fact that  $G$  is a 1-flow network. The same fact is contradicted

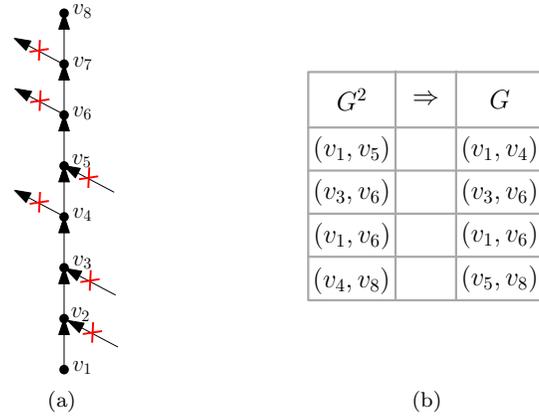


Figure 9: Illustration for the proof of Lemma 4. (a) Edges  $(v_i, v_{i+1})$ ,  $1 \leq i \leq 7$  are in  $G$ . The stroked out edges symbolize the fact that in- or outdegree of the vertex is 1. (b) Edges of  $G^2$  (first column) that imply the existence of the edges of  $G$  (third column).

by assuming that both  $\text{indeg}(v_i) > 1$  and  $\text{outdeg}(v_i) > 1$ , thus the statement follows.  $\square$

We are now ready to prove the following

**Statement 2** *It holds in  $G$  that  $\text{indeg}(v_2) = \text{indeg}(v_3) = \text{indeg}(v_5) = 1$  and  $\text{outdeg}(v_4) = \text{outdeg}(v_6) = \text{outdeg}(v_7) = 1$ .*

**Proof:** Observe that, if  $\text{outdeg}(v_3) = 1$  and  $\text{indeg}(v_6) = 1$  then  $G^2$  cannot contain the edge  $(v_3, v_6)$ . Thus either  $\text{outdeg}(v_3) > 1$  or  $\text{indeg}(v_6) > 1$ . Using Observation 1 and Statement 1 we have the following chain of double implications which can be constructed either from fact that  $\text{outdeg}(v_3) > 1$  or from the fact that  $\text{indeg}(v_6) > 1$ :  $\text{indeg}(v_6) > 1 \xleftrightarrow{\text{Stat.1}} \text{outdeg}(v_6) = 1 \xleftrightarrow{\text{Obs.1}} \text{outdeg}(v_5) > 1 \xleftrightarrow{\text{Stat.1}} \text{indeg}(v_5) = 1 \xleftrightarrow{\text{Obs.1}} \text{indeg}(v_4) > 1 \xleftrightarrow{\text{Stat.1}} \text{outdeg}(v_4) = 1 \xleftrightarrow{\text{Obs.1}} \text{outdeg}(v_3) > 1 \xleftrightarrow{\text{Stat.1}} \text{indeg}(v_3) = 1$ .

The observation that  $\text{indeg}(v_2) = \text{outdeg}(v_7) = 1$  concludes the proof of the statement.  $\square$

From Statement 2 and the fact that  $G^2 = K_8$  we infer that the following edges exist in  $G$  (see Figure 9 for more detail):  $(v_1, v_4)$ ,  $(v_3, v_6)$ ,  $(v_1, v_6)$ ,  $(v_5, v_8)$ . The same statement implies that either  $(v_3, v_7)$  or  $(v_3, v_8)$  is in  $G$ . However,  $(v_3, v_7)$  and  $(v_7, v_8)$ , is a subdivision of  $(v_3, v_8)$ . Thus  $G$  contains either  $(v_3, v_8)$  or its subdivision. Similarly, by Statement 2, either  $(v_1, v_7)$  or  $(v_2, v_8)$  or  $(v_1, v_8)$  is in  $G$ . Since the pairs  $(v_1, v_7)$ ,  $(v_7, v_8)$  and  $(v_1, v_2)$ ,  $(v_2, v_8)$  are both subdivisions of  $(v_1, v_8)$ , we infer that  $G$  contains either  $(v_1, v_8)$  or its subdivision.

So, we infer that  $G$  contains either the following edges or their subdivisions:  $(v_1, v_4)$ ,  $(v_3, v_4)$ ,  $(v_5, v_4)$ ,  $(v_1, v_6)$ ,  $(v_3, v_6)$ ,  $(v_5, v_6)$ ,  $(v_1, v_8)$ ,  $(v_3, v_8)$ ,  $(v_5, v_8)$ .

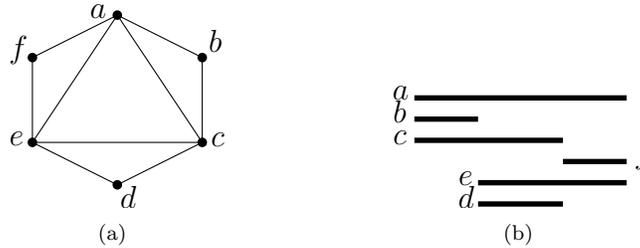


Figure 10: (a) Graph  $S_3$  of Lemma 5. (b) StB1 representation of  $S_3$ .

Thus,  $G$  is non-planar since  $\{v_1, v_3, v_5\}$  and  $\{v_4, v_6, v_8\}$  form a subdivision of  $K_{3,3}$  in  $G$ .  $\square$

Let  $S_3$  denote the graph consisting of a cycle of length 6 with an inscribed triangle (Figure 10.a).

**Lemma 5**  $S_3$  is a planar StB1 graph and is not the square of a 1-flow network.

**Proof:**

A StB1 representation of  $S_3$  is shown in Figure 10(b). In the following we show that there exists no 1-flow network  $G$ , such that  $G^2 = S_3$ . For the sake of contradiction assume such a  $G$  exists. We first assume that  $G$  does not contain all the edges of the external face of  $S_3$ . Without loss of generality assume that  $ab$  is not in  $G$ . Then both  $bc$  and  $ac$  must be in  $G$ . Moreover they must be appropriately directed. Assume that they are directed as  $(b, c)$  and  $(c, a)$  ( $(c, b)$  and  $(a, c)$ , respectively). Then edge  $dc$  is not in  $G$ , since  $(d, c)$  would induce  $(d, a)$  (resp.  $(d, b)$ ) in  $G^2$ , while  $(c, d)$  would induce  $(b, d)$  (resp.  $(a, d)$ ). Thus both edges  $ec$  and  $ed$  must be in  $G$ . Edge  $ec$  must be oriented as  $(e, c)$  (resp.  $(c, e)$ ), otherwise edge  $(b, e)$  (resp.  $(e, b)$ ) is in  $G^2$ . Thus,  $(d, e) \in G$  (resp.  $(e, d) \in G$ ). Similarly, we conclude that  $(a, e) \in G$  (resp.  $(e, a) \in G$ ), and therefore we get a cycle  $ace$  in  $G$ , which is a contradiction to the upward condition of 1-flow networks.

Now, assume that  $G$  contains all the edges of the outer face. We distinguish cases based on the length of the directed paths contained in the outer face. If the longest path has length one then none of the edges  $ae, ac, ec$  are induced in  $G^2$  by outer edge paths, and so at least one must be in  $G$ . But, any orientation of this edge creates an additional edge in  $G^2$ , which does not belong to  $S_3$ .

If there exists a path of length three we get a contradiction, since one of its length two subpaths induces an edge not in  $S_3$ .

Assume there exists a single path of length two, and no path of length three. Then the middle vertex of the path must be  $b, d$ , or  $f$ , otherwise the path induces an edge not in  $S_3$ . Without loss of generality assume that the path is  $abc$ . Then  $fa$  is oriented as  $(a, f)$  and  $dc$  as  $(d, c)$ . Any orientation of  $fe$  and  $ed$  either introduces a path of length three (above case) or two paths of length two (the next case).

Finally, assume there are two paths of length two. They must share a vertex, otherwise one of them induces an edge not in  $S_3$ , and they must be oriented opposite, otherwise a path of length three exists. Without loss of generality we can assume that they are either paths  $efa$  and  $cba$ , or paths  $afe$  and  $abc$ . In case of  $efa$  and  $cba$ , edges  $ed$  and  $cd$  must be oriented as  $(e, d)$  and  $(c, d)$ . Thus edge  $ec$  must be in  $G$ . But any orientation of  $ec$  induces an edge in  $G^2$  that is not in  $S_3$ . Similar facts hold for paths  $afe$  and  $abc$ .  $\square$

## 6 Conclusion and Open Problems

In this paper we investigated the relation of bar 1-visibility graphs with other classes of graphs that are “close to planar” by proving: (i) All 1-planar graphs are WeB1, (ii) All WeB1 graphs are quasi-planar, and (iii) All 1-flow<sup>2</sup> (but not all 2-flow<sup>2</sup>) graphs are WeB1. While these results provide some insight on the class of bar 1-visibility graphs it would be interesting to provide a complete characterization of WeB1 or StB1 graphs. Regarding the relation of WeB1 and  $k$ -flow<sup>2</sup> graphs, what can we say about the squares of planar digraphs, where for each vertex  $v$ , either  $\min\{\text{indeg}(v), \text{outdeg}(v)\} = 1$ , or  $\text{indeg}(v) = \text{outdeg}(v) = 2$  (except for  $v = s, t$ )?

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