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Crossing-Optimal Acyclic HP-Completion for Outerplanar *st*-Digraphs

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Abstract

Given an embedded planar acyclic digraph G, we define the problem of *acyclic hamiltonian path completion with crossing minimization (acyclic-HPCCM)* to be the problem of determining a *hamiltonian path completion set* of edges such that, when these edges are embedded on G, they create the smallest possible number of edge crossings and turn G to a hamiltonian acyclic digraph. Our results include:

1. We provide a characterization under which a planar st-digraph G is hamiltonian.
2. For an outerplanar st-digraph G, we define the *st-Polygon decomposition of* G and, based on its properties, we develop a linear-time algorithm that solves the acyclic-HPCCM problem.
3. For the class of planar st-digraphs, we establish an equivalence between the acyclic-HPCCM problem and the problem of determining an upward 2-page topological book embedding with minimum number of spine crossings. We obtain (based on this equivalence) for the class of outerplanar st-digraphs, an upward topological 2-page book embedding with minimum number of spine crossings.

To the best of our knowledge, it is the first time that edge-crossing minimization is studied in conjunction with the acyclic hamiltonian completion problem and the first time that an optimal algorithm with respect to spine crossing minimization is presented for upward topological book embeddings.

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1 Introduction

In the *hamiltonian path completion problem* (for short, *HP-completion*) we are given a graph G (directed or undirected) and we are asked to identify a set of edges (referred to as an *HP-completion set*) such that, when these edges are embedded on G they turn it to a hamiltonian graph, that is, a graph containing a hamiltonian path.[1] The resulting hamiltonian graph G' is referred to as the *HP-completed graph* of G. When we treat the HP-completion problem as an optimization problem, we are interested in an HP-completion set of minimum size.

When the input graph G is a planar embedded digraph, an HP-completion set for G must be naturally extended to include an embedding of its edges on the plane, yielding to an embedded HP-completed digraph G'. In general, G' is not planar, and thus, it is natural to attempt to minimize the number of edge crossings of the embedding of the HP-completed digraph G' instead of the size of the HP-completion set. We refer to this problem as the *HP-completion with crossing minimization problem* (for short, *HPCCM*).

When the input digraph G is acyclic, we can insist on HP-completion sets which leave the HP-completed digraph G' acyclic. We refer to this version of the problem as the *acyclic HP-completion problem*.

A *k-page book* is a structure consisting of a line, referred to as *spine*, and of k half-planes, referred to as *pages*, that have the spine as their common boundary. A *book embedding* of a graph G is a drawing of G on a book such that the vertices are aligned along the spine, each edge is entirely drawn on a single page, and edges do not cross each other. If we are interested only in two-dimensional constructions/drawings we have to concentrate on 2-page book embeddings and to allow spine crossings. These embeddings are also referred to as 2-page *topological* book embeddings.

For acyclic digraphs, an upward book embedding can be considered to be a book embedding in which the spine is vertical and all edges are drawn monotonically increasing in the upward direction. As a consequence, in an upward book embedding of an acyclic digraph the vertices appear along the spine in topological order.

The results on topological book embeddings that appear in the literature focus on the number of spine crossings per edge required to book-embed a graph on a 2-page book. However, approaching the topological book embedding problem as an optimization problem, it makes sense to also try to minimize the total number of spine crossings.

In this paper, we introduce the problem of *acyclic hamiltonian path completion with crossing minimization* (for short, *acyclic-HPCCM*) for planar embedded acyclic digraphs. To the best of our knowledge, this is the first time that edge-crossing minimization is studied in conjunction with the acyclic HP-completion problem. Then, we provide a characterization under which a planar

[1] In the literature, a *hamiltonian graph* is traditionally referred to as a graph containing a hamiltonian cycle. In this paper, we refer to a hamiltonian graph as a graph containing a hamiltonian path.

st-digraph is hamiltonian. For an outerplanar *st*-digraph G, we define the *st-Polygon decomposition of* G and, based on the decomposition's properties, we develop a linear-time algorithm that solves the acyclic-HPCCM problem.

In addition, for the class of planar *st*-digraphs, we establish an equivalence between the acyclic-HPCCM problem and the problem of determining an upward 2-page topological book embedding with a minimal number of spine crossings. Based on this equivalence, we obtain for the class of outerplanar *st*-digraphs an upward topological 2-page book embedding with minimum number of spine crossings. Again, to the best of our knowledge, this is the first time that an optimal algorithm with respect to spine crossing minimization is presented for upward topological book embeddings.

1.1 Problem Definition

Let $G = (V, E)$ be a graph. Throughout the paper, we use the term *"graph"* when we refer to both directed and undirected graphs. We use the term *"digraph"* when we want to restrict our attention to directed graphs. We assume familiarity with basic graph theory [9, 14]. A *hamiltonian path* of G is a path that visits every vertex of G exactly once. Determining whether a graph has a hamiltonian path or circuit is NP-complete [12]. The problem remains NP-complete for cubic planar graphs [12], for maximal planar graphs [34] and for planar digraphs [12]. It can be trivially solved in polynomial time for planar acyclic digraphs.

Given a graph $G = (V, E)$, directed or undirected, a non-negative integer $k \leq |V|$ and two vertices $s,\ t \in V$, *the hamiltonian path completion (HPC)* problem asks whether there exists a superset E' of E such that $|E' \setminus E| \leq k$ and the graph $G' = (V, E')$ has a hamiltonian path from vertex s to vertex t. We refer to G' and to the set of edges $E' \setminus E$ as the *HP-completed graph* and the *HP-completion set* of graph G, respectively. We assume that all edges of an HP-completion set are part of the Hamiltonian path of G', since otherwise they can be removed. When G is a directed acyclic graph, we can insist on HP-completion sets which leave the HP-completed digraph also acyclic. We refer to this version of the problem as the *acyclic HP-completion problem*. The hamiltonian path completion problem is NP-complete [11]. For acyclic digraphs the HPC problem is solved in polynomial time [18].

A *drawing* Γ of graph G maps every vertex v of G to a distinct point $p(v)$ on the plane and each edge $e = (u, v)$ of G to a simple open curve joining $p(u)$ with $p(v)$. A drawing in which every edge (u, v) is a simple open curve monotonically increasing in the vertical direction is an *upward drawing*. A drawing Γ of graph G is *planar* if no two distinct edges intersect. Graph G is called *planar* if it admits a planar drawing Γ. Given a planar drawing Γ of a planar graph G, the set of points of the plane that can be connected by a curve that does not intersect any vertex or edge of the drawing are said to belong to the same *face*. Each face of a drawing can be indicated by the sequence of edges that surround it.

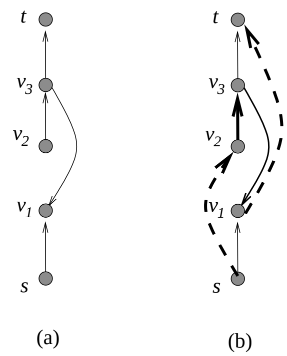

Figure 1: An acyclic digraph and a minimum acyclic HP-completion set that is not a minimum HP-completion set. Edge (v_1, v_2) forms a minimum HP-completion set.

An embedding of a planar graph G is the equivalence class of planar drawings of G that define the same set of faces or, equivalently, of face boundaries. A planar graph together with the description of a set of faces F is called an *embedded planar graph*.

Let $G = (V, E)$ be an embedded planar graph, E' be a superset of edges containing E, and $\Gamma(G')$ be a drawing of $G' = (V, E')$. When the deletion from $\Gamma(G')$ of the edges in $E' \setminus E$ induces the embedded planar graph G, we say that $\Gamma(G')$ *preserves the embedded planar graph* G.

Definition 1 *Given an embedded planar graph $G = (V, E)$, directed or undirected, a non-negative integer c, and two vertices $s,\ t \in V$, the* hamiltonian path completion with edge crossing minimization (HPCCM) problem *asks whether there exists a superset E' of E and a drawing $\Gamma(G')$ of graph $G' = (V, E')$ such that (i) G' has a hamiltonian path from vertex s to vertex t, (ii) $\Gamma(G')$ has at most c edge crossings, and (iii) $\Gamma(G')$ preserves the embedded planar graph G.*

We refer to the version of the HPCCM problem where the input is an acyclic digraph and we are interested in HP-completion sets which leave the HP-completed digraph also acyclic as the *acyclic-HPCCM* problem. Figure 1.a shows an acyclic digraph that has a minimum *acyclic* HP-completion set consisting of two edges (shown dashed in Figure 1.b) which is not a minimum HP-completion set.

Over the set of all HP-completion sets for a graph G, and over all of their different drawings that preserve G, any set with a minimum number of edge-crossings is called a *crossing-optimal HP-completion set*.

Note that an acyclic HP-completion set of minimum size is not necessarily a crossing-optimal HP-completion set. This fact is demonstrated in Figure 2. For the non-triangulated outerplanar *st*-digraph of Figure 2.a, every acyclic HP-completion set of size 1 creates 1 edge crossing (see Figure 2.b) while, it is possible to obtain an acyclic HP-completion set of size 2 without any crossings (see Figure 2.c).

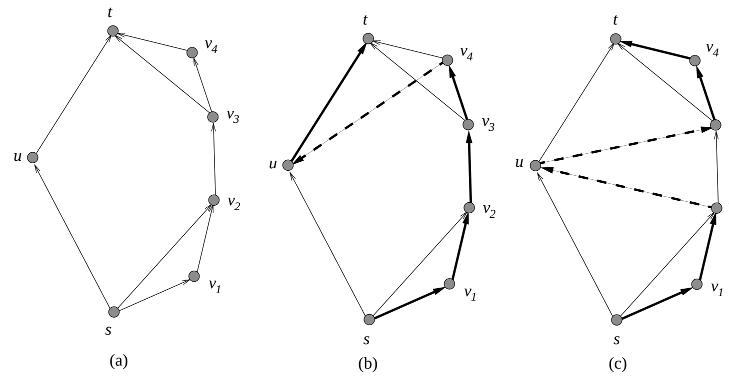

Figure 2: An acyclic digraph that has a crossing-optimal HP-completion set of size 2 that creates no crossings. Any HP-completion set of size 1 creates 1 crossing.

Let $G = (V, E)$ be an embedded planar graph, let E_c be an HP-completion set of G and let $\Gamma(G')$ of $G' = (V, E \cup E_c)$ be a drawing with c crossings that preserves G. The graph G_c induced from drawing $\Gamma(G')$ by inserting a new vertex at each edge crossing and by splitting the edges involved in the edge-crossing is referred to as the *HP-extended graph of G with respect to* $\Gamma(G')$ (see Figure 3).

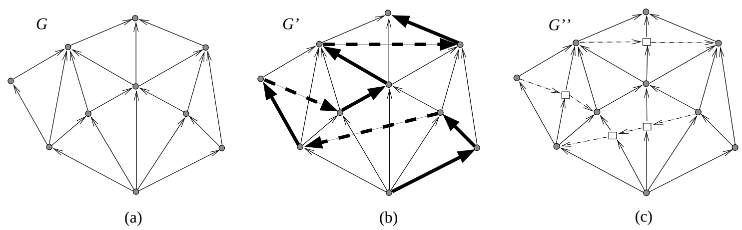

Figure 3: (a) A planar embedded digraph G. (b) A drawing $\Gamma(G')$ of an HP-completed digraph G' of G. The edges of the hamiltonian path of G' appear bold, with the edges of the HP-completion set shown dashed. (c) The HP-extended digraph G'' of G with respect to $\Gamma(G')$. The newly inserted vertices appear as squares.

In this paper, we present a linear time algorithm for solving the acyclic-HPCCM problem for outerplanar *st*-digraphs. A planar graph G is *outerplanar* if there exists a planar drawing of G such that all the vertices of G appear on the boundary of the same face (which is usually drawn as the external face). Let $G = (V, E)$ be a digraph. A vertex of G with in-degree equal to zero is called a *source*, while, a vertex of G with out-degree equal to zero is called a *sink*. An *st-digraph* is an acyclic digraph with exactly one source and exactly one sink. Traditionally, the source and the sink of an *st*-digraph are denoted by

s and t, respectively. An st-digraph which is planar (respectively outerplanar) and, in addition, it is embedded on the plane so that both of its source and its sink appear on the boundary of its external face, is referred to as a *planar st-digraph* (respectively an *outerplanar st-digraph*). It is known that a planar st-digraph admits a planar upward drawing [19, 7]. In the rest of the paper, all st-digraphs will be drawn upward.

1.2 Related Work

For acyclic digraphs, the acyclic-HPC problem has been studied in the literature in the context of partially ordered sets (posets) under the terms *Linear Extensions* and *Jump Number*. Each acyclic digraph G can be treated as a poset P. A linear extension of P is a total ordering $L = \{x_1 \dots x_n\}$ of the elements of P such that $x_i < x_j$ in L whenever $x_i < x_j$ in P. We denote by $L(P)$ the set of all linear extensions of P. A pair (x_i, x_{i+1}) of consecutive elements of L is called a *jump in* L if x_i is not comparable to x_{i+1} in P. Denote the number of jumps of L by $s(P, L)$. Then, the *jump number* of P, $s(P)$, is defined as $s(P) = \min\{s(P, L) : L \in L(P)\}$. A linear extension $L \in L(P)$ is called *optimal* if $s(P, L) = s(P)$. The *jump number problem* is to find $s(P)$ and to construct an optimal linear extension of P.

From the above definitions, it follows that an optimal linear extension of a poset P (or its corresponding acyclic digraph G), is identical to an acyclic HP-completion set E_c of minimum size for G, and its jump number is equal to the size of E_c. This problem has been widely studied, in part due to its applications to scheduling. It has been shown to be NP-hard even for bipartite ordered sets [28] and for the class of interval orders [24]. Up to our knowledge, its computational classification is still open for lattices. Nevertheless, polynomial time algorithms are known for several classes of ordered sets. For instance, efficient algorithms are known for series-parallel orders [4], N-free orders [29], cycle-free orders [6], orders of width two [3], orders of bounded width [5], bipartite orders of dimension two [31] and K-free orders [30]. Brightwell and Winkler [2] showed that counting the number of linear extensions is ♯P-complete. An algorithm that generates all linear extensions of a poset in constant amortized time, that is in time $\mathcal{O}(|L(P)|)$, was presented by Pruesse and Ruskey [27]. Later on, Ono and Nakano [26] presented an algorithm which generates each linear extension in "worst case" constant time.

With respect to related work on book embeddings, Yannakakis [35] has shown that planar graphs have a book embedding on a 4-page book and that there exist planar graphs that require 4 pages for their book embedding. Thus, book embeddings for planar graphs are, in general, three-dimensional structures. If we are interested only on two-dimensional structures we have to concentrate on 2-page book embeddings and to allow spine crossings. In the literature, the book embeddings where spine crossings are allowed are referred to as *topological book embeddings* [10]. It is known that every planar graph admits a 2-page topological book embedding with only one spine crossing per edge [8].

For acyclic digraphs and posets, *upward book embeddings* have also been studied in the literature [1, 15, 16, 17, 25]. An upward book embedding can be considered to be a book embedding in which the spine is vertical and all edges are drawn monotonically increasing in the upward direction. The minimum number of pages required by an upward book embedding of a planar acyclic digraph is unbounded [15], while the minimum number of pages required by an upward planar digraph is not known [1, 15, 25]. Giordano et al. [13] studied *upward topological book embeddings* of embedded upward planar digraphs, i.e., topological 2-page book embedding where all edges are drawn monotonically increasing in the upward direction. They have showed how to construct in linear time an upward topological book embedding for an embedded triangulated planar *st*-digraph with at most one spine crossing per edge. Given that (i) upward planar digraphs are exactly the subgraphs of planar *st*-digraphs [7, 19] and (ii) embedded upward planar digraphs can be augmented to become triangulated planar *st*-digraphs in linear time [13], it follows that any embedded upward planar digraph has a topological book embedding with one spine crossing per edge.

We emphasize that the presented bibliography is in no way exhaustive. The topics of *hamiltonian paths*, *linear orderings* and *book embeddings* have been studied for a long time and an extensive body of literature has been accumulated.

1.3 Our Results

Preliminary versions of the results presented in this paper have appeared in [22] and [23]. In [20, 22] we reported a linear time algorithm that solves the acyclic-HPCCM problem for the class of outerplanar *triangulated st*-digraph provided that *each edge of the initial graph can be crossed at most once* by the edge of the crossing-optimal HP-completion set. Figure 4.a gives an example of an outerplanar triangulated *st*-digraph for which an HP-completion set with smaller number of crossings can be found if there is no restriction on the number of crossings per edge. In particular, the *st*-digraph becomes hamiltonian by adding one of the following completion sets: $A = \{(v_4, u_1)\}$, $B = \{()u_8, v_1\}$ or $C = \{(u_3, v_1), (v_4, u_4)\}$ (see Figures 4.b-d). Each of sets A and B creates 5 crossings with one crossing per edge of G while, set C creates 4 crossings with at most 2 crossings per edge of G. In addition to relaxing the restriction of at most one crossing per edge of the *st*-digraph, the algorithm presented in this paper does not require its input outerplanar *st*-digraph to be triangulated, extending in this way the class of graphs for which we are able to compute a crossing-optimal HP-completion set.

In this paper, we show that (i) for any *st*-Polygon (i.e., an outerplanar *st*-digraph with no edge connecting its two opposite sides) there is always a crossing-optimal acyclic HP-completion set of size at most 2 (Section 3.1, Theorem 2), and, (ii) any crossing-optimal acyclic HP-completion set for an outerplanar *st*-digraph G creates at most 2 crossings per edge of G (Section 3.3, Theorem 4). Based on these properties and the introduced *st*-Polygon decomposition of an outerplanar *st*-digraph (Section 3.2), we derive a linear time algorithm that solves the acyclic-HPCCM problem for outerplanar *st*-digraphs.

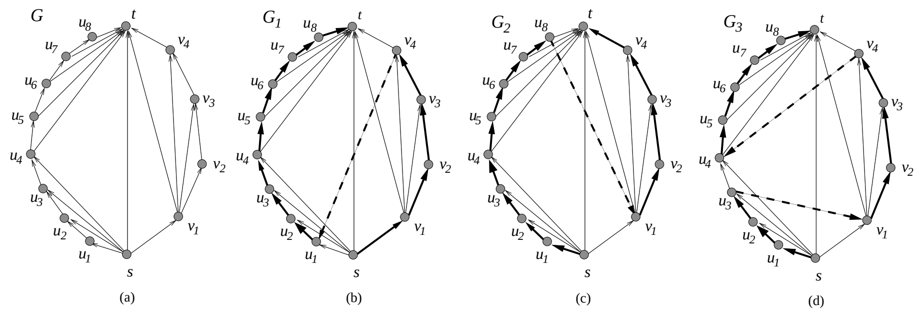

Figure 4: Two crossings per edge are required in order to minimize the total number of crossings. The edges of the HP-completion sets appear dashed. The resulting hamiltonian paths are shown in bold.

We also establish an equivalence between the acyclic-HPCCM problem and the problem of determining an upward 2-page topological book embedding with a minimal number of spine crossings. Based on this equivalence and the algorithm presented in the paper, we can obtain for the class of outerplanar *st*-digraphs an upward topological 2-page book embedding with minimum number of spine crossings. This is the first time that an optimal algorithm with respect to spine crossing minimization is presented for upward topological book embeddings without restrictions to the number of crossings per edge.

Recently, the acyclic-HPCCM problem has been solved efficiently for the classes of N-free and bounded-width upward planar digraphs [21].

2 Hamiltonian Planar *st*-Digraphs

In this section, we develop a necessary and sufficient condition for a planar *st*-digraph to be hamiltonian. The provided characterization will be later on used in the development of crossing-optimal HP-completion sets for outerplanar *st*-digraphs.

It is well known [33] that for every vertex v of a planar *st*-digraph, its incoming (outgoing) incident edges appear consecutively around v. For any vertex v, we denote by $Left(v)$ (respectively $Right(v)$) the face to the left (respectively to the right) of the leftmost (respectively rightmost) incoming and outgoing edges incident to v. For any edge $e = (u, v)$, we denote by $Left(e)$ (respectively $Right(e)$) the face to the left (respectively to the right) of edge e as we move from u to v. The *dual* of an *st*-digraph G, denoted by G^*, is a digraph such that: (i) there is a vertex in G^* for each face of G; (ii) for every edge $e \neq (s, t)$ of G, there is an edge $e^* = (f, g)$ in G^*, where $f = Left(e)$ and $g = Right(e)$; (iii) edge (s^*, t^*) is in G^*. The following lemma is a direct consequence of Lemma 7 of Tamassia and Preparata [32].

Lemma 1 *Let u and v be two vertices of a planar st-digraph such that there is no directed path between them in either direction. Then, in the dual G^* of*

G there is either a path from $Right(u)$ to $Left(v)$ or a path from $Right(v)$ to $Left(u)$. □

The following lemma demonstrates a property of planar *st*-digraphs.

Lemma 2 *Let G be a planar st-digraph that does not have a hamiltonian path. Then, there exist two vertices in G that are not connected by a directed path in either direction.*

Proof: Let P be a longest path from s to t and let a be a vertex that does not belong to P. Since G does not have a hamiltonian path, such a vertex always exists. Let s' be the last vertex in P such that there exists a path $P_{s' \rightsquigarrow a}$ from s' to a with no vertices in P. Similarly, define t' to be the first vertex in P such that there exists a path $P_{a \rightsquigarrow t'}$ from a to t' with no vertices in P. Since G is acyclic, s' appears before t' in P (see Figure 5). Note that s' (respectively t') might be vertex s (respectively t). From the construction of s' and t' it follows that any vertex b, distinct from s' and t', that is located on path P between vertices s' and t', is not connected to vertex a in either direction. Thus, vertices a and b satisfy the property of the lemma.

Note that such a vertex b always exists. If this was not the case, then path P would contain edge (s', t'). Then, path P could be extended by replacing (s', t') by path $P_{s' \rightsquigarrow a}$ followed by path $P_{s' \rightsquigarrow a}$. This would lead to new path P' from s to t that is longer than P, and this would be a contradiction since P was assumed to be of maximum length. □

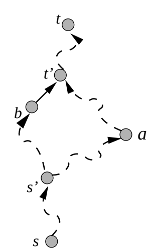

Figure 5: The subgraph used in the proof of Lemma 2. Vertices a and b are not connected by a path in either direction.

Every face of a planar *st*-digraph consists of two sides, each of them directed from its source to its sink. When one side of the face is a single edge and the other side (the longest) contains exactly one vertex, the face is referred to as a *triangle* (see Figure 6). In the case where the longest edge contains more than one vertex, the face is referred to as a *generalized triangle* (see Figure 7). We call both a triangle and a generalized triangle *left-sided* (respectively *right-sided*) if its left (respectively right) side is its longest side, i.e., it contains at least one vertex.

The outerplanar *st*-digraph of Figure 8 is called a *strong rhombus*. It consists of two generalized triangles (one left-sided and one right-sided) which have their

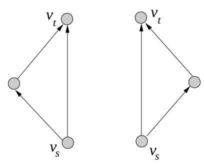

Figure 6: Left and right-sided embedded triangles.

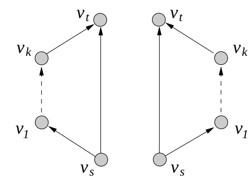

Figure 7: Left and right-sided embedded generalized triangles.

(v_s, v_t) edge in common. The edge (v_s, v_t) of a strong rhombus is referred to as its *median* and is drawn in the interior of its drawing. The outerplanar *st*-digraph resulting by deleting the median of a strong rhombus is referred to as a *weak rhombus*. Thus, a weak rhombus is an outerplanar *st*-digraph consisting of a single face that has at least one vertex at each of its sides (see Figure 9). We use the term *rhombus* to refer to either a strong or a weak rhombus. The following

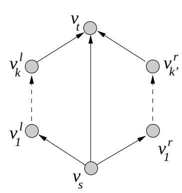

Figure 8: A strong rhombus.

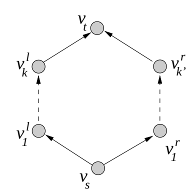

Figure 9: A weak rhombus.

theorem provides a characterization of *st*-digraphs that have a hamiltonian path.

Theorem 1 *Let G be a planar st-digraph. Then G has a hamiltonian path if and only if G does not contain any rhombus (strong or weak) as a subgraph.*

Proof: $(\Rightarrow)$ We assume that G has a hamiltonian path and we show that it

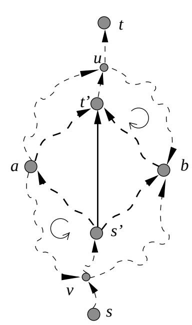

Figure 10: The subgraph containing a rhombus which is used in the proof of Theorem 1. In the case of a weak rhombus, edge (s', t') is not present.

contains no rhombus (strong or weak) as an embedded subgraph. For the sake of contradiction, assume first that G contains a strong rhombus with vertices s' (its source), t' (its sink), a (on its left side) and b (on its right side) (see Figure 10). Then, vertices a and b of the strong rhombus are not connected by a directed path in either direction. To see this, assume without loss of generality that there was a path connecting a to b. Then, this path has to lie outside the rhombus and intersect either the path from t' to t at a vertex u or the path from s to s' at a vertex v. In either case, there must exist a cycle in G, contradicting the fact that G is acyclic.

Assume now, for the sake of contradiction again, that G contains a weak rhombus characterized by vertices s', t', a, and b. Then, by using the same argument as above, we conclude that vertices a and b of the weak rhombus are not connected by a directed path that lies outside the rhombus in either direction. Note also that vertices a and b cannot be connected by a path that lies in the internal of the weak rhombus since the weak rhombus consists, by definition, of a single face.

So, we have shown that vertices a and b of the rhombus (strong or weak) are not connected by a directed path in either direction, and thus, there cannot exist any hamiltonian path in G, a clear contradiction.

($\Leftarrow$) We assume that G contains neither a strong nor a weak rhombus as an embedded subgraph and we prove that G has a hamiltonian path. For the sake of contradiction, assume that G does not have a hamiltonian path. Then, from Lemma 2, it follows that there exist two vertices u and v of G that are not connected by a directed path in either direction. From Lemma 1, it then follows that there exists in the dual G^* of G a directed path from either $Right(u)$ to $Left(v)$, or from $Right(v)$ to $Left(u)$. Without loss of generality, assume that the path in the dual G^* is from $Right(u)$ to $Left(v)$ (see Figure 11.a) and let $f_0,\ f_1,\ \ldots,\ f_k$ be the faces the path passes through, where $f_0 = Right(u)$ and $f_k = Left(v)$. We denote the path from $Right(u)$ to $Left(v)$ by $P_{u,v}$. Note that each face of digraph G and therefore of path $P_{u,v}$ is a generalized triangle, because we supposed that G does not contain any weak rhombus.

Note that path $P_{u,v}$ can exit face f_0 only through the solid edge (see Figure 11.a). The path then enters a new face and, in the rest of the proof, we construct the sequence of faces it goes through.

The next face f_1 of the path, consists of the solid edge of face f_0 and some other edges. There are two possible cases to consider for face f_1.

Case 1: Face f_1 is left-sided. Then, path $P_{u,v}$ enters f_1 through one of the edges on its left side (see Figures 11.b, 11.c and 11.d for possible configurations). Observe that, since f_1 is left-sided, f_1 has only one outgoing edge in G^*. Thus, in all of these cases, the only edge through which path $P_{u,v}$ can leave f_1 is the single edge on the right side of the generalized triangle f_1.

Case 2: The face f_1 is right-sided. Then the only edge through which the path $P_{u,v}$ can enter f_1 is the only edge of its left side (see Figure 11.e). Note that in this case, f_0 and f_1 form a strong rhombus. Thus, this case cannot occur, since we assumed that G has no strong rhombus as an embedded subgraph.

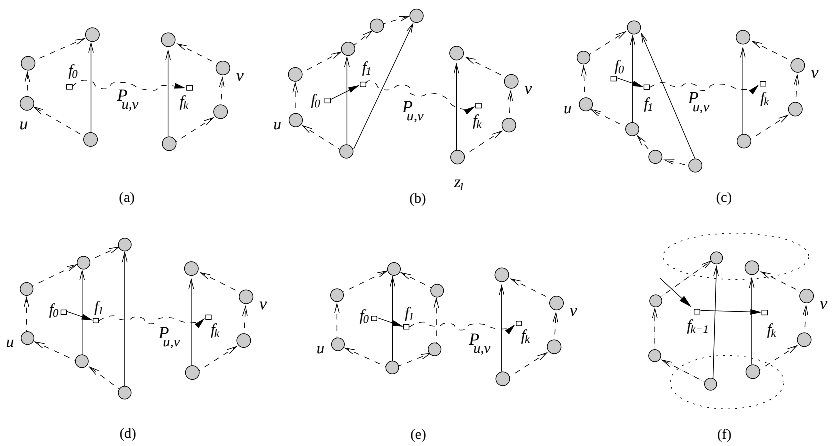

Figure 11: The different cases occurring in the construction of path $P_{u,v}$ as described in the proof of Theorem 1.

A characteristic property of the first case that allows to further continue the identification of the faces path $P_{u,v}$ goes through is that there is *a single* edge that exits face f_1. Thus, we can continue identifying the faces path $P_{u,v}$ passes through, and build a unique sequence $f_0,\ f_1,\ \ldots,\ f_{k-1}$ in this way. Note that all of these faces are left-sided (otherwise, G contains a strong rhombus).

At the end, path $P_{u,v}$ has to leave the left-sided face f_{k-1} and enter the right-sided face f_k. As the only way to enter a right-sided face is to cross the single edge on its left side, we have that the single edge on the right side of f_{k-1} and the single edge on the left side of f_k coincide forming a strong rhombus (see Figure 11.f). This is a clear contradiction since we assumed that G has no strong rhombus as an embedded subgraph. □

3 Optimal Acyclic Hamiltonian Path Completion for Outerplanar st-digraphs

In this section we present an algorithm that computes a crossing-optimal acyclic HP-completion set for an outerplanar *st*-digraph. Let $G = (V^l \cup V^r \cup \{s,t\}, E)$ be an outerplanar *st*-digraph, where s is its source, t is its sink and the vertices in V_l (respectively V_r) are located on the left (respectively right) side of the boundary of the external face. Let $V^l = \{v_1^l,\ \ldots, v_k^l\}$ and $V^r = \{v_1^r,\ \ldots, v_m^r\}$, where the subscripts indicate the order in which the vertices appear on the left (right) side of the external boundary. By convention, the source and the sink are considered to lie on both the left and the right sides of the external boundary. Observe that each face of G is also an outerplanar *st*-digraph. We refer to an edge that has both of its end-vertices on the same side of G as an *one-sided* edge. All remaining edges are referred to as *two-sided* edges. The edges exiting the source and the edges entering the sink are treated as one-sided edges.

The following lemma presents an essential property of an acyclic HP-completion set of an outerplanar *st*-digraph G.

Lemma 3 *An acyclic HP-completion set of an outerplanar st-digraph* $G = (V^l \cup V^r \cup \{s,t\}, E)$ *induces a hamiltonian path that visits the vertices of* V_l *(respectively* V_r*) in the order they appear on the left side (respectively right side) of* G.

Proof: Let E_c be an acyclic HP-completion set for G and let G_c be the induced HP-completed acyclic digraph. Consider two vertices v_1 and v_2 that appear, in this order, on the same side (left or right) of G. Then, in G there is a path P_{v_1,v_2} from v_1 to v_2 since each side of an outerplanar *st*-digraph is a directed path from its source to its sink. For the sake of contradiction, assume that v_2 appears before v_1 in the hamiltonian path induced by the acyclic HP-completion set of G. Then, the hamiltonian path contains a sub-path P_{v_2,v_1} from v_2 to v_1. Thus, paths P_{v_1,v_2} and P_{v_2,v_1} form a cycle in G_c. This is a contradiction, since G_c is acyclic. □

3.1 st-Polygons

A *strong st-Polygon* is an outerplanar *st*-digraph that has at least one vertex at each side and always contains edge (v_s, v_t) connecting its source v_s to its sink v_t (see Figure 12). Edge (v_s, v_t) is referred to as its *median* and it always lies in the interior of its drawing. As a consequence, in a strong *st*-Polygon no edge connects a vertex on its left side to a vertex on its right side. The outerplanar *st*-digraph that results from the deletion of the median of a strong *st*-Polygon is referred to as a *weak st-Polygon* (see Figure 13). We use the term *st-Polygon* to refer to both a strong and a weak *st*-Polygon. Observe that an *st*-Polygon has at least 4 vertices.

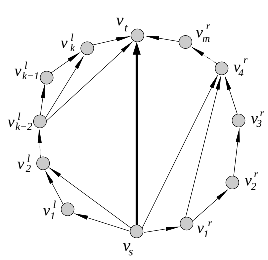

Figure 12: A strong *st*-Polygon.

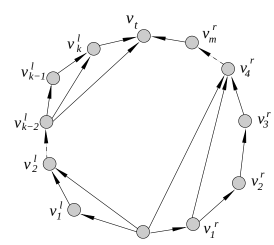

Figure 13: A weak *st*-Polygon.

Consider an outerplanar *st*-digraph G and one of its embedded subgraphs G_p that is an *st*-Polygon (strong or weak). G_p is called a *maximal st-Polygon* if it cannot be extended (and still remains an *st*-Polygon) by the addition of more vertices to its external boundary. In Figure 14, the *st*-Polygon $G_{a,d}$ with vertices a (source), b, c, d (sink), e, and f on its boundary is not maximal

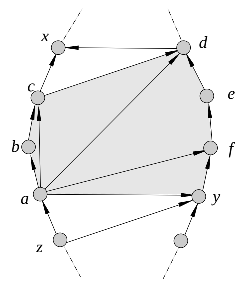

Figure 14: The *st*-Polygon with vertices a (source), b, c, d (sink), e, f, and y on its boundary is maximal.

since the subgraph $G'_{a,d}$ obtained by adding vertex y to it is still an *st*-Polygon. However, the *st*-Polygon $G'_{a,d}$ is maximal since the addition of either vertex x or z to it does not yield another *st*-Polygon.

Observe that an *st*-Polygon that is a subgraph of an outerplanar *st*-digraph G fully occupies a *"strip"* of it that is limited by two edges (one adjacent to its source and one to its sink), each having its endpoints at different sides of G. We refer to these two edges as the *limiting edges* of the *st*-Polygon. Note that the limiting edges of an *st*-Polygon that is an embedded subgraph of an outerplanar graph are sufficient to define it. In Figure 14, the maximal *st*-Polygon with vertex a as its source and vertex d as its sink is limited by edges (a, y) and (c, d).

Lemma 4 *An st-Polygon contains exactly one rhombus.*

Proof: Let G_p be a weak *st*-Polygon. By definition it contains a weak rhombus. Suppose that this is not the only weak rhombus contained in G_p and let R be a second one. As G_p is an outerplanar graph and does not contain edges connecting its two opposite sides, we have that all the vertices of R must lie on the same side of G_p, say its left side. But then we have that the sink of R is another sink in G_p or that the source of R is another source of G_p (see Figure 15). This contradicts the fact that G_p is an *st*-Polygon. Suppose now that R is a strong rhombus. This case also leads to a contradiction, as R can be converted to a weak rhombus by deleting its median.

If G_p is a strong *st*-Polygon, then by the same argument we show that G_p cannot contain a second rhombus (strong or weak). □

The following lemmata are concerned with a crossing-optimal acyclic HP-completion set for a single *st*-Polygon. They state that there exists a crossing-optimal acyclic HP-completion set containing at most two edges.

Lemma 5 *Let* $R = (V^l \cup V^r \cup \{s, t\}, E)$ *be an st-Polygon. Let* P *be an acyclic HP-completion set for* R *such that* $|P| = 2\mu + 1$, $\mu \in \mathbb{N}$. *Then, there exists another acyclic HP-completion set* P' *for* R *such that* $|P'| = 1$ *and the edge in* P' *creates at most as many crossings with the edges of* R *as the edges in* P *do.*

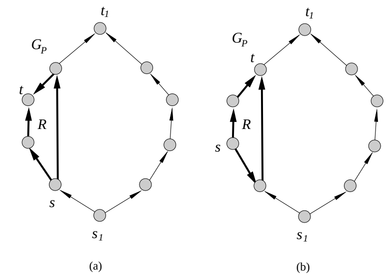

Figure 15: Two possible ways for the embedding of a second rhombus into an *st*-Polygon. Both lead to a configuration that contradicts the definition of an *st*-Polygon.

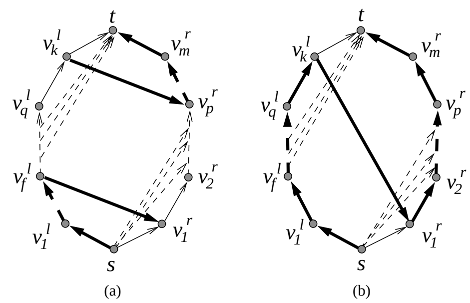

Figure 16: An acyclic HP-completion set of odd size for an *st*-Polygon and an equivalent acyclic HP-completion set of size 1.

In addition, the hamiltonian paths induced by P and P' have in common their first and last edges.

Proof: First observe that, as a consequence of Lemma 3, any acyclic HP-completion set for R does not contain any one-sided edge. Thus, all $2\mu + 1$ edges of P are two-sided edges. Moreover, since P contains an odd number of edges, both the first and the last edge of P have the same direction[2]. Without loss of generality, let the lowermost edge of P be directed from left to right (see Figure 16(a)). By Lemma 3, it follows that the destination of the lowermost edge of P is the lowermost vertex on the right side of R (i.e., vertex v_1^r) while the origin of the topmost edge of P is the topmost vertex of the left side of R (i.e., vertex v_k^l).

Observe that $P' = \{(v_k^l, v_1^r)\}$ is an acyclic HP-completion set for R. The induced hamiltonian path is $(s \dashrightarrow v_k^l \rightarrow v_1^r \dashrightarrow t)$.[3]

[2]Two two-sided edges of an *st*-Polygon are sail to have the *same direction* if their origins lie at the same side of the *st*-Polygon. Otherwise, they are said to have *opposite directions*.

[3]A dashed-arrow "$\dashrightarrow$" indicates a path that is on the left or the right side of an *st*-Polygon (or outerplanar graph) and might contain intermediate vertices.

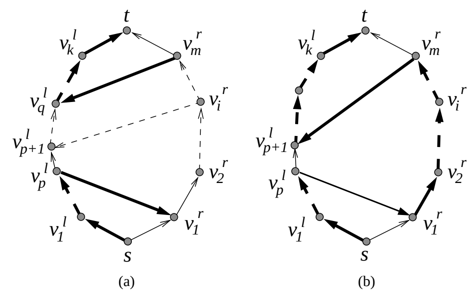

Figure 17: An acyclic HP-completion set of even size for an st-Polygon and an equivalent acyclic HP-completion set of size 2.

In order to complete the proof, we show that edge (v_k^l, v_1^r) does not cross more edges of R than the edges in P do. To see that, observe that edge (v_k^l, v_1^r) crosses all edges in set $\{(s,v) : v \in V^r \setminus \{v_1^r\}\}$ as well as all edges in set $\{(v,t) : v \in V^l \setminus \{v_m^l\}\}$, provided they exist (see Figure 16(b)). However, the edges in these two sets are also crossed by the lowermost and the topmost edges in P, respectively. Thus, edge (v_k^l, v_1^r) creates at most as many crossings with the edges of R as the edges in P do. Observe also that the hamiltonian paths induced by P and P' have in common their first and last edges. □

Lemma 6 *Let $R = (V^l \cup V^r \cup \{s,t\}, E)$ be an st-Polygon. Let P be an acyclic HP-completion set for R such that $|P| = 2\mu,\ \mu \in \mathbb{N},\ \mu \geq 1$. Then, there exists another acyclic HP-completion set P' for R such that $|P'| = 2$ and the edges in P' create at most as many crossings with the edges of R as the edges in P do. In addition, the hamiltonian paths induced by P and P' have in common their first and last edges.*

Proof: As in the case of an HP-completion set of odd size (Lemma 5), the 2μ edges in P are two-sided edges. Moreover, since P contains an even number of edges, the first and the last edge of P have opposite direction. Without loss of generality, let the lowermost edge in P be directed from left to right (see Figure 17.a). By Lemma 3, it follows that the destination of the lowermost edge in P is the lowermost vertex on the right side of R (i.e., vertex v_1^r) while the origin of the topmost edge in P is the topmost vertex of the right side of R (i.e., vertex v_m^r). Let the lowermost edge in P be (v_p^l, v_1^r). Then, again from Lemma 3, it follows that the HP-completion set P also contains edge (v_i^r, v_{p+1}^l) for some $1 < i \leq m$. If $i = m$, then P contains exactly 2 edges and lemma is trivially true. So, we consider the case where $i < m$.

Observe that, for the case where $|P| > 3$, the set of edges $P' = \{(v_p^l, v_1^r), (v_m^r, v_{p+1}^l)\}$ is an acyclic HP-completion set for R (see Figure 17.b). The induced hamiltonian path is $(s \dashrightarrow v_p^l \rightarrow v_1^r \dashrightarrow v_m^r \rightarrow v_{p+1}^l \dashrightarrow t)$.

In order to complete the proof, we show that edges (v_p^l, v_1^r) and (v_m^r, v_{p+1}^l) do not cross more edges of R than the edges in P do. The edges of E that are crossed by the two edges in P' can be classified in the following disjoint groups:

a) *Edges having their origin below edge* (v_p^l, v_1^r) *and their destination above edge* (v_m^r, v_{p+1}^l). All of these edges are crossed by both edges in P'. But, they are also crossed by at least edges (v_p^l, v_1^r) and (v_i^r, v_{p+1}^l) in P.

b) *Edges having their origin below edge* (v_p^l, v_1^r) *and their destination between edges* (v_p^l, v_1^r) *and* (v_m^r, v_{p+1}^l). All of these edges are only crossed by edge (v_p^l, v_1^r) in P'. But, (v_p^l, v_1^r) also belongs in P.

c) *Edges having their origin between edges* (v_p^l, v_1^r) *and* (v_m^r, v_{p+1}^l) *and their destination above edge* (v_m^r, v_{p+1}^l). All of these edges are only crossed by edge (v_m^r, v_{p+1}^l) in P'. But, they are also crossed by at least the topmost edge (v_m^r, v_q^l) in P.

Thus, the edges in P' create at most as many crossings with the edges of R as the edges of P do. Observe also that the hamiltonian paths induced by P and P' have in common their first and last edges. □

The following theorem follows directly from Lemma 5 and Lemma 6.

Theorem 2 *Any st-Polygon has a crossing optimal acyclic HP-completion set of size at most 2.* □

3.2 *st*-Polygon decomposition of an outerplanar *st*-digraph

Lemma 7 *Let* $G = (V^l \cup V^r \cup \{s,t\}, E)$ *be an outerplanar st-digraph and* $e = (s', t') \in E$ *be an arbitrary edge. Denote by* V *the vertex set of* G. *If* $O(V)$ *time is available for the preprocessing of* G, *we can decide in* $O(1)$ *time whether* e *is a median edge of some strong st-Polygon. Moreover, the two vertices (in addition to* s' *and* t'*) that define a maximal strong st-Polygon having edge* e *as its median can also be computed in* $O(1)$ *time.*

Proof: We can preprocess graph G in linear time so that for each of its vertices we know the first and last (in clock-wise order) in-coming and out-going edges. Observe that an one-sided edge (u, v) is a median of a strong *st*-Polygon if and only if the following hold (see Figure 18.a):

a) u and v are not successive vertices of the side of G.

b) u has a two-sided outgoing edge.

c) v has a two-sided incoming edge.

Similarly, observe that a two-sided edge (u, v) with $u \in V^R$ (respectively $u \in V^L$) is a median of a strong *st*-Polygon if and only if the following hold (see Figure 18.b):

a) u has a two-sided outgoing edge that is clock-wise before (respectively after) (u, v).

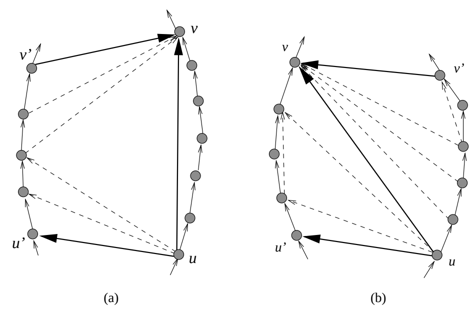

Figure 18: *st*-Polygons with one-sided and two-sided medians. The median and the two edges that bound the *st*-Polygons are shown in bold.

b) v has a two-sided incoming edge that is clock-wise before (respectively after) (u, v).

All of the above conditions can be trivially tested in $O(1)$ time. Then, the two remaining vertices that define the maximal strong *st*-Polygon having (u, v) as its median can be found in $O(1)$ time and, moreover, the strong *st*-Polygon can be reported in time proportional to its size. □

Lemma 8 *Let $G = (V^l \cup V^r \cup \{s, t\}, E)$ be an outerplanar st-digraph and f a face with source u and sink v. Denote by V the vertex set of G. If $O(V)$ time is available for the preprocessing of G, we can decide in $O(1)$ time whether f is a weak rhombus. Moreover, the two vertices (in addition to u and v) that define a maximal weak st-Polygon that contains f can be also computed in $O(1)$ time.*

Proof: By definition, a weak rhombus is a face that has at least one vertex on each of its sides. Thus, we can test whether face f is a weak rhombus in $O(1)$ time, if for each face the lists of vertices on its left and right sides are available.

As it was noted in the previous proof, we can preprocess graph G in linear time so that for each of its vertices we know its first and last (in clock-wise order) in-coming and out-going edges. Then, the two remaining vertices that define the maximal weak *st*-Polygon having f as a subgraph can be found in $O(1)$ time and it can be reported in time proportional to its size. For example, in Figure 19 where vertices u and v are both on the right side, the limiting edges of the maximal weak *st*-Polygon are the first outgoing edge from u and the last incoming edge to v.

□

Observe also that, as we extend a weak (strong) rhombus to finally obtain the maximal weak (respectively, strong) *st*-Polygon that contains it, we include all edges that are outgoing from u and incoming to v. During this procedure, all faces attached to the rhombus are generalized triangles.

Lemma 9 *The maximal st-Polygons contained in an outerplanar st-digraph G are mutually area-disjoint.*

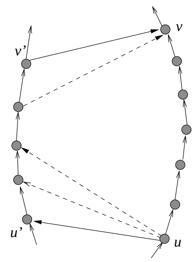

Figure 19: The weak rhombus with u and v as its source and sink, respectively, and the maximal st-Polygon containing it.

Proof:

We first observe that a maximal st-Polygon cannot fully contain another one. If it does, then we would have a maximal st-Polygon containing two rhombi, which is impossible due to Lemma 4.

For the sake of contradiction, assume that two st-Polygons P_1 and P_2 have a partial overlap. We denote by $(s_1, u_1^l, \ldots, u_k^l, u_1^r, \ldots, u_m^r, t_1)$ and $(s_2, v_1^l, \ldots, v_k^l, v_1^r, \ldots, v_m^r, t_2)$ the vertices of P_1 and P_2, respectively. Throughout the proof we refer to Figure 20.

Due to the assumed partial overlap of P_1 and P_2, an edge of one of them, say P_1, must be contained within the other (say P_2). Below we show that none of the two possible upper limiting edges (u_k^l, t_1) and (u_m^r, t_1) of P_1 can be contained in P_2.

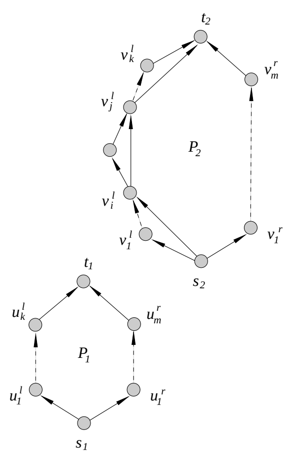

Figure 20: Two st-Polygons from the proof of Lemma 9

We have to consider three cases.

Case 1: *One of the edges (u_k^l, t_1) and (u_m^r, t_1) of P_1 coincide with an internal edge of P_2 connecting s_2 with a vertex v_i^l on its left side (the case where it is on* *its right side is symmetric).* Edge (u^l_k, t_1) cannot coincide with (s_2, v^l_i), since then, edge (u^r_m, t_1) has to be inside P_2 and therefore to connect the left side of P_2 with its right side. This is a contradiction since P_2 is an st-Polygon and it cannot contain any such edge.

Now assume that edge (u^r_m, t_1) of P_1 coincides with edge (s_2, v^l_i) of P_2, Then, edge (s_2, v^l_1) is inside P_1 and joins its right with its left side, which is again impossible since P_1 is an st-Polygon.

Case 2: *One of edges (u^l_k, t_1) and (u^r_m, t_1) of P_1 coincides with an internal edge of P_2 connecting two vertices on its same side.* Let it again be the left side and denote the edge by (v^l_i, v^l_j). Assume first that (u^r_m, t_1) coincides with (v^l_i, v^l_j). As graph P_2 is outerplanar, we have that all the remaining vertices of P_1 have to be placed above vertex v^l_i and below vertex v^l_j on the left side of P_2. Therefore P_1 is fully contained in P_2, which is impossible.

Assume now that (u^l_k, t_1) coincides with edge (v^l_i, v^l_j). Then, edge (u^r_m, t_1) of P_1 coincides with edge $(v^l_{i'}, v^l_j)$ of P_2, $i' < i$. This is impossible as it was covered in the above paragraph. Note also that $v^l_{i'}$ cannot be s_2 since this configuration was shown to be impossible in Case 1.

Case 3: *One of edges (u^l_k, t_1) and (u^r_m, t_1) of P_1 coincides with an internal edge of P_2 connecting the vertex on its side (suppose again on its left side) with sink t_2.* Let this edge be denoted by (v^l_j, t_2). Suppose first that (u^l_k, t_1) coincides with (v^l_j, t_2). If vertex u^r_m is on the right side of P_2 then P_1 is not maximal as P_1 can be extended (and still remain an st-Polygon) by including in it vertices v^l_j to v^l_k. So, assume that u^r_m is on the left side of P_2. Then, as covered in Case 2, P_1 must be fully contained in P_2 which leads to a contradiction.

Assume now that edge (u^r_m, t_1) coincides with edge (v^l_j, t_2). Due to the outer-planarity of P_2, we have again that all the vertices of P_1 have to be placed above v^l_j and below t_2 on the left side of P_2. So P_1 is again fully contained in P_2, leading again to a contradiction.

We have managed to show that none of edges (u^l_k, t_1) and $(u^r_m, 1_1)$ is contained in P_2. Therefore, there can be no partial overlap between P_1 and P_2. □

Denote by $\mathcal{R}(G)$ the set of all maximal st-Polygons of an outerplanar st-digraph G. Observe that not every vertex of G belongs to one of its maximal st-Polygons. We refer to the vertices of G that are not part of any maximal st-Polygon as *free vertices* and we denote them by $\mathcal{F}(G)$. Also observe that an ordering can be imposed on the maximal st-Polygons of an outerplanar st-digraph G based on the ordering of the area disjoint strips occupied by each st-Polygon. The vertices which do not belong to any st-Polygon are located in the area between the strips occupied by consecutive st-Polygons.

Lemma 10 *Let R_1 and R_2 be two consecutive maximal st-Polygons of an outerplanar st-digraph G which do not share an edge and let $V_f \subseteq \mathcal{F}(G)$ be the set* *of free vertices lying between R_1 and R_2. Denote by (u, t_1) and (s_2, v) the upper limiting edge and the lower limiting edge of R_1 and R_2, respectively. For the embedded subgraph G_f of G induced by the vertices of $V_f \cup \{u, t_1, s_2, v\}$ it holds:*

a) G_f is an outerplanar st-digraph having vertices u and v as its source and sink, respectively.

b) G_f is hamiltonian.

Proof: We first show that statement (a) is true, that is, G_f is an outerplanar st-digraph having vertices u and v as its source and sink, respectively. Without loss of generality, assume that the limiting edge (s_2, v) of the upper maximal st-Polygon R_2 is directed towards the right side of the outerplanar st-digraph G. We consider cases based on whether R_1 and R_2 share a common vertex.

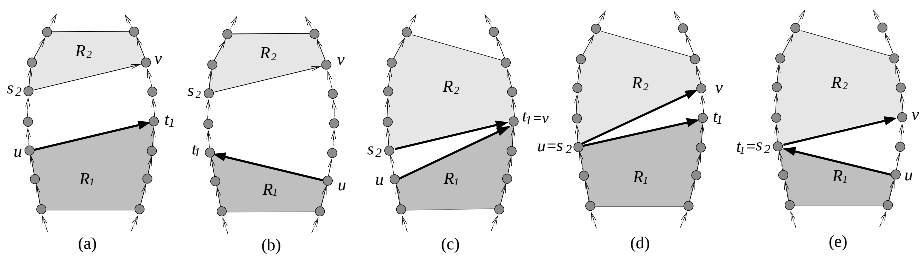

Figure 21: The configurations of two st-Polygons used in the proof of Lemma 10.

Case 1: R_1 ***and*** R_2 ***share no common vertex.*** Based on the direction of the limiting edge (u, t_1) we can further distinguish the following two cases:

Case 1a: (u, t_1) ***is directed towards the right side of*** G***.***
See Figure 21.a.

Case 1b: (u, t_1) ***is directed towards the left side of*** G***.***
See Figure 21.b.

In both of the above cases, we observe that G_f forms an st-Polygon (or, a rhombus) which contradicts the fact that R_1 and R_2 are consecutive maximal st-Polygons. Thus, Case 1 cannot occur.

Case 2: R_1 ***and*** R_2 ***share one common vertex.*** First observe that the limiting edge (u, t_1) of R_1 is directed towards the left side of G. To see that, assume for the sake of contradiction that edge (u, t_1) is directed towards the right side of G. If v coincides with t_1 (see Figure 21.c) then the st-Polygon R_1 could be extended (and still remain an st-Polygon) by adding to it the area between the two polygons R_1 and R_2, contradicting the fact that R_1 is maximal. If u coincides with s_2 (see Figure 21.d) then the st-Polygon R_2 could be extended (and still remain an st-Polygon) by adding to it the area between the two polygons R_1 and R_2, contradicting the fact that R_2 is maximal. Thus, the limiting edge (u, t_1) of R_1 is directed towards the left side of G and s_2 coincides with t_1 (see Figure 21.e). The rest of the proof is a special case of Case 1b (where no vertices of V_f exist on the left side of G).

So G_f is an outerplanar st-digraph with the source u and the sink v. Note also that G_f does not contain a rhombus. If it does, then it would be an st-Polygon, contradicting the fact that R_1 and R_2 are consecutive maximal st-Polygons. Then, from Theorem 1 it follows that G_f is hamiltonian. □

Lemma 11 *Let R_1 and R_2 be two consecutive maximal st-Polygons of an outerplanar st-digraph G that share a common edge. Let t_1 be the sink of R_1 and s_2 be the source of R_2. Then, edge (s_2, t_1) is their common edge.*

Proof: Let the upper limiting edge of R_1 be edge (u, t_1) and the lower limiting edge of R_2 be edge (s_2, v). Since these are the only two edges that can coincide, we conclude that v coincides with t_1 and u coincides with s_2. Thus, edge (s_2, t_1) is the edge shared by R_1 and R_2. □

Lemma 12 *Let G be an outerplanar st-digraph. Let R_1 and R_2 be two G's consecutive maximal st-Polygons and let $V_f \subseteq \mathcal{F}(G)$ be the set of free vertices lying between R_1 and R_2. Then, the following statements hold:*

a) For any pair of vertices u, $v \in V_f$ there is either a path from u to v or from v to u.

b) For any vertex $v \in V_f$ there are a path from the sink of R_1 to v and a path from v to the source of R_2.

c) If $V_f = \emptyset$, then there is a path from source of R_1 to the source of R_2.

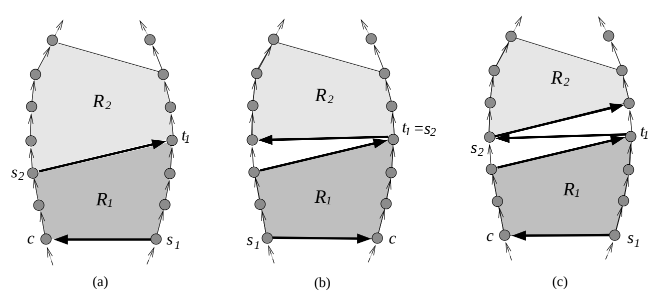

Figure 22: The configurations of adjacent st-Polygons of an outerplanar st-digraph.

Proof:

a) From Lemma 10 we have that the subgraph G_f of G (as defined in the proof of Lemma 10) is hamiltonian. Thus, all vertices in V_f are connected by a directed path.

b) Follows directly from Lemma 10.

c) Note that there are 3 configuration in which no free vertex exists between two consecutive *st*-Polygons (see Figures 22.a-c). Denote by s_1 and s_2 the sources of R_1 and R_2, respectively. If s_1 and s_2 lie on the same side of G then the claim is obviously true since G is an outerplanar *st*-digraph. If they belong to opposite sides of G, observe that the lower limiting edge (s_1, c) of R_1 leads to the side of G which contains s_2. Since there is a path from c to s_2, it follows that there is a path from s_1 to s_2.

□

We refer to the source vertex s_i of each maximal *st*-Polygon $R_i \in \mathcal{R}(G)$, $1 \leq i \leq |\mathcal{R}(G)|$ as the *representative* of R_i and we denote it by $r(R_i)$. We also define the representative of a free vertex $v \in \mathcal{F}(G)$ to be v itself, i.e. $r(v) = v$. For any two distinct elements $x,\ y\ \in \mathcal{R}(G) \cup \mathcal{F}(G)$, we define the relation $\angle_p$ as follows: $x \angle_p y$ *if and only if there exists a path from* $r(x)$ *to* $r(y)$.

Lemma 13 *Let G be an n-vertex outerplanar st-digraph. Then, relation $\angle_p$ defines a total order on the elements $\mathcal{R}(G) \cup \mathcal{F}(G)$. Moreover, this total order can be computed in $O(n)$ time.*

Proof: The fact that $\angle_p$ is a total order on $\mathcal{R}(G) \cup \mathcal{F}(G)$ follows from Lemma 12. The order of the elements of $\mathcal{R}(G) \cup \mathcal{F}(G)$ can be easily derived by the numbers assigned to the representatives of the elements (i.e., to vertices of G) by a topological sort of the vertices of G. To complete the proof, recall that an n-vertex acyclic planar graph can be topologically sorted in $O(n)$ time. □

Definition 2 *Given an outerplanar st-digraph G, the st-*Polygon decomposition $\mathcal{D}(G)$ *of G is defined to be the total order $\{o_1,\ \ldots, o_\lambda\}$ induced by relation $\angle_p$ on its maximal st-Polygons and its free vertices, that is, o_i, $1 \leq i \leq \lambda$, is either a maximal st-Polygon or a free vertex of G and $o_i \angle_p o_{i+1}$, $1 \leq i < \lambda$.*

The following theorem follows directly from Lemma 7, Lemma 8 and Lemma 13.

Theorem 3 *An st-Polygon decomposition of an n-vertex outerplanar st-digraph G can be computed in $O(n)$ time.*

3.3 Properties of a crossing-optimal acyclic HP-completion set

In this section, we present three properties of crossing-optimal acyclic HP-completion sets for an outerplanar *st*-digraph that will be taken into account by our algorithm. Let $G = (V^l \cup V^r \cup \{s, t\}, E)$ be an outerplanar *st*-digraph and $\mathcal{D}(G) = \{o_1,\ \ldots, o_\lambda\}$ its *st*-Polygon decomposition. As G_i we denote the graph induced by the vertices of elements $o_1, \ldots, o_i$, $i \leq \lambda$.

Property 1 *Let $G = (V^l \cup V^r \cup \{s, t\}, E)$ be an outerplanar st-digraph. Then, no edge of E is crossed by more than 2 edges of a crossing-optimal acyclic HP-completion set for G.*

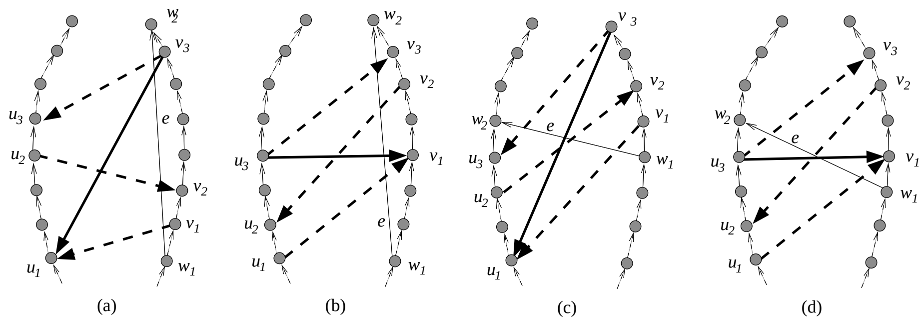

Figure 23: The configurations of crossing edges used in the proof of Property 1.

Proof: For the sake of contradiction, assume that P_{opt} is a crossing-optimal acyclic HP-completion set for G, the edges of which cross some edge $e = (w_1, w_2)$ of G three times. We will show that we can obtain an acyclic HP-completion set for G that induces a smaller number of crossings than P_{opt}, a clear contradiction. We assume that all edges of P_{opt} participate in the hamiltonian path of G; otherwise they can be discarded.

We distinguish two cases based on whether edge e is one-sided or two-sided.

Case 1: ***The edge e is one-sided.*** Suppose without loss of generality that e is on the right side. We further distinguish two cases based on the orientation of the edge, say e_1, which appears first on the hamiltonian path of G (out of the 3 edges crossing edge e).

Case 1a: ***Edge e_1 is directed from right to left.*** Let e_1 be edge (v_1, u_1) and let (u_2, v_2) and (v_3, u_3) be the other two edges on the hamiltonian path which cross e (see Figure 23.a). It is clear that these three edges have alternating direction. Observe that the path $P_{v_1,u_3} = (v_1 \rightarrow u_1 \dashrightarrow u_2 \rightarrow v_2 \dashrightarrow v_3 \rightarrow u_3)$ is a sub-path of the hamiltonian path of G. Also, by Lemma 3, vertex u_2 is immediately below vertex u_3 on the left side of G and vertex v_2 is immediately above vertex v_1 on the right side of G.

Now, we show that the substitution of path P_{v_1,u_3} of the hamiltonian path of G by path $P'_{v_1,u_3} = (v_1 \rightarrow v_2 \dashrightarrow v_3 \rightarrow u_1 \dashrightarrow u_2 \rightarrow u_3)$ results in a reduction of the total number of crossings by at least 2. Thus, there exists an HP-completion set that crosses edge e only once and causes 2 crossings less with edges of G compared to P_{opt}, a clear contradiction.

Let us examine the edges of G that are crossed by the new edge (v_3, u_1). These edges can be grouped as follows: (i) The one-sided edges on the right side of G that have their source below v_3 and their sink above v_3. Note that these edges are also crossed by edge (v_3, u_3). In addition, the edges that belong to this group and have their origin below w_1 and their sink above w_2 are crossed by all three edges (v_1, u_1), (u_2, v_2) and (v_3, u_3) in the original HP-completion set. (ii) The two-sided edges that have their source below w_1 on the right side of G and their sink above v_1 on the left side of G. These edges are also crossed by at least edge (v_1, u_1) (and possibly by one or both of edges (u_2, v_2) and (v_3, u_3)). (iii)The one-sided edges on the left side of G that have their source below u_1 and their sink above u_1. Note that these edges are also crossed by at least edge (v_1, u_1) (and possibly by one or both of edges (u_2, v_2) and (v_3, u_3)). (iv) The two sided edges that have their source below u_1 on the left side of G and their sink above w_2 on the right side of G. These edges are also crossed by all three edges (v_1, u_1) (u_2, v_2) and (v_3, u_3). Thus, we have shown that edge (v_3, u_1) crosses at most as many edges of G as the three edges (v_1, u_1), (u_2, v_2), (v_3, u_3) taken together.

Case 1b: ***Edge*** e_1 ***is directed from left to right.*** Let e_1 be edge (u_1, v_1) and let (v_2, u_2) and (u_3, v_3) be the next two edges on the hamiltonian path which cross e (see Figure 23.b). Observe that the path $P_{u_1,v_3} = (u_1 \rightarrow v_1 \dashrightarrow v_2 \rightarrow u_2 \dashrightarrow u_3 \rightarrow v_3)$ is a sub-path of the hamiltonian path of G. Also, by Lemma 3, vertex u_2 is immediately above vertex u_1 on the left side of G and vertex v_2 is immediately below vertex v_3 on the right side of G. By arguing in a way similar to that of Case 1a, we can show that the substitution of path P_{u_1,v_3} of the hamiltonian path of G by path $P'_{u_1,v_3} = (u_1 \rightarrow u_2 \dashrightarrow u_3 \rightarrow v_1 \dashrightarrow v_2 \rightarrow v_3)$ results in a reduction of the total number of crossings by at least 2.

Case 2: ***Edge*** e ***is two-sided.*** Assume without loss of generality that e is directed from right to left. We again distinguish two cases based on the orientation of the edge, say e_1, which appears first on the hamiltonian path of G (out of the 3 edges crossing edge e).

Case 2a: ***Edge*** e_1 ***is directed from right to left.*** The proof is identical to that of Case 1a.

Case 2b: ***Edge*** e_1 ***is directed from left to right.*** The proof is identical to that of Case 1b.

□

Property 2 *Let* $G = (V^l \cup V^r \cup \{s, t\}, E)$ *be an outerplanar st-digraph and let* $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ *be its st-Polygon decomposition. Then, there exists a crossing-optimal acyclic HP-completion set for G such that, for every maximal st-Polygon* $o_i \in \mathcal{D}(G)$, $i \leq \lambda$, *the HP-completion set does not contain any edge that crosses the upper limiting edge of* o_i *and leaves* G_i.

Proof: Let $e = (x, t_i)$ be the upper limiting edge of o_i and assume without loss of generality that it is directed from right to left. Also assume a crossing-optimal acyclic HP-completion set P_{opt} that violates the stated property, that is, it contains an edge $\tilde{e} = (u, v)$, $u \in G_i$, that crosses the limiting edge e. Based on Lemma 3, we conclude that edge $\tilde{e}$ is a two-sided edge, otherwise the vertices of a single side appear out of order in the hamiltonian path induced by P_{opt}. We distinguish two cases based on direction of the two-sided edge $\tilde{e}$.

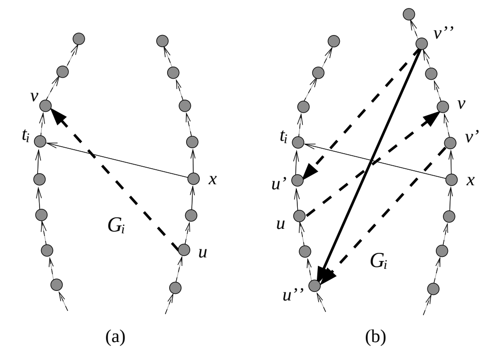

Figure 24: The configurations of crossing edges used in the proof of Property 2.

Case 1: ***Edge*** $\tilde{e} = (u, v)$ ***is directed from right to left.*** See Figure 24.a. By Lemma 3, in the hamiltonian path induced by P_{opt} vertex x is visited after vertex u . So, in the resulting HP-completed digraph, there must be a path from v to x which, together with (x, t_i) and the path $(t_i \dashrightarrow v)$ on the left side of G forms a cycle. This contradicts the fact that P_{opt} is an acyclic HP-completion set.

Case 2: ***Edge*** $\tilde{e} = (u, v)$ ***is directed from left to right.*** See Figure 24.b. Denote by v' the vertex positioned immediately below vertex v (note that v' may coincide with x) and by u' the vertex that is immediately above u (note that u' may coincide with t_i).

Consider the hamiltonian path induced by P_{opt}. By Lemma 3 it follows that before crossing to the right side of G using edge (u, v) it had visited all vertices on the right side which are placed below v, and thus, there is an edge $(v', u'') \in P_{\text{opt}}$, where u'' is some vertex below u on the left side of G. Now note that, by Lemma 3, vertex u' has to appear in the hamiltonian path after vertex u, and thus, there exists an edge $(v'', u') \in P_{\text{opt}}$ where v'' is a vertex above v on the right side of G.

By arguing in a way similar to that of Property 1, we can show that the substitution of path $P_{v',u'} = (v' \to u'' \dashrightarrow u \to v \dashrightarrow v'' \to u')$ of the hamiltonian path of G by path $P'_{v',u'} = (v' \dashrightarrow v'' \to u'' \dashrightarrow u')$ does not result in an increase of the number of edge crossings. More specifically, when v' does not coincide with x and/or u' does not coincide with t_i, the resulting new path causes at least one less crossing, contradiction the optimality of P_{opt}. In the case where v' coincides with x and u' coincides with t_i and the two hamiltonian paths cause the same number of crossings, the new HP-completion set has the desired property, that is, none of its edges crosses the limiting edge (x, t_i) and leaves G_i. □

Property 3 *Let* $G = (V^l \cup V^r \cup \{s, t\}, E)$ *be an outerplanar st-digraph and let* $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ *be its st-Polygon decomposition. Then, in every crossing-optimal acyclic HP-completion set for* G *and for every maximal st-Polygon* $o_i \in$

$\mathcal{D}(G)$, $i \leq \lambda$, at most one edge crosses the upper limiting edge of o_i.

Proof: Let edge $e = (x, t_i)$ be the upper limiting edge of o_i. Without loss of generality assume that it is directed from the right to the left side of G, and let v be the vertex immediately above x on the right side of G and u be the vertex immediately below t_i on the left side of G. By Property 1, we have that the edges of a crossing-optimal acyclic HP-completion set for G do not cross e three or more times.

For the sake of contradiction assume that there is a crossing-optimal acyclic HP-completion set P_{opt} for G that crosses edge e twice. Let $e_1, e_2 \in P_{\text{opt}}$ be the edges which cross e. Clearly, these two edges cross e in the opposite direction and do not cross each other. Let e_1 be the edge that crosses e and leaves G_i. Observe that e_1 has opposite direction to that of e, otherwise a cycle is created. Then, since e_1 does not cross e_2, edge e_1 does not coincide with (u, v). However, for the case where $e_1 \neq (u, v)$, we established in the proof of Property 2 (Case 2) that we are always able to build an acyclic HP-completion set that induced less crossings than P_{opt}, a clear contradiction.[4] □

The following theorem states that there always exists a crossing-optimal acyclic HP-completion set for outerplanar *st*-digraphs that has certain properties. The algorithm which we present in the next section, focuses only on an HP-completion set satisfying these properties.

Theorem 4 *Let $G = (V^l \cup V^r \cup \{s, t\}, E)$ be an outerplanar st-digraph and let $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-Polygon decomposition. Then, there exists a crossing-optimal acyclic HP-completion set P_{opt} for G such that it satisfies the following properties:*

a) Each edge of E is crossed by at most two edges of P_{opt}.

b) Each upper limiting edge e_i of any maximal st-Polygon o_i, $i \leq \lambda$, is crossed by at most one edge of P_{opt}. Moreover, the edge crossing e_i, if any, enters G_i.

Proof: Follows directly from Properties 1, 2 and 3. □

3.4 The Algorithm

The algorithm for obtaining a crossing-optimal acyclic HP-completion set for an outerplanar *st*-digraph G is a dynamic programming algorithm based on the *st*-Polygon decomposition $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ of G. The following lemmata allow us to compute a crossing-optimal acyclic HP-completion set for an *st*-Polygon and to obtain a crossing-optimal acyclic HP-completion set for G_{i+1} by combining an optimal solution for G_i with an optimal solution for o_{i+1}.

Let G be an outerplanar *st*-digraph. We denote by $S(G)$ the hamiltonian path on the HP-extended digraph of G obtained when a crossing-optimal HP-completion set is added to G. Note that if we are only given $S(G)$ we can infer

[4] The proof is identical and for this reason it is not repeated

the size of the HP-completion set and the number of edge crossings. Denote by $c(G)$ the number of edge crossings caused by the HP-completion set inferred by $S(G)$. If we are restricted to Hamiltonian paths that enter the sink of G from a vertex on the left (respectively right) side of G, then we denote the corresponding size of HP-completion set by $c(G, L)$ (respectively $c(G, R)$). Obviously, $c(G) = \min\{c(G, L),\ c(G, R)\}$. Moreover, the notation can be extended so that we denote by $c^i(G, L)$ ($c^i(G, R)$) the number of crossings for HP-completion sets that contain exactly i edges, provided they exist. By Theorem 2, we know that the size of a crossing-optimal acyclic HP-completion set for an st-Polygon is at most 2. This notation that restricts the size of the HP-completion set will be used only for st-Polygons and thus, only the terms $c^1(G, L)$, $c^1(G, R)$, $c^2(G, L)$ and $c^2(G, R)$ will be utilized.

We use the operator $\oplus$ to indicate the concatenation of two paths. By convention, the hamiltonian path of a single vertex is the vertex itself.

Lemma 14 *Let $o = (V^l \cup V^r \cup \{s, t\}, E)$ be an n-vertex st-Polygon. A crossing-optimal acyclic HP-completion set for o and the corresponding number of crossings can be computed in $O(n)$ time.*

Proof: From Lemma 5 and Lemma 6 it follows that it is sufficient to look

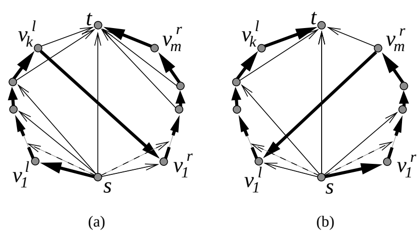

Figure 25: The two single-edge HP-completion sets of an st-Polygon.

through all HP-completion sets with one or two edges in order to find a crossing-optimal acyclic HP-completion set. Let $V^l = \{v_1^l,\ \ldots, v_k^l\}$ and $V^r = \{v_1^r,\ \ldots, v_m^r\}$, where the subscripts indicate the order in which the vertices appear on the left (right) boundary of o. Suppose that $I : V \times V \rightarrow \{0, 1\}$ is an indicator function such that $I(u, v) = 1$ if and only if $(u, v) \in E$.

The only two possible HP-completion sets consisting of exactly one edge are $\{(v_k^l, v_1^r)\}$ and $\{(v_m^r, v_1^l)\}$.

Edge (v_k^l, v_1^r) crosses all edges connecting t with vertices in $V_l \setminus \{v_k^l\}$, the median (provided it exists), and all edges connecting s with vertices in $V_r \setminus \{v_1^R\}$ (see Figure 25.a). It follows that:

$$c^1(o, R) = I(s, t) + \sum_{i=2}^{k-1} I(v_i^\ell, t) + \sum_{i=2}^{m-1} I(s, v_i^r).$$

Similarly, edge (v_m^r, v_1^l) crosses all edges connecting t with vertices in $V_r \setminus \{v_m^r\}$, the median (provided it exists), and all edges connecting s with vertices in $V_l \setminus \{v_1^l\}$ (see Figure 25.b). It follows that:

$$c^1(o,L) = I(s,t) + \sum_{i=2}^{m-1} I(v_i^r,t) + \sum_{i=2}^{k-1} I(s,v_i^\ell).$$

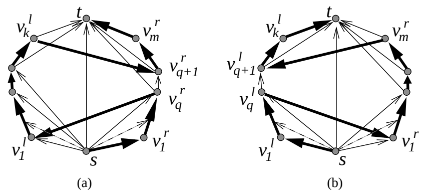

Figure 26: The two-edge HP-completion sets of an *st*-Polygon.

Consider now an acyclic HP-completion set of size 2. Assume that the lowermost edge leaves vertex v_q^r on the right side of o (see Figure 26.a). Then, it must enter vertex v_1^l. Moreover, the second edge of the acyclic HP-completion set must leave vertex v_k^l and enter vertex v_{q+1}^r. Thus, the HP-completion set is $\{(v_q^r, v_1^l),\ (v_k^l, v_{q+1}^r)\}$ and, as we observe, it can be put into correspondence with edge (v_q^r, v_{q+1}^r) on the right side of o. In addition, we observe that the hamiltonian path enters t from the right side. An analogous situation occurs when the lowermost edge leaves the left side of o (see Figure 26.b).

We denote by $c_q^2(o,R)$ the number of crossings caused by the completion set associated with the edge originating at the q^{th} lowermost vertex on the right side of o. Similarly we define $c_q^2(o,L)$. $c_q^2(o,R)$ can be computed as follows:

$$\begin{aligned} c_q^2(o,R) \quad = \quad & 2 \cdot I(s,t) + \textstyle\sum_{i=1}^{k-1} I(v_i^\ell,t) + \sum_{i=2}^{k} I(s,v_i^\ell) + 2 \cdot \sum_{i=1}^{q-1} I(v_i^r,t) + \\ & 2 \cdot \textstyle\sum_{i=q+2}^{m} I(s,v_i^r) + I(v_q^r,t) + I(s,v_{q+1}^r) \end{aligned}$$

Then, the optimal solution where the hamiltonian path terminates on the right side of o can be taken as the minimum over all $c_q^2(o,R),\ 1 \le q \le m-1$:

$$c^2(o,R) = \min_{1 \le q \le m-1} \{c_q^2(o,R)\}.$$

Similarly, $c_q^2(o,L)$ can be computed as follows:

$$\begin{aligned} c_q^2(o,L) \quad = \quad & 2 \cdot I(s,t) + \textstyle\sum_{i=1}^{m-1} I(v_i^r,t) + \sum_{i=2}^{m} I(s,v_i^r) + 2 \cdot \sum_{i=1}^{q-1} I(v_i^\ell,t) + \\ & 2 \cdot \textstyle\sum_{i=q+2}^{k} I(s,v_i^\ell) + I(v_q^\ell,t) + I(s,v_{q+1}^\ell). \end{aligned}$$

Then, the optimal solution where the hamiltonian path terminates on the left side of o can be taken as the minimum over all $c_q^2(o,R),\ 1 \le q \le k-1$:

$$c^2(o,L) = \min_{1 \le q \le k-1} \{c_q^2(o,L)\}.$$

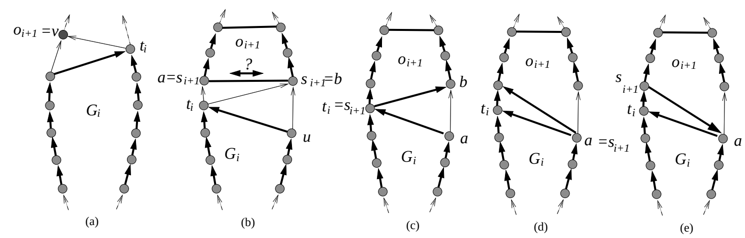

Figure 27: The configurations used in the proof of Lemma 15.

So, now, the number of crossings that corresponds to the optimal solution can be computed as follows:

$$c(o) = \min\{c^1(o,L), c^1(o,R), c^2(o,L), c^2(o,R)\}.$$

It is clear that $c^1(o,R)$ and $c^1(o,L)$ can be computed in time $O(n)$. It is also easy to see that any $c^2_q(o,R)$ can be computed from $c^2_{q-1}(o,R)$ in constant time, while $c^2_1(o,R)$ can be computed in time $O(n)$. Therefore, $c^2(o,R)$, as well as $c^2(o,L)$, can be computed in linear time. Thus, we conclude that a crossing-optimal acyclic HP-completion set for any n-vertex st-Polygon o and its corresponding number of crossings can be computed in $O(n)$ time. □

Let $\mathcal{D}(G) = \{o_1, \ldots, o_\lambda\}$ be the st-Polygon decomposition of G, where element o_i, $1 \le i \le \lambda$, is either an st-Polygon or a free vertex. Recall that, we denote by G_i, $1 \le i \le \lambda$, the graph induced by the vertices of elements $o_1, \ldots, o_i$. Graph G_i is also an outerplanar st-digraph. The same holds for the subgraph of G that is induced by any number of consecutive elements of $\mathcal{D}(G)$.

Lemma 15 *Let G be an outerplanar st-digraph and $\mathcal{D}(G) = \{o_1, \ldots, o_\lambda\}$ be its st-Polygon decomposition. Consider any two consecutive elements o_i and o_{i+1} of $\mathcal{D}(G)$ that share at most one vertex. Then, the following statements hold:*

(i) $S(G_{i+1}) = S(G_i) \oplus S(o_{i+1})$*, and*

(ii) $c(G_{i+1}) = c(G_i) + c(o_{i+1})$.

Proof: We proceed to prove first statement (i). There are three cases to consider in which 2 consecutive elements of $\mathcal{D}(G)$ share at most 1 vertex.

Case 1: ***Element*** $o_{i+1} = v$ ***is a free vertex*** (see Figure 27.a). By Lemma 12, if o_i is either a free vertex or an st-Polygon, there is an edge connecting the sink of o_i to v. Also observe that if v was not the last vertex of $S(G_{i+1})$ then the crossing-optimal HP-completion set had to include an edge from v to some vertex of G_i. This is impossible since it would create a cycle in the HP-extended digraph of $S(G_{i+1})$.

Algorithm 1: ACYCLIC-HPC-CM(G)

input : An Outerplanar st-digraph $G(V^l \cup V^r \cup \{s,t\}, E)$.
output : The minimum number of edge crossings $c(G)$ resulting from the addition of a crossing-optimal acyclic HP-completion set to G.

1. Compute the st-Polygon decomposition $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ of G;

2. For each element $o_i \in \mathcal{D}(G)$, $1 \leq i \leq \lambda$, compute $c^1(o_i, L)$, $c^1(o_i, R)$ and $c^2(o_i, L)$, $c^2(o_i, R)$:
   if o_i is a free vertex, **then** $c^1(o_i, L) = c^1(o_i, R) = c^2(o_i, L) = c^2(o_i, R) = 0$.
   if o_i is an st-Polygon, **then** $c^1(o_i, L)$, $c^1(o_i, R)$, $c^2(o_i, L)$, $c^2(o_i, R)$ are computed based on Lemma 14.

3. **if** o_1 is a free vertex, **then** $c(G_1, L) = c(G_1, R) = 0$;
   else $c(G_1, L) = \min\{c^1(o_1, L), c^2(o_1, L)\}$ and
   $c(G_1, R) = \min\{c^1(o_1, R), c^2(o_1, R)\}$;

4. For $i = 1 \ldots \lambda - 1$, compute $c(G_{i+1}, L)$ and $c(G_{i+1}, R)$ as follows:
   if o_{i+1} is a free vertex, **then**
   $c(G_{i+1}, L) = c(G_{i+1}, R) = \min\{c(G_i, L), c(G_i, R)\}$;

   else-if o_{i+1} is an st-Polygon sharing **at most** one vertex with G_i, **then**
   $c(G_{i+1}, L) = \min\{c(G_i, L), c(G_i, R)\} + \min\{c^1(o_{i+1}, L), c^2(o_{i+1}, L)\}$;
   $c(G_{i+1}, R) = \min\{c(G_i, L), c(G_i, R)\} + \min\{c^1(o_{i+1}, R), c^2(o_{i+1}, R)\}$;
   else { o_{i+1} is an st-Polygon sharing **exactly** two vertices with G_i},
   if $t_i \in V^l$, **then**
   $c(G_{i+1}, L) = \min\{c(G_i, L) + c^1(o_{i+1}, L) + 1, c(G_i, R) + c^1(o_{i+1}, L),$
   $c(G_i, L) + c^2(o_{i+1}, L), c(G_i, R) + c^2(o_{i+1}, L)\}$
   $c(G_{i+1}, R) = \min\{c(G_i, L) + c^1(o_{i+1}, R), \ c(G_i, R) + c^1(o_{i+1}, R),$
   $c(G_i, L) + c^2(o_{i+1}, R) + 1, \ c(G_i, R) + c^2(o_{i+1}, R)\}$
   else { $t_i \in V^r$ }
   $c(G_{i+1}, L) = \min\{c(G_i, L) + c^1(o_{i+1}, L), \ c(G_i, R) + c^1(o_{i+1}, L),$
   $c(G_i, L) + c^2(o_{i+1}, L), \ c(G_i, R) + c^2(o_{i+1}, L) + 1\}$
   $c(G_{i+1}, R) = \min\{c(G_i, L) + c^1(o_{i+1}, R), \ c(G_i, R) + c^1(o_{i+1}, R) + 1,$
   $c(G_i, L) + c^2(o_{i+1}, R), \ c(G_i, R) + c^2(o_{i+1}, R)\}$

5. **return** $c(G) = \min\{c(G_\lambda, L), c(G_\lambda, R)\}$

Case 2: ***Element*** o_{i+1} ***is an*** st***-Polygon that shares no common vertex with*** G_i (see Figure 27.b). Without loss of generality, assume that the sink of G_i is located on its left side. We first observe that edge (t_i, s_{i+1}) exists in G. If s_{i+1} is on the left side of G, we are done. Note that there can be no other vertex between t_i and s_{i+1} in this case, because then o_i and o_{i+1} would not be consecutive. If s_{i+1} is on the right side of G, realize that the area between two st-Polygons o_i and o_{i+1} cannot be free of edges, as it is a weak st-Polygon. Note also that the edge (u, a) cannot exist in G, since, if it existed, the area between the two polygons would be a strong st-Polygon with (u, a) as its median. Thus, that area can only contain the edge (t_i, s_{i+1}). Thus, as indicated in Figure 27.b, each of the end-vertices of the lower limiting edge of o_{i+1} can be its source. Since edge (t_i, s_{i+1}) exists, the solution $S(o_{i+1})$ can be concatenated to $S(G_i)$ and yield a valid hamiltonian path for G_{i+1}. Now notice that in $S(G_{i+1})$ all vertices of G_i have to be placed before the vertices of o_{i+1}. If this was not the case, then the crossing-optimal HP-completion set had to include an edge from a vertex v of o_{i+1} to some vertex u of G_i. This is impossible since it would create a cycle in the HP-extended digraph of $S(G_{i+1})$.

Case 3: ***Element*** o_{i+1} ***is an*** st***-Polygon that shares one common vertex with*** G_i (see Figure 27.c). Without loss of generality, assume that the sink t_i of G_i is located on its left side. Firstly, notice that the vertex shared by G_i and o_{i+1} has to be vertex t_i. To see that, let a be the upper vertex at the right side of G_i. Then, edge (a, t_i) exists since t_i is the sink of G_i. For the sake of contradiction assume that a was the vertex shared between G_i and o_{i+1}. If a was also the source of o_{i+1} (see Figure 27.d) then o_{i+1} wouldn't be maximal (edge (a, t_i) should also belong to o_{i+1}). If s_{i+1} was on the left side (see Figure 27.e), then a cycle would be formed involving edges (t_i), (t_1, s_{i+1}) and (s_{i+1}, a), which is impossible since G is acyclic. Thus, the vertex shared by G_i and o_{i+1} has to be vertex t_i. Secondly, observe that t_i must coincide with vertex s_{i+1} (see Figure 27.c). If s_{i+1} coincided with vertex b, then the st-Polygon o_i wouldn't be maximal since edge (b, t_i) should also belong to o_i. We conclude that t_i coincides with s_{i+1} and, thus, the solution $S(o_{i+1})$ can be concatenated to $S(G_i)$ and yield a valid hamiltonian path for G_{i+1}. To complete the proof for this case, we can show by contradiction (on the acyclicity of G; as in Case 2) that in $S(G_{i+1})$ all vertices of G_i have to be placed before the vertices of o_{i+1}.

Now observe that statement (ii) is trivially true since, in all three cases, the hamiltonian paths $S(G_i)$ and $S(o_{i+1})$ were concatenated by using at most one additional edge of graph G. Since G is planar, no new crossings are created. □

Lemma 16 *Let G be an outerplanar st-digraph and $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ be its st-Polygon decomposition. Consider any two consecutive elements o_i and o_{i+1} of $\mathcal{D}(G)$ that share an edge. Then, the following statements hold:*

1. $t_i \in V^l \Rightarrow c(G_{i+1}, L) = \min\{ c(G_i, L) + c^1(o_{i+1}, L) + 1,\ c(G_i, R) + c^1(o_{i+1}, L),\ c(G_i, L) + c^2(o_{i+1}, L),\ c(G_i, R) + c^2(o_{i+1}, L)\}.$

2. $t_i \in V^l \Rightarrow c(G_{i+1}, R) = \min\{ c(G_i, L) + c^1(o_{i+1}, R),\ c(G_i, R) + c^1(o_{i+1}, R),\ c(G_i, L) + c^2(o_{i+1}, R) + 1,\ c(G_i, R) + c^2(o_{i+1}, R)\}.$

3. $t_i \in V^r \Rightarrow c(G_{i+1}, L) = \min\{ c(G_i, L) + c^1(o_{i+1}, L),\ c(G_i, R) + c^1(o_{i+1}, L),\ c(G_i, L) + c^2(o_{i+1}, L),\ c(G_i, R) + c^2(o_{i+1}, L) + 1\}.$

4. $t_i \in V^r \Rightarrow c(G_{i+1}, R) = \min\{ c(G_i, L) + c^1(o_{i+1}, R),\ c(G_i, R) + c^1(o_{i+1}, R) + 1,\ c(G_i, L) + c^2(o_{i+1}, R),\ c(G_i, R) + c^2(o_{i+1}, R)\}.$

Proof: We first show how to build hamiltonian paths that infer HP-completion sets of the specified size. For each of the statements, there are four cases to

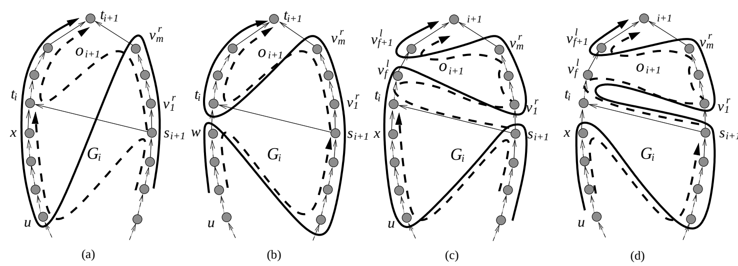

Figure 28: The hamiltonian paths for statement (1) of Lemma 16.

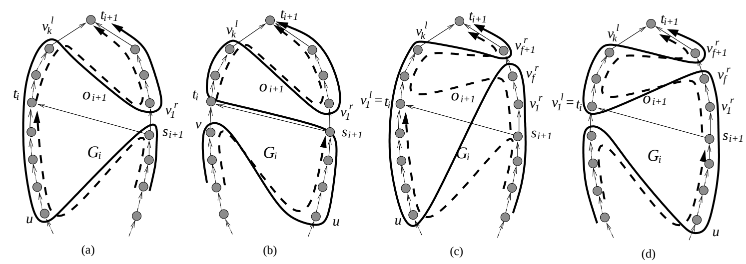

Figure 29: The hamiltonian paths for statement (2) of Lemma 16.

consider. The minimum number of crossings, is then determined by taking the minimum over the four sub-cases.

(1) $$t_i \in V^l \Rightarrow c(G_{i+1}, L) = \min \quad \{c(G_i, L) + c^1(o_{i+1}, L) + 1, c(G_i, R) + c^1(o_{i+1}, L) \\ c(G_i, L) + c^2(o_{i+1}, L), c(G_i, R) + c^2(o_{i+1}, L)\}$$

Case 1a. *The hamiltonian path enters t_i from a vertex on the left side of G_i and the size of the HP-completion of G_{i+1} is one.* Figure 28.a shows the hamiltonian paths for G_i (lower dashed path) and o_{i+1} (upper dashed path) as well as the resulting hamiltonian path for G_{i+1} (shown in bold). From the figure, it follows that $c(G_{i+1}, L) = c(G_i, L) + c^1(o_{i+1}, L) + 1$. To see that, just follow the edge (v^r_m, u) that becomes part of the completion set of G_{i+1}. Edge (v^r_m, u) is involved in as many edge crossings as edge (v^r_m, t_i) (the only edge in the HP-completion set of o_{i+1}), plus as many edge crossings as edge (s_{i+1}, u) (an edge in the HP-completion set of G_i), plus one (1) edge crossing of the lower limiting edge of o_{i+1}.

Case 1b. *The hamiltonian path reaches t_i from a vertex on the right side of G_i and the size of the HP-completion of G_{i+1} is one.* Figures 28.b shows the resulting path. From the figure, it follows that $c(G_{i+1}, L) = c(G_i, R) + c^1(o_{i+1}, L)$, that is, the simple concatenation of two solutions.

Case 1c. *The hamiltonian path enters t_i from a vertex on the left side of G_i and the size of the HP-completion of G_{i+1} is two.* Figure 28.c shows the resulting path. From the figure, it follows that $c(G_{i+1}, L) = c(G_i, L) + c^2(o_{i+1}, L)$,

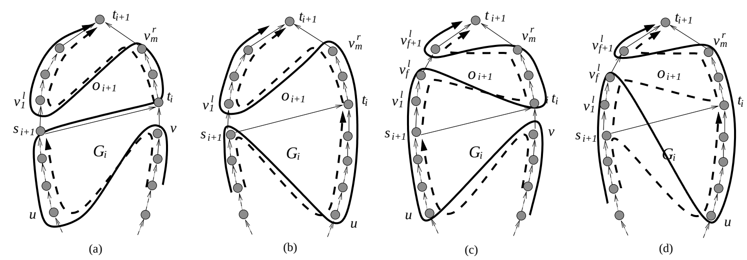

Figure 30: The hamiltonian paths for statement (3) of Lemma 16.

which is just concatenation of two solutions.

Case 1d. *The hamiltonian path reaches t_i from a vertex on the right side of G_i and the size of the HP-completion of G_{i+1} is two.* Figure 28.d shows the resulting pathes. From the figure, it follows that $c(G_{i+1}, L) = c(G_i, R) + c^2(o_{i+1}, L)$, which is again a simple concatenation of two solutions.

$$\textbf{(2)} \quad t_i \in V^l \Rightarrow c(G_{i+1}, R) = \min \begin{array}{r} \{c(G_i, L) + c^1(o_{i+1}, R), c(G_i, R) + c^1(o_{i+1}, R) \\ c(G_i, L) + c^2(o_{i+1}, R) + 1, c(G_i, R) + c^2(o_{i+1}, R)\} \end{array}$$

Case 2a. *The hamiltonian path enters t_i from a vertex on the left side of G_i and the size of the HP-completion of G_{i+1} is one.* Figure 29.a shows the resulting path. From the figure, it follows that $c(G_{i+1}, R) = c(G_i, L) + c^1(o_{i+1}, R)$, that is, a simple concatenation of the two solutions.

Case 2b. *The hamiltonian path reaches t_i from a vertex on the right side of G_i and the size of the HP-completion of G_{i+1} is one.* Figure 29.b shows the resulting path. From the figure, it follows that $c(G_{i+1}, R) = c(G_i, R) + c^1(o_{i+1}, R)$, that is, a simple concatenation of the two solutions.

Case 2c. *The hamiltonian path enters t_i from a vertex on the left side of G_i and the size of HP-completion set of G_{i+1} is two.* Figure 29.c shows the resulting path. From the figure, it follows that $c(G_{i+1}, R) = c(G_i, L) + c^2(o_{i+1}, R) + 1$. Note that the added edge (v_i^r, u) creates one more crossing than the number of crossings caused by edges (v_i^r, v_1^l), (s_{i+1}, u) taken together. The additional crossing is due to the crossing of the lower limiting edge of o_{i+1}.

Case 2d. *The hamiltonian path reaches t_i from a vertex on the right side of G_i and the size of HP-completion set of G_{i+1} is two.* Figure 29.d shows the resulting path. From the figure, it follows that $c(G_{i+1}, R) = c(G_i, R) + c^2(o_{i+1}, R)$, that is, a simple concatenation of the two solutions.

The proofs for statements (3) and (4) are symmetric to those of statements (2) and (1), respectively. Figures 30 and 31 show how to construct the corresponding hamiltonian paths in each case.

In order to complete the proof, we need to also show that the constructed hamiltonian paths which cause the stated number of crossings are optimal. The basic idea of the proof is the following: we assume $P_{G_{i+1}}^{\text{opt}}$ is a crossing-optimal

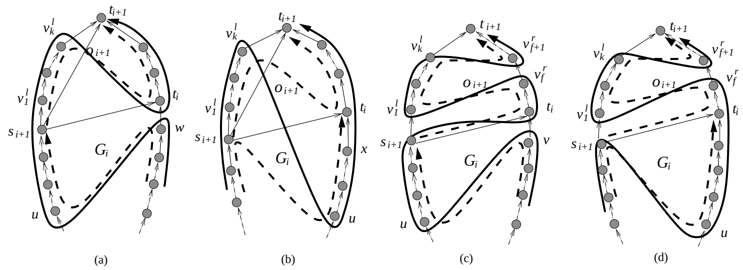

Figure 31: The hamiltonian paths for statement (4) of Lemma 16.

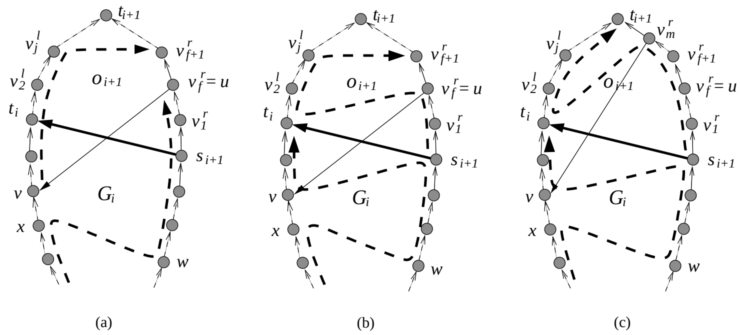

Figure 32: The configuration used in proving that the Hamiltonian paths obtained as described in Lemma 16 are optimal.

solution for G_{i+1} and, based on $P^{\text{opt}}_{G_{i+1}}$, we identify two solutions P_{G_i} and $P_{o_{i+1}}$ for G_i and o_{i+1}, respectively, such that (i) they are crossing-optimal and (ii) $P^{\text{opt}}_{G_{i+1}}$ can be obtained from P_{G_i} and $P_{o_{i+1}}$ as one of the four cases in the statement of the Lemma. We proceed by distinguishing cases based on the structure of $P^{\text{opt}}_{G_{i+1}}$.

We denote by $\tilde{e}$ the common edge of G_i and o_{i+1}. Note that $\tilde{e}$ is the lower limiting edge of o_{i+1} (refered in the rest of the proof as, simply, *limiting edge*).

Without loss of generality, we assume that the limiting edge $\tilde{e}$ is directed from the right to the left side of the outerplanar graph. From Lemma 11 it follows that $\tilde{e} = (s_{i+1}, t_i)$. So, by our assumption, s_{i+1} is on the right side, while t_i is on the left.

Case 1. $P^{\text{opt}}_{G_{i+1}}$ ***contains an edge*** e ***crossing the limiting edge*** $\tilde{e}$***.*** By Property 3, there can be only one such edge in $P^{\text{opt}}_{G_{i+1}}$, denote it by $e = (u, v)$. By Property 2, we can assume that e does not leave G_i and, thus, vertex u is above edge (s_{i+1}, t_i) while v is below it. In addition, u must be on the right side while v is on the left. To see that, simply observe that if u was on the left side and v was on the right or, if they were both on the same side, then a cycle would be created.

Next, we construct two solutions, P_{G_i} and $P_{o_{i+1}}$ for G_i and o_{i+1}, respectively, from the given optimal solution $P_{G_{i+1}}^{\text{opt}}$ and show that they are both also optimal.

Suppose, as showed in Figure 32.a, that $u = v_f^r$ and let x be the vertex positioned just below vertex v. Since there can exist only one edge in $P_{G_{i+1}}^{\text{opt}}$ crossing the limiting edge $\tilde{e}$ (by Property 2), $P_{G_{i+1}}^{\text{opt}}$ approaches vertex u from the right side. Follow the hamiltonian path backwards from u on the right side and let w be the last vertex we visit before we switch to the left side (see Figure 32.a). By Lemma 3, the vertex on the left side has to be located just below v, that is, it coincides with vertex x. Consider now the structure of $P_{G_{i+1}}^{\text{opt}}$ following edge e. It surely continues on the left side and leaves the left side above t_i, since e is the only edge connecting G_i and o_{i+1}. In Figure 32.a the solution $P_{G_{i+1}}^{\text{opt}}$ is shown by a dashed bold line. Now, we can set path P_{G_i} and $P_{o_{i+1}}$ as follows: P_{G_i} is identical to $P_{G_{i+1}}^{\text{opt}}$ till vertex x on the left side of G_i (see Figure 32.b), then it switches to the right, as $P_{G_{i+1}}^{\text{opt}}$ does, to vertex w and it continues on the right side till vertex s_{i+1}, then it switches to the left to vertex v and continues on the left side till vertex t_i. $P_{o_{i+1}}$ starts on the right side at vertex s_{i+1} and continues till vertex u, then it switches to the left side to vertex t_i and continues identical to $P_{G_{i+1}}^{\text{opt}}$. Note that if vertex u coincides with the last vertex of the right side, then solution $P_{o_{i+1}}$ terminates on the left side (see Figure 32.c). Observe now that $P_{G_{i+1}}^{\text{opt}}$ can be obtained from P_{G_i} and $P_{o_{i+1}}$ by cases 1.a and 2.c of the Lemma (Figures: 28.a and 29.c).

Next we show that P_{G_i} and $P_{o_{i+1}}$ are the optimal solutions for G_i and o_{i+1}, respectively. Suppose first that $P_{o_{i+1}}$ is not an optimal solution. Then, there is a solution $P'_{o_{i+1}}$ which has a smaller number of crossings than $P_{o_{i+1}}$. So, if we combine $P'_{o_{i+1}}$ with P_{G_i} using one of the rules of the Lemma we get a better solution than $P_{G_{i+1}}^{\text{opt}}$, a contradiction. Similarly, suppose that P_{G_i} is not an optimal solution and let P'_{G_i} which one with a smaller number of crossings. Then, by combining P'_{G_i} with $P_{o_{i+1}}$ we get a better solution than $P_{G_{i+1}}^{\text{opt}}$, a contradiction.

Case 2. $P_{G_{i+1}}^{\text{opt}}$ ***does not contain any edge crossing the limiting edge*** $\tilde{e}$**.** In this case, by Lemma 3, $P_{G_{i+1}}^{\text{opt}}$ visits first all the vertices of G_i and then all the vertices of o_{i+1}. We will split $P_{G_{i+1}}^{\text{opt}}$ into two paths P_{G_i} and $P_{o_{i+1}}$ which are solutions for G_i and o_i, respectively.

Case 2a. *The last vertex of* $P_{G_{i+1}}^{\text{opt}}$ *before* t_i *is on the left side.* Hence, the last visited vertex before t_i is the vertex placed just below the vertex t_i on the left side. Set P_{G_i} to be the subpath of $P_{G_{i+1}}^{\text{opt}}$ terminating to vertex t_i and $P_{o_{i+1}}$ to consist of edge (s_{i+1}, t_i) followed by the subpath of $P_{G_{i+1}}^{\text{opt}}$ starting from vertex t_i. Now, note that $P_{G_{i+1}}^{\text{opt}}$ can be obtained from P_{G_i} and $P_{o_{i+1}}$ by cases 1.c and 2.a (see Figures: 28.c and 29.a). It is easy to see that P_{G_i} and $P_{o_{i+1}}$ are optimal. If we suppose that one of P'_{G_i} or $P'_{o_{i+1}}$ is better than P_{G_i} or $P_{o_{i+1}}$, respectively, then, combining P'_{G_i} with $P_{o_{i+1}}$, or P_{G_i} with $P'_{o_{i+1}}$ gives us a better solution, a clear contradiction.

Case 2b. *The last vertex of* $P_{G_{i+1}}^{\text{opt}}$ *before* t_i *is on the right side.* We distinguish two cases based on the vertex before t_i:

- *The vertex before t_i is s_{i+1}.* This case corresponds to cases 1.d and 2.b (see Figures: 28.d and 29.b). The figures describe how to construct optimal solutions for G_i and o_{i+1} from $P^{\text{opt}}_{G_{i+1}}$. Proving that these solutions are optimal, proceeds in a way identical of that in Case 2a.

- *The vertex before t_i is a vertex v^r_i on the right side which is above s_{i+1}.* This case corresponds to the cases 1.b and 2.d (see Figures: 28.b and 29.d). The figures describe how to construct optimal solutions for G_i and o_{i+1} from $P^{\text{opt}}_{G_{i+1}}$.

Note that, there is not necessarily a unique hamiltonian path that yields an optimal solution. Since in our construction we apply a "minimum" operator, more than one of the involved hamiltonian paths yield the same number of crossings, and thus, we might have more than one different equivalent (with respect to edge crossings) hamiltonian paths. □

Algorithm 1 is a dynamic programming algorithm, based on Lemmata 15 and 16, which computes the minimum number of edge crossings $c(G)$ resulting from the addition of a crossing-optimal HP-completion set to an outerplanar st-digraph G. The algorithm can be easily extended to also compute the corresponding hamiltonian path $S(G)$.

Theorem 5 *Given an n-vertex outerplanar st-digraph G, a crossing-optimal HP-completion set for G and the corresponding number of edge-crossings can be computed in $O(n)$ time.*

Proof: Algorithm 1 computes the number of crossings in an acyclic HP-completion set. Note that it can be easily extended so that it computes the actual hamiltonian path (and, as a result, the acyclic HP-completion set). To achieve this, we only need to store in an auxiliary array the term that resulted to the minimum values in Step 4 of the algorithm, together with the endpoints of the edge that is added to the HP-completion set for each st-Polygon in the st-Polygon decomposition $\mathcal{D}(G) = \{o_1, \ \ldots, o_\lambda\}$ of G. The correctness of the algorithm follows immediately from Lemmata 15 and 16.

From Lemma 7 and Theorem 3, it follows that Step 1 of the algorithm needs $O(n)$ time. The same holds for Step 2 (due to Lemma 14). Step 3 is an initialization step that needs $O(1)$ time. Finally, Step 4 requires $O(\lambda)$ time. In total, the running time of Algorithm 1 is $O(n)$. Observe that $O(n)$ time is enough to also recover the acyclic HP-completion set. □

4 Spine Crossing Minimization for Upward Topological 2-Page Book Embeddings of Outerplanar st-Digraphs

In this section, we establish for the class of planar st-digraphs an equivalence (through a linear time transformation) between the acyclic-HPCCM problem

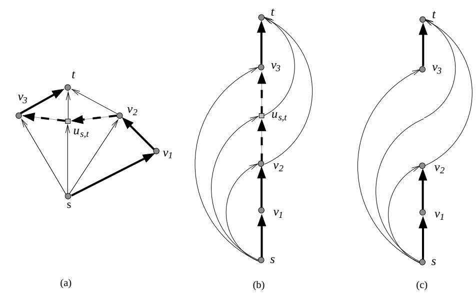

Figure 33: (a) A drawing of an HP-extended digraph for an *st*-digraph G. The dotted segments correspond to the single edge (v_2, v_3) of the HP-completion set for G. (b)An upward topological 2-page book embedding of G_c with its vertices placed on the spine in the order they appear on a hamiltonian path of G_c. (c)An upward topological 2-page book embedding of G.

and the problem of obtaining an upward topological 2-page book embeddings with minimum number of spine crossings. We exploit this equivalence to develop an optimal (with respect to spine crossings) book embedding for outerplanar *st*-digraphs.

Theorem 6 *Let $G = (V, E)$ be an n-vertex planar st-digraph. G has a crossing-optimal HP-completion set E_c with Hamiltonian path $P = (s = v_1, v_2, \dots, v_n = t)$ such that the corresponding optimal drawing $\Gamma(G')$ of $G' = (V, E \cup E_c)$ has c crossings **if and only if** G has an optimal (with respect to the number of spine crossings) upward topological 2-page book embedding with c spine crossings where the vertices appear on the spine in the order $\Pi = (s = v_1, v_2, \dots, v_n = t)$.*

Proof: We show how to obtain from an HP-completion set with c edge crossings an upward topological 2-page book embedding with c spine crossings and vice versa. It then follows that a crossing-optimal HP-completion set for G with c edge crossings corresponds to an optimal upward topological 2-page book embedding with the same number of spine crossings.
"$\Rightarrow$" We assume that we have an HP-completion set E_c that satisfies the conditions stated in the theorem. Let $\Gamma(G')$ of $G' = (V, E \cup E_c)$ be the corresponding drawing that has c crossings and let $G_c = (V \cup V_c, E' \cup E'_c)$ be the acyclic HP-extended digraph of G with respect to $\Gamma(G')$. V_c is the set of new vertices placed at each edge crossing. E' and E'_c are the edge sets resulting from E and E_c, respectively, after splitting their edges involved in crossings and maintaining their orientation (see Figure 33(a)). Note that G_c is also a planar *st*-digraph.

Observe that in $\Gamma(G')$ we have no crossing involving two edges of G. If this was the case, then $\Gamma(G')$ would not preserve G. Similarly, in $\Gamma(G')$ we have no crossing involving two edges of the HP-completion set E_c. If this was the case, then G_c would contain a cycle.

The hamiltonian path P on G' induces a hamiltonian path P_c on the HP-extended digraph G_c. This is due to the facts that: (i) all edges of E_c are used in the hamiltonian path P and (ii) all vertices of V_c correspond to crossings involving edges of E_c. We use the hamiltonian path P_c to construct an upward topological 2-page book embedding for graph G with exactly c spine crossings. We place the vertices of G_c on the spine in the order of hamiltonian path P_c, with vertex $s = v_1$ being the lowest. Since the HP-extended digraph G_c is a planar st-digraph with vertices s and t on the external face, each edge of G_c appears either to the left or to the right of the hamiltonian path P_c. We place the edges of G_c on the left (respectively right) page of the book embedding if they appear to the left (respectively right) of path P_c. The edges of P_c are drawn on the spine (see Figure 33(b)). Later on they can be moved to any of the two book pages.

Note that all edges of E_c appear on the spine. Consider any vertex $v_c \in V_c$. Since v_c corresponds to a crossing between an edge of E and an edge of E_c, and the edges of E'_c incident to it have been drawn on the spine, the two remaining edges of E' correspond to (better, they are parts of) an edge $e \in E$ and drawn on different pages of the book. By removing vertex v_c and merging its two incident edges of E' we create a crossing of edge e with the spine. Thus, the constructed book embedding has as many spine crossings as the number of edge crossings of HP-completed graph G' (see Figure 33(c)).

It remains to show that the constructed book embedding is upward. It is sufficient to show that the constructed book embedding of G_c is upward. For the sake of contradiction, assume that there exists a downward edge $(u, w) \in E'_c$. By construction, the fact that w is drawn below u on the spine implies that there is a path in G_c from w to u. This path, together with edge (u, w) forms a cycle in G_c, a clear contradiction since G_c is acyclic.

"$\Leftarrow$" Assume that we have an upward 2-page topological book embedding of st-digraph G with c spine crossings where the vertices appear on the spine in the order $\Pi = (s = v_1, v_2, \ldots, v_n = t)$. Then, we construct an HP-completion set E_c for G that satisfies the condition of the theorem as follows: $E_c = \{(v_i, v_{i+1}) \mid 1 \leq i < n \text{ and } (v_i, v_{i+1}) \notin E\}$, that is, E_c contains an edge for each consecutive pair of vertices of the spine that (the edge) was not present in G. By adding/drawing these edges on the spine of the book embedding we get a drawing $\Gamma(G')$ of $G' = (V, E \cup E_c)$ that has c edge crossings. This is due to the fact that all spine crossing of the book embedding are located, (i) at points of the spine above vertex s and below vertex t, and (ii) at points of the spine between consecutive vertices that are not connected by an edge. By inserting at each crossing of $\Gamma(G')$ a new vertex and by splitting the edges involved in the crossing while maintaining their orientation, we get an HP-extended digraph G_c. It remains to show that G_c is acyclic. For the sake of contradiction, assume that G_c contains a cycle. Then, since graph G is acyclic, each cycle of G_c must contain a segment resulting from the splitting of an edge in E_c. Given that in $\Gamma(G')$ all vertices appear on the spine and all edges of E_c are drawn upward, there must be a segment of an edge of G that is downward in order to close the cycle. Since, by construction, the book embedding of G is a sub-drawing of $\Gamma(G')$, one of its edges (or just a segment of it) is downward. This is a clear contradiction since we assume that the topological 2-page book embedding of G is upward. □

Theorem 7 *Given an n-vertex outerplanar st-digraph G, an upward 2-page topological book embedding for G with minimum number of spine crossings and the corresponding number of spine-crossings can be computed in $O(n)$ time.*

Proof: By Theorem 6 we know that by solving the acyclic-HPCCM problem on G, we can deduce the wanted upward 2-page topological book embedding. By Theorem 5, the acyclic-HPCCM problem can be solved in $O(n)$ time. □

5 Conclusions and Open Problems

We have studied the problem of acyclic-HPCCM and we have presented a linear time algorithm that computes a crossing-optimal acyclic HP-completion set, and hence an upward topological book embedding with a minimum number of spine crossings, for outerplanar *st*-digraphs.

We emphasize that there exist outerplanar *st*-digraphs that can be upward book embedded with a constant number of spine crossings for which the previously known algorithm [13] required $O(n)$ spine crossings. This is demonstrated by the graph of Figure 4.a (slightly modified so that at its left side it has $O(n)$ vertices) that can be upward book embedded with only four spine crossings since the corresponding crossing-optimal HP-completion set creates four crossings (see Figure 4.d). Figure 4.b (respectively Figure 4.c) shows the HP-completion set that corresponds to the upward topological book embedding produced by the algorithm presented in [13] if the "left-to-right" dual (respectively the "rigth-to-left" dual) is used. It is clear that the number of edge crossings, and hence the number of spine crossings in the corresponding upward topological book embedding, of the modified graph (with $O(n)$ vertices on its left side) is $O(n)$.

The outerplanar *st*-digraphs studied in this paper is the first class of *st*-digraphs for which we were able to determine a crossing-optimal HP-completion set. While recently, the acyclic-HPCCM problem has been solved efficiently for classes of N-free and bounded-width upward planar digraphs [21], the complexity of the general case still remains an open problem.

The outerplanar *st*-digraphs studied in this paper, as well as the N-free and bounded-width upward planar digraphs [21] are the first classes of *st*-digraphs for which we were able to determine a crossing-optimal HP-completion set. A natural research direction is to study the acyclic-HPCCM problem on the larger class of *st*-digraphs. For this case, no polynomial time algorithm is known.

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