

On the Maximum Independent Set Problem in Subclasses of Planar Graphs

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Abstract

The MAXIMUM INDEPENDENT SET problem is known to be NP-hard in the class of planar graphs. In the present paper, we study its complexity in hereditary subclasses of planar graphs. In particular, by combining various techniques, we show that the problem is polynomially solvable in the class of $S_{1,2,k}$ -free planar graphs, generalizing several previously known results. $S_{1,2,k}$ is the graph consisting of three induced paths of lengths 1, 2 and k , with a common initial vertex.

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1 Introduction

An *independent set* (also called *stable set*) in a graph G is a subset of pairwise non-adjacent vertices. The MAXIMUM INDEPENDENT SET problem (IS for short) is that of finding in a graph an independent set of maximum cardinality. The problem is known to be NP-hard in general. Moreover, it remains NP-hard even under substantial restrictions, for instance, for triangle-free graphs [20], $K_{1,4}$ -free graphs [19], and planar graphs of degree at most three [9]. On the other hand, for graphs in some particular classes such as perfect graphs or claw-free graphs, the problem can be solved in polynomial time. We will call a class of graphs X *IS-easy* if the IS problem admits a polynomial-time solution for graphs in X .

This paper is focused on complexity issues related to the maximum independent set problem in planar graphs. While some problems that are intractable for general graphs are solvable in polynomial time in planar graphs (e.g. MAX CLIQUE, MAX CUT [11]), this is not the case for the IS problem. As mentioned above, the problem remains hard even for planar graphs of degree at most three [9]. It is therefore interesting to study the maximum independent set problem in proper subclasses of planar graphs. This topic has been a frequent subject of investigation in the literature (see e.g. [18, 14, 13]).

All the above mentioned classes possess the property that with any graph G they contain all induced subgraphs of G . Such classes are called *hereditary*. This family of graph classes is of particular interest, since hereditary (and only hereditary) classes admit a uniform description in terms of forbidden induced subgraphs, which in turn provides a systematic way to study various graph problems. For a set \mathcal{F} of graphs, we say that a graph G is \mathcal{F} -free if it does not contain an induced subgraph isomorphic to a member of \mathcal{F} . Our objective is to distinguish conditions on the set \mathcal{F} that would imply polynomial-time solvability or NP-hardness of the IS problem in the class of \mathcal{F} -free planar graphs. In this respect, a promising direction is suggested by the following theorem proved by Alekseev [1].

Let \mathcal{S} denote the set of graphs each connected component of which is of the form $S_{i,j,k}$ (see Figure 1), where the values of $i, j, k \geq 0$ may depend on the component.

Theorem 1 ([1]) *Let X be the class of graphs defined by a set \mathcal{F} of forbidden induced subgraphs. If \mathcal{F} is finite and contains no graph from \mathcal{S} , then the MAXIMUM INDEPENDENT SET problem is NP-hard in X .*

In particular, the NP-hardness of the problem in triangle-free and $K_{1,4}$ -free graphs can be obtained as a corollary of Theorem 1. Theorem 1 implies that, unless $P=NP$, the class of \mathcal{F} -free graphs (for a finite \mathcal{F}) can be IS-easy only if the set \mathcal{F} of forbidden induced subgraphs contains a graph from \mathcal{S} .

Theorem 1 remains valid even if the input is restricted to planar graphs of degree at most three. More formally:

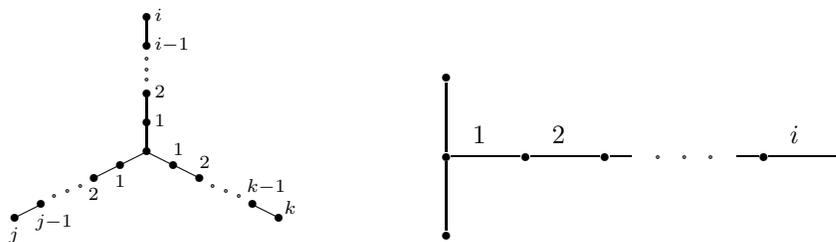


Figure 1: Graphs $S_{i,j,k}$ (left) and H_i (right)

Theorem 2 *Let \mathcal{F} be a finite set of graphs and let X be the class of \mathcal{F} -free planar graphs of degree at most three. If \mathcal{F} contains no graph from \mathcal{S} , then the MAXIMUM INDEPENDENT SET problem is NP-hard in X .*

Proof: The proof is a slight modification of the proof of Theorem 1, so we present only the main ideas. Theorem 1 was obtained using a reduction from the IS problem in graphs of degree at most three, in two steps: First, by performing sufficiently many double subdivisions of the edges of the input graph, it can be shown that the IS problem is NP-hard for $(C_1, \dots, C_k, H_1, \dots, H_k)$ -free graphs of degree at most three (for any fixed $k \geq 3$), where C_i and H_i denote the cycle of length i and the graph on the right of Figure 1, respectively. Second, it can be shown that for every \mathcal{F} as in the theorem there is a $k \geq 3$ such that the class of \mathcal{F} -free graphs of degree at most three contains the class of $(C_1, \dots, C_k, H_1, \dots, H_k)$ -free graphs of degree at most three.

Subdividing edges preserves planarity, therefore one could perform the same reduction from the IS problem in planar graphs of degree at most three, to show that the IS problem is NP-hard in $(C_1, \dots, C_k, H_1, \dots, H_k)$ -free planar graphs of degree at most three. The second part of the argument is the same as in the original proof. The claimed result follows. \square

For planar graphs, this result implies a similar conclusion as Theorem 1 does for general graphs: Unless $P=NP$, the class of \mathcal{F} -free planar graphs (for a finite \mathcal{F}) can be IS-easy only when \mathcal{F} contains a graph from \mathcal{S} . Note that for infinite \mathcal{F} , this need not be the case (as shown by the classes of forests, bipartite graphs, or perfect graphs). Henceforth, we will assume that \mathcal{F} is finite.

A classical result of this type is the polynomial-time solution to the IS problem for claw-free (i.e., $S_{1,1,1}$ -free) graphs, obtained independently by Minty [19], Sbihi [21], and Lovász and Plummer [15]. This result has been further extended by Alekseev to fork-free (i.e., $S_{1,1,2}$ -free) graphs [3]. Other examples include P_4 -free graphs (also known as *cographs*) [5], and $(S_{1,1,1} + K_2)$ -free graphs [17], where by $A + B$ we denote the disjoint union of graphs A and B .

Clearly, whenever X is an IS-easy class of graphs, the class of planar graphs in X is IS-easy. In some cases however, the IS-easiness of a class relies on the planarity assumption. For example, this is the case for the class of P_k -free

planar graphs, i.e., planar graphs excluding a path P_k on k vertices as an induced subgraph. In general, for $k \geq 5$, the complexity of the problem in P_k -free graphs is unknown. On the other hand, for P_k -free *planar graphs*, the following result holds.

Proposition 1 *For any $k \geq 2$, the MAXIMUM INDEPENDENT SET problem admits a linear-time solution for P_k -free planar graphs.*

Proof: It suffices to solve the problem for connected graphs in the class. Every connected P_k -free graph is of diameter at most $k - 2$, and the treewidth of planar graphs is bounded above by a function of their diameter [8, 6]. It follows that the treewidth of P_k -free planar graphs is bounded above by a constant. A linear-time algorithm for the IS problem in P_k -free planar graphs now follows as the problem is solvable in linear-time on graphs of bounded treewidth (see e.g. [4] for a proof of a more general statement). \square

Our contribution. Our main result is the following theorem.

Theorem 3 *For any $k \geq 2$, the MAXIMUM INDEPENDENT SET problem is polynomially solvable for $S_{1,2,k}$ -free planar graphs.*

This result is interesting for two reasons. First, we extend some known polynomial-time results for the IS problem in subclasses of planar graphs, such as P_k -free planar, $S_{1,1,2}$ -free planar and $(S_{1,1,1} + K_2)$ -free planar graphs. Secondly, our solution combines two approaches to the IS problem which, to the best of our knowledge, have so far only been used separately. These are the *augmenting graph method* and the *decomposition by clique separators*. The former has been used to develop polynomial-time solutions to the IS problem e.g. in claw-free [19] and fork-free graphs [3], while the latter provides efficient solutions to the IS problem in $(P_5, co-(P_2 + P_3))$ -free graphs [2] and chordal graphs (it follows from a result of Dirac [7] that every connected chordal graph without clique separators is a complete graph).

As can be seen from the proofs, the result of Theorem 3 can be extended to a more general setting: we can replace the planarity assumption by the condition “the input graph does not contain a $K_{3,3}$ -minor.”

Organization. The paper is structured as follows. In Section 2 we present the necessary background, which will be needed in Section 4. In Section 3, we develop a reduction of the IS problem from $S_{1,2,k}$ -free planar graphs to $S_{1,2,2}$ -free planar graphs, by means of bounding the treewidth. Finally, Section 4 is devoted to the solution to the IS problem in $S_{1,2,2}$ -free planar graphs. The solution combines the technique of finding augmenting graphs with a reduction to 2-connected components.

Notation and definitions. All graphs considered are finite, simple and undirected. We use standard graph terminology and customary notation. As usual,

P_n and C_n denote the chordless path and the chordless cycle on n vertices, respectively. K_n denotes the complete graph on n vertices, and $K_{m,n}$ the complete bipartite graph with parts of size m and n . For a vertex $x \in V(G)$, we denote by $N(x)$ the neighborhood of x , i.e., the set of vertices adjacent to x . The *degree* of x is the size of its neighborhood. For a set $A \subseteq V(G)$, we denote by $N(A)$ the set $\cup_{a \in A} \{u \in N(a) : u \notin A\}$, and for sets $A, B \subseteq V(G)$ we denote $N_B(A) := N(A) \cap B$. Instead of $N_B(\{v\})$, we simply write $N_B(v)$.

The *diameter* of a connected graph G is the maximum distance between two vertices of the graph. The *distance* between two subsets U, W of vertices of a connected graph G is defined as $\min\{d(u, w) : u \in U, w \in W\}$, where $d(u, w)$ stands for the *length* (i.e., the number of edges) of a shortest u - w path in G . A *cut-vertex* of a connected graph G is a vertex whose removal disconnects the graph. A *2-connected graph* is a connected graph with no cut-vertices. A *2-connected component* of a graph G is a maximal induced subgraph of G that is 2-connected.

2 Preliminaries: the Method of Augmenting Graphs

In this section, we briefly review the method of augmenting graphs, including the notion of a redundant set introduced in [16].

Let G be a graph and I an independent set in G . We will call the vertices of I *white* and the remaining vertices of G *black*.

Definition 1 *An augmenting graph for I in G is an induced bipartite subgraph $H = (W, B, E)$ of G , where $W \cup B$ is a bipartition of its vertex set and E is the set of its edges, such that $|B| > |W|$, $W \subseteq I$, $B \subseteq V(G) \setminus I$, and $N(B) \cap I \subseteq W$. A bipartite graph H will be called *augmenting* if there is a graph G and an independent set I of G such that H is augmenting for I in G .*

If H is augmenting for I , we also say that I *admits* the augmenting graph. Clearly if $H = (W, B, E)$ is an augmenting graph for I , then I is not a maximum independent set in G , since the set $I' = (I - W) \cup B$ is independent and $|I'| > |I|$. We will say that the set I' is obtained from I by *H -augmentation*. Conversely, if I is not a maximum independent set, and I' is an independent set such that $|I'| > |I|$, then the subgraph of G induced by the set $(I - I') \cup (I' - I)$ is augmenting for I . Therefore, the following key result holds.

Theorem of augmenting graphs. *An independent set I in a graph G is maximum if and only if there are no augmenting graphs for I .*

This theorem suggests the following general approach to find a maximum independent set in a graph G : begin with any independent set I in G and as long as I admits an augmenting graph H , apply *H -augmentation* to I . From the NP-hardness of the MAXIMUM INDEPENDENT SET problem and the Theorem of

augmenting graphs we conclude that the problem of finding augmenting graphs is generally NP-hard. However, for graphs in particular classes, such as $S_{1,1,1}$ -free [19] or $S_{1,1,2}$ -free graphs [3], it can be solved efficiently. To simplify the problem, we first observe that, without loss of generality, we may restrict our attention to those augmenting graphs that are minimal.

Definition 2 *An augmenting graph H for a set I is called minimal if no proper induced subgraph of H is augmenting for I . A bipartite graph H will be called minimal augmenting if there is a graph G and an independent set I of G such that H is minimal augmenting for I in G .*

The following lemma characterizes minimal augmenting graphs (which are then easily seen to be connected).

Lemma 1 ([16]) *An augmenting graph $H = (W, B, E)$ is minimal augmenting if and only if $|W| = |B| - 1$, and every nonempty subset $A \subseteq W$ satisfies $|A| < |N(A)|$.*

For a polynomial-time implementation of the augmenting graph approach in a class of graphs X , one has to

- (a) characterize all minimal augmenting graphs in X ,
- (b) develop a polynomial-time procedure for detecting them.

Point (a) above can be simplified by means of the following notion introduced in [16].

Definition 3 *In an augmenting graph $H = (W, B, E)$, a subset of vertices U satisfying $|U \cap W| = |U \cap B|$ will be called redundant if H contains no edges between black vertices of U and vertices of $H - U$.*

It was proved in [16] that, for the sake of a polynomial-time implementation of the augmenting graph approach, augmenting graphs that contain a redundant set of bounded size (i.e., of size not exceeding a certain constant) are irrelevant. The problem of finding such graphs can be reduced in polynomial time to the problem of finding augmenting graphs without small redundant sets. Therefore, we do not even need to characterize minimal augmenting graphs in X that contain small redundant sets; they can be safely omitted from the characterization mentioned in point (a) above.

3 Reduction to $S_{1,2,2}$ -free Planar Graphs

In this section, we show that in order to develop a polynomial-time solution to the IS problem in planar $S_{1,2,k}$ -free graphs, it suffices to solve the case $k = 2$. In our reduction to the latter case, we use the fact that the MAXIMUM (WEIGHT) INDEPENDENT SET problem in a hereditary class of graphs can be restricted,

without loss of generality, to 2-connected graphs in the class. This follows from a more general statement that allows us to consider only graphs without clique separators. (A *clique separator* in a connected graph G is a clique C in G whose removal disconnects G .) The corresponding algorithmic tool is called *decomposition by clique separators* and has proved useful in developing algorithms for several graph optimization problems (see the papers by Whitesides [23] and Tarjan [22] for the weighted, and the paper by Alekseev [2] for the unweighted case).

We will thus restrict our attention to 2-connected graphs. The following auxiliary result will prove useful for our reduction.

Lemma 2 *For any $k \geq 2$, the diameter of every 2-connected $S_{1,2,k}$ -free planar graph G that contains an induced copy of $S_{1,2,2}$ is at most $2k + 4$.*

Proof: Consider an induced copy F of $S_{1,2,2}$ in a 2-connected $S_{1,2,k}$ -free planar graph G . Let $V(F) = \{a, b, c, d, e, f\}$ and $E(F) = \{ab, ac, ad, ce, df\}$. We will show that no vertex in G has distance greater than k from $V(F)$. In turn, this will imply that no vertex in G has distance greater than $k + 2$ from a , the vertex of degree 3 in F . By the triangle inequality, this will imply that the diameter of G is at most $2k + 4$.

For a positive integer j , let us denote by V_j the set of all vertices in G at distance j from $V(F)$. Our goal is to show that $V_{k+1} = \emptyset$.

Assume, for contradiction, that there is a vertex v at distance $k + 1$ from $V(F)$. Let $P = (v_0, v_1, \dots, v_{k+1})$ be a shortest $V(F)$ - v path in G with $v_0 \in V(F)$, $v = v_{k+1}$ and $v_i \in V_i$ for all $1 \leq i \leq k + 1$. We distinguish several cases with respect to the distance in G between v_1 and a .

Case 1. $d(v_1, a) = 3$. This means that v_1 is adjacent to one of $\{e, f\}$ (or both), but not to any of $\{a, b, c, d\}$. Without loss of generality, we may assume that $v_0 = e$. Now, if v_1 is adjacent to f , then an $S_{1,2,k}$ arises on the vertex set $V(P) \cup \{c, f\}$, which is impossible. Otherwise, the vertex set $V(P) \cup V(F)$ induces an $S_{1,2,k+3}$, again a contradiction.

Case 2. $d(v_1, a) = 2$. In this case, v_1 is adjacent to one (or more) of $\{b, c, d\}$, but not to a . We distinguish two subcases.

2.1. v_1 is adjacent to b . In this case, we may assume that $v_0 = b$. Then v_1 is not adjacent to e (otherwise an $S_{1,2,k}$ arises on the vertex set $V(P) \cup \{a, e\}$). Similarly, v_1 is not adjacent to f . Also, v_1 is not adjacent to c (or there is an induced $S_{1,2,k}$ on $V(P) \cup \{c, e\}$), and similarly v_1 is not adjacent to d . But now, an $S_{1,2,k+2}$ arises on $V(P) \cup \{a, c, d, e\}$. Contradiction.

2.2. v_1 is not adjacent to b . Without loss of generality, we may assume that $v_0 = c$. We see that v_1 is adjacent to e (otherwise an $S_{1,2,k+1}$ arises on the vertex set $V(P) \cup \{a, b, e\}$). Next, v_1 is not adjacent to f (or there is an induced $S_{1,2,k}$ on $V(P) \cup \{a, f\}$). Moreover, v_1 is not adjacent to d (or there is an induced $S_{1,2,k}$ on $V(P) \cup \{d, f\}$). But now, an $S_{1,2,k+2}$ arises on $V(P) \cup \{a, b, d, f\}$. Contradiction.

Case 3. $d(v_1, a) = 1$. Without loss of generality, we may assume that $v_0 = a$. We distinguish three subcases.

3.1. v_1 is not adjacent to b . Then v_1 is not adjacent to e (otherwise an $S_{1,2,k}$ arises on the vertex set $V(P) \cup \{b, e\}$), and by symmetry, v_1 is not adjacent to f . Next, we see that v_1 is adjacent to c (or there is an induced $S_{1,2,k+1}$ on $V(P) \cup \{b, c, e\}$), and, similarly, v_1 is adjacent to d . But now, an $S_{1,2,k}$ arises on $V(P) \setminus \{a\} \cup \{c, d, e\}$. Contradiction.

3.2. v_1 is adjacent to b and not adjacent to any of $\{c, e\}$. In this case, v_1 is not adjacent to f (otherwise an $S_{1,2,k}$ arises on the vertex set $V(P) \cup \{c, f\}$). Next, v_1 is adjacent to d (or there is an induced $S_{1,2,k+1}$ on $V(P) \cup \{c, d, e\}$). But now, an $S_{1,2,k}$ arises on $V(P) \setminus \{a\} \cup \{b, d, f\}$. Contradiction.

3.3. v_1 is adjacent to b , to at least one of $\{c, e\}$, and to at least one of $\{d, f\}$. Since G is 2-connected, it contains an induced $V(F)$ - v path $P' = (u_0, u_1, \dots, u_l)$ with $u_0 \in V(F)$ and $l \geq k + 1$ that does not contain v_1 . Without loss of generality, we may assume that P' is a shortest $V(F)$ - v path in $G - v_1$. Since P' cannot correspond to any of the already considered cases (with v_1 replaced by u_1), we conclude that u_1 is adjacent to b , to at least one of $\{c, e\}$, and to at least one of $\{d, f\}$. However, it is now easy to see that G contains $K_{3,3}$ as a minor (on the vertex set $V(F) \cup \{v_1, u_1\}$). This contradiction completes this case and the proof of the lemma. \square

Recall that the treewidth of a planar graph is bounded above by a function of its diameter [8, 6]. Since the IS problem is solvable in linear time on graphs of bounded treewidth [4], the following result holds.

Corollary 1 *For any $k \geq 2$, the MAXIMUM INDEPENDENT SET problem for $S_{1,2,k}$ -free planar graphs is polynomially equivalent to the MAXIMUM INDEPENDENT SET problem for $S_{1,2,2}$ -free planar graphs.*

4 The Maximum Independent Set Problem in the Class of $S_{1,2,2}$ -free Planar Graphs

Let X denote the class of $S_{1,2,2}$ -free planar graphs. In this section, we present a polynomial-time solution to the MAXIMUM INDEPENDENT SET problem for graphs in X . The main ingredients of our solution are the technique of augmenting graphs and reduction to 2-connected components.

To apply the augmenting graph technique, we have to characterize the minimal augmenting graphs in our class. We start by showing that minimal augmenting graphs in the class cannot contain vertices of arbitrarily high degree.

Lemma 3 *The maximum degree of minimal augmenting graphs in X is bounded by a constant.*

Proof: Let H be a minimal augmenting graph in X . The proof consists of two parts. First, we prove that *no black vertex of H has degree more than 9*.

Assume that H contains a black vertex x of degree 10 or more. By Lemma 1 and Hall's Theorem [12], we know that the subgraph $H - x$ has a perfect matching M . For a vertex $v \in V(H - x)$, we denote by $m(v)$ the unique vertex such

that $\{v, m(v)\}$ is an edge in M . Also, let A denote the set of neighbors of x , and $A' = \{m(v) : v \in A\}$. Since H does not contain a $K_{3,3}$ as a subgraph, we have $|A \cap N(u) \cap N(v)| \leq 2$ for all pairs of distinct vertices $u, v \in A'$.

Claim: A' contains at most one vertex with 5 or more neighbors in A .

Suppose, for contradiction, that A' contains two distinct vertices, say u and v , such that $|N(u) \cap A| \geq 5$ and $|N(v) \cap A| \geq 5$. Let $A_0 = N(u) \cap N(v) \cap A$ and $A_1 = A \setminus A_0$. Then $|A_0| \leq 2$, which implies $|A_1 \cap N(u)| \geq 3$. Let w denote a neighbor of u in A_1 , different from $m(u)$, and let $w' = m(w)$. Since u and w' have at most two common neighbors in A , there is a vertex in $A_1 \cap N(u)$, say z , that is non-adjacent to w' . Note that w and z are non-adjacent to v , since, by definition, A_1 contains no common neighbor of u and v . But now, the vertices $\{x, z, w, w', v', v\}$, where $v' \in (A \cap N(v)) \setminus N(w')$, induce a copy of $S_{1,2,2}$ in H . This contradiction shows the claim.

Therefore, A' contains a subset A'' of at least 9 vertices, each of which has at most 4 neighbors in A . Let $u, v \in A''$. Clearly A contains a vertex nonadjacent to both u and v . To avoid an induced $S_{1,2,2}$, we conclude that either $N(u) \cap A \subseteq N(v) \cap A$ or $N(v) \cap A \subseteq N(u) \cap A$. Therefore, the vertices of A'' admit an ordering $u_1, u_2, \dots, u_{|A''|}$ such that $N(u_{i+1}) \cap A \subseteq N(u_i) \cap A$ for each i . But then $N_A(u_1) \cap N_A(u_2) \supseteq \{m(u_2), m(u_3), m(u_4)\}$, which leads to an induced $K_{3,3}$ in H . This contradiction completes the first part of our proof.

Now let us show that *if H contains no black vertex of degree more than $k \geq 2$, then the degree of each white vertex is at most $4k - 3$.*

Assume that H contains a white vertex x of degree more than $4k - 3$, while no black vertex of H has degree more than $k \geq 2$. Fix an arbitrary neighbor b of x . As before, the subgraph $H - b$ has a perfect matching M . For a subset $U \subset V(H - b)$ of vertices of the same color, we denote by $m(U)$ the set of vertices of the opposite color matched with vertices of U with respect to M . For a vertex $a \in V(H - b)$, let $m(a) := m(\{a\})$. Denote $A_1 := N(x) \setminus \{b, m(x)\}$ and $A_2 := m(A_1) \setminus N(m(x))$.

Since $m(x)$ has at most $k - 1$ neighbors in the set $m(A_1)$, it follows that $|A_2| \geq 3k - 3$. Now, fix an arbitrary vertex $a \in A_2$, and let $A_3 = A_2 \setminus N(m(a))$. We see that $|A_3| \geq 2k - 2$.

Note that a is adjacent to all vertices in $m(A_3)$, since otherwise the vertices $x, m(x), m(a), a$ together with any vertex $v \in m(A_3)$ non-adjacent to a and its neighbor $m(v)$ induce an $S_{1,2,2}$ in H .

Since H does not contain an induced $K_{3,3}$, every vertex of A_3 has at most two neighbors in $m(A_3)$. Now, fix an arbitrary vertex $a' \in A_3$. In particular, given $|A_3| \geq 2k - 2$ and the bound on the degree of black vertices, this implies that there is a vertex $a'' \in A_3$ which shares no neighbor with a' in the set $m(A_3)$. But now an $S_{1,2,2}$ arises on the vertex set $\{x, m(x), m(a'), a', m(a''), a''\}$. This contradiction completes the proof of the lemma. \square

Now we proceed to a characterization of minimal augmenting graphs in our class that are of bounded vertex degree. To this end, we introduce two families of graphs that generalize paths and cycles. The *duplication* of a vertex v of a graph G results in a graph obtained from G by introducing a new vertex v' with

$$N(v') = N(v).$$

Definition 4 A strip is a graph obtained from a path by repeatedly (zero or more times) duplicating vertices. A bracelet is a graph obtained in the same manner from a cycle.

Lemma 4 There are only finitely many minimal augmenting graphs in X , different from strips and bracelets.

Proof: Let $H = (W, B, E)$ be a minimal augmenting graph in X . Let l denote a fixed (large enough) integer. There are only finitely many connected graphs of bounded vertex degree that are P_l -free. Therefore, we may assume that a longest induced path $P = (v_1, \dots, v_r)$ in H satisfies $r \geq l$. If $H = P$, then H is a strip. For the rest of the proof, assume that H is different from P . Consider any vertex v outside P and which has a neighbor on P .

Recall that by definition, H is bipartite. In particular, H contains no triangles, which implies that v cannot have two consecutive neighbors on P . We will now show that v has at most two neighbors on H . Let $N_P(v) = \{v_{i_1}, \dots, v_{i_p}\}$ with $i_1 < \dots < i_p$.

If $p \geq 4$, then the vertices $\{v, v_{i_1}, v_{i_2}, v_{i_2+1}, v_{i_4}, v_{i_4-1}\}$ induce a copy of $S_{1,2,2}$ in H .

Suppose that $p = 3$. If $i_3 \leq r - 2$, then the vertices $\{v_{i_3}, v_{i_3-1}, v, v_{i_1}, v_{i_3+1}, v_{i_3+2}\}$ induce a copy of $S_{1,2,2}$ in H . It follows that $i_3 \in \{r - 1, r\}$. By symmetry, $i_1 \in \{1, 2\}$. Since r is large enough, we may assume that $i_3 \geq i_2 + 4$. But now, an induced copy of $S_{1,2,2}$ in H arises on the vertices $\{v_{i_2}, v_{i_2-1}, v, v_{i_3}, v_{i_2+1}, v_{i_2+2}\}$.

This shows that v has at most two neighbors on P . If v has two neighbors on P , say v_i and v_j , then either $|i - j| = 2$ or $i = 1$ and $j = r$, since otherwise an induced $S_{1,2,2}$ arises (by similar arguments as above). If v has exactly one neighbor v_i on P then either $i = 2$ or $i = r - 1$, since otherwise either P is not a longest path or H contains an induced $S_{1,2,2}$.

The above discussion enables us to conclude that every vertex of H outside P has a neighbor on P . (If not, then one could find an induced $S_{1,2,2}$ in H with the help of a vertex v as in the above paragraph and a neighbor of v that has no neighbors on P .)

Next, we observe that there must be a vertex outside P with exactly two neighbors in $V(P)$. Assume to the contrary that every vertex outside P has exactly one neighbor in P , either v_2 or v_{r-1} . In particular, both endpoints of P belong to B . (For instance, if $v_1 \in W$, then the fact that H is minimal augmenting implies that v_1 has a neighbor different from v_2 .) Consequently, every vertex outside P is black. However, assuming that $H \neq P$, it follows that $|B| > |B \cap V(P)| = |W \cap V(P)| + 1 = |W| + 1$, contrary to the fact that H is minimal augmenting.

Suppose now that there is a vertex v with $N_P(v) = \{v_1, v_r\}$. Then $V(P) \cup \{v\}$ induce a cycle in H , say C . To see that H must be a bracelet, consider an induced bracelet Q in H which contains C and has as many vertices as possible.

If $H \neq Q$, then Q has a neighbor $u \in V(H) \setminus V(Q)$. Let $x \in N_Q(u)$. For a vertex $z \in V(Q)$, let us denote by $Q(z)$ the subset of vertices of Q with the same neighborhood in Q as z . Without loss of generality we may assume that $x \in V(C)$. Let y_1 and y_2 be the neighbors of x on C , and let further z_1 (resp. z_2) be the neighbor of y_1 (resp. y_2) on C different from x . From our previous observations and from the fact that C contains a longest induced path of H , we conclude that u has no more than two neighbors on C . If u is adjacent to neither of $\{z_1, z_2\}$, then H contains an $S_{1,2,2}$ centered at x , a contradiction. Therefore, $N_C(u) = \{x, z'\}$ with $z' \in \{z_1, z_2\}$. We may assume $z' = z_1$.

Suppose $x' \in Q(x)$ is a non-neighbor of u . Then H contains an $S_{1,2,2}$, centered at z_1 (and containing u, y_1 and x'). Hence u is adjacent to all vertices of $Q(x)$. A symmetric argument shows that u is also adjacent to all vertices of $Q(z_1)$. Also, $N_Q(u) \subseteq Q(x) \cup Q(z_1)$, since otherwise we are in one of the previously considered cases which led to a contradiction. Hence equality holds, and u has the same set of neighbors as y_1 . But now $V(Q) \cup \{u\}$ induces a bracelet Q' with $|V(Q')| > |V(Q)|$, contradicting the choice of Q .

The last remaining case is such that for every longest induced path P of H and for every $v \in N(P)$ with exactly two neighbors on P , the neighbors of v on P are at distance 2. Fix a longest induced path $P = (v_1, \dots, v_r)$ of H . To see that H must be a strip, consider an induced strip Q in H which contains P and has as many vertices as possible. If $H \neq Q$, then Q has a neighbor $u \in V(H) \setminus V(Q)$. Without loss of generality we may assume that $N_Q(u) \supseteq \{v_i, v_{i+2}\}$ for some $i \in \{1, \dots, r-2\}$. Similarly as above, let us denote by $Q(z)$ the subset of vertices of Q each of which has the same neighborhood in Q as z .

Suppose $v' \in Q(v_i)$ is a non-neighbor of u . Then H contains an $S_{1,2,2}$, centered either at v_{i-1} or at v_{i+2} (depending on whether $i \geq 4$ or not). Hence u is adjacent to all vertices of $Q(v_i)$. A symmetric argument shows that u is also adjacent to all vertices of $Q(v_{i+2})$. By the assumption of this case, $N_Q(u) = Q(v_i) \cup Q(v_{i+2})$. Hence u has the same set of neighbors as v_{i+1} . But now $V(Q) \cup \{u\}$ induces a strip Q' with $|V(Q')| > |V(Q)|$, contradicting the choice of Q . The lemma follows. \square

Lemma 4 reduces the problem of finding augmenting graphs in the class under consideration to finding augmenting strips and bracelets. Now we provide a further specification of the structure of augmenting graphs in our class. To this end, let us introduce some more notations and definitions.

Definition 5 *Let us call two vertices in a graph G similar, or twins, if they have the same neighborhood in G .*

Clearly, similarity is an equivalence relation. Note that every equivalence class is an independent set. For a vertex $v \in V(G)$, we denote by C_v the equivalence class containing v .

Definition 6 *Given a graph G and a vertex $v \in V(G)$,*

- *the thickness of v is the cardinality of C_v ;*

- the thickness of G is the maximum thickness of any vertex of G .

The following lemma specifies the structure of minimal augmenting strips and bracelets in our class in terms of their thickness.

Lemma 5 *If $H = (W, B, E)$ is a minimal augmenting strip or bracelet in X , then H is either*

- a strip of thickness at most 2, or
- a bracelet obtained from an even cycle by the duplication of exactly one vertex.

Proof: If $H = K_{2,3}$, the lemma is true. Assume now that $H \neq K_{2,3}$.

Suppose that the thickness of H is at least 3. By Lemma 1, no set of pairwise similar white vertices A can have cardinality more than 2 (else $A \cup N(A)$ would contain a $K_{3,4}$). Therefore, there is a set of pairwise similar black vertices B' such that $|B'| \geq 3$. By the $K_{3,3}$ -minor-freeness and connectedness, we have $1 \leq |N(B')| \leq 2$. Denote $A = W \setminus N(B')$. If A is empty, then $H = K_{1,3}$, contradicting the minimality of H . Therefore, A is nonempty and satisfies $|A| \geq |W| - 2$ and $|N(A)| \leq |B| - 3$. Together with Lemma 1, this implies $|A| \geq |N(A)|$, contradicting the minimality of H again. Thus, we conclude that thickness of H is at most two, which proves the lemma in case when H is a strip.

Assume now that H is a bracelet. Since no cycles are augmenting, H must contain a vertex x of thickness 2. Since H is planar, no neighbor of x has thickness 2 or more (or H would contain a subdivision of $K_{3,3}$). Therefore, x has exactly 2 neighbors, both of thickness 1. Next, observe that x must be black, since otherwise $A := C_x$ would violate the inequality $|N(A)| > |A|$. Hence, all white vertices have thickness 1, and since $|B| = |W| + 1$, there can only be one black vertex of thickness more than 1. The lemma follows. \square

Our next step is to show that some of the augmenting graphs revealed in the above lemma can be neglected, as they contain redundant sets. Again, we start with definitions.

Definition 7 *In a strip H ,*

- an endpoint is a vertex that belongs to a longest induced path P in H and has degree 1 in P ;
- a pair of twins $\{u, u'\}$ is said to be inner if u and u' are at distance at least 4 from every endpoint of H .

In the following lemma, an *augmenting chain* is an augmenting graph isomorphic to a path.

Lemma 6 *Let $H \in X$ be a minimal augmenting strip or bracelet with $|V(H)| \geq 19$. Then either H is a strip containing an inner pair of twins, or H contains a redundant set $U \subseteq V(H)$ of size at most 18 such that $H - U$ is an augmenting chain.*

Proof: If H is a bracelet, then it follows from Lemma 5 that H contains a redundant set $U \subseteq V(H)$ of size 4 such that $H - U$ is an augmenting chain.

Now let H be a strip and let $P = (v_1, \dots, v_l)$ be a longest induced path in H . Also, let a_i denote the thickness of v_i , for $i \in \{1, \dots, l\}$. By Lemma 5, $a_i \leq 2$ for any i . Thus, if $l \leq 9$ then $|V(H)| \leq 18$, and therefore, in what follows we assume that $l \geq 10$.

If $a_i = 2$ for some $i \in \{5, \dots, l - 4\}$, then H contains an inner pair of twins. Now assume $a_i = 1$ for $5 \leq i \leq l - 4$. Denote by $x = v_i$ the black vertex satisfying $i = \min\{i' : 1 \leq i' \leq 6, a_{i'} = a_{i'+1} = \dots = a_6 = 1\}$. Note that such a vertex exists since $l \geq 10$. Symmetrically, let $y = v_j$ denote the black vertex satisfying $j = \max\{j' : l - 5 \leq j' \leq l : a_{l-5} = a_{l-4} = \dots = a_{j'} = 1\}$. Also, denote by H' the path connecting x to y in H , and by U the remaining vertices of H . It is not difficult to see that U is a redundant set of size at most 18 and $H - U$ is an augmenting chain. \square

In [10], a polynomial-time algorithm was developed for finding augmenting chains in $S_{1,2,3}$ -free graphs. Since every $S_{1,2,2}$ -free graph is also $S_{1,2,3}$ -free, we conclude, using Lemma 6 above and the algorithm from [10], that the IS problem in $S_{1,2,2}$ -free planar graphs can be completely solved by augmentation, *unless the input graph contains a minimal augmenting strip with an inner pair of twins.*

Luckily, it turns out that even in this case, we can still reduce the problem to augmentation by a double transformation of the input graph G . First, we shrink every class C of similar vertices in G to a single vertex and assign to this vertex the weight equal to $|C|$, obtaining in this way a weighted graph G' . Obviously, a maximum independent set in G corresponds to a maximum weight independent set in G' and vice versa. To solve the problem for G' , we first decompose it into 2-connected components, and then for each 2-connected component of G' we implement a reverse transformation by expanding every vertex with weight ω to a class of similar vertices of cardinality ω . It will be shown later that every 2-connected graph transforms in this way into an unweighted graph without strips with inner twins.

We now describe these transformations in detail. For the input graph G , we denote by \mathcal{C} the set of all similarity classes, i.e., classes of vertices with the same neighborhood. For each similarity class $C \in \mathcal{C}$, we fix an arbitrary member of C and denote it by v_C .

Transformation 1 (From unweighted to weighted) $\phi_1 : \bar{G} \mapsto (\hat{G}, \hat{\omega})$

INPUT: An induced subgraph \bar{G} of G .

OUTPUT: The ordered pair $(\hat{G}, \hat{\omega})$, where:

\hat{G} is the subgraph of G , induced by the set $\{v_C : C \in \mathcal{C}, C \cap V(\bar{G}) \neq \emptyset\}$, and $\hat{\omega}$ is the collection of vertex weights of \hat{G} , given by $\hat{\omega}(v_C) = |C \cap V(\bar{G})|$ for all $v_C \in V(\hat{G})$.

Transformation 2 (From weighted to unweighted) $\phi_2 : (\hat{G}, \hat{\omega}) \mapsto \bar{G}$

INPUT: An ordered pair $(\hat{G}, \hat{\omega})$, where:

\hat{G} is an induced subgraph of G of the form $\hat{G} = G[\{v_C : C \in \mathcal{C}'\}]$ for some nonempty subset of equivalence classes $\mathcal{C}' \subseteq \mathcal{C}$, and

$\hat{\omega}$ is a collection of integer vertex weights of \hat{G} satisfying $1 \leq \hat{\omega}(v_C) \leq |C|$ for all $C \in \mathcal{C}'$.

OUTPUT: The subgraph \bar{G} of G , induced by the vertex set $F = \bigcup_{C \in \mathcal{C}'} F_C$ where, for each $C \in \mathcal{C}'$, F_C is an arbitrary subset of C of cardinality $\hat{\omega}(v_C)$.¹

It is easy to see that these two transformations are inverse to each other. The importance of these transformations for our solution is due to the following result.

Lemma 7 *Let \bar{G} be an induced subgraph of G that contains a minimal augmenting strip with inner twins, and let $(\hat{G}, \hat{\omega}) = \phi_1(\bar{G})$. Then \hat{G} contains a cut-vertex.*

Proof: Let $H = (W, B; E)$ be a minimal augmenting strip with an inner twin $\{u, u'\}$ in \bar{G} . By definition, H is an induced subgraph of \bar{G} and hence of G .

First, we notice that u has a neighbor of thickness 2 in H . If not, then, according to Lemma 1, we conclude that $u, u' \in B$. Deleting the vertices $\{u, u'\}$ from H results in two disjoint strips, say $H_i = (W_i, B_i, E_i)$ for $i = 1, 2$. Since $\{u, u'\}$ is an inner pair of twins of H , the sets $A_i := W_i \setminus N(u)$ (for $i = 1, 2$) are nonempty. But now, it follows from Lemma 1 that

$$\begin{aligned} |B| &= |N(A_1)| + |N(A_2)| + 2 \\ &\geq (|A_1| + 1) + (|A_2| + 1) + 2 \\ &= |W_1| + |W_2| + 2 \\ &= |W| + 2 = |B| + 1, \end{aligned}$$

a contradiction.

Therefore, there is a pair of twins $\{v, v'\}$ in H such that $uv, uv', u'v, u'v' \in E(G)$. Consider the 4-cycle C induced by the vertices $\{v, v', u, u'\}$. We claim that C is a separating set of G . Indeed, since $\{u, u'\}$ is an inner pair of twins in H , we may consider two vertices $x \in V(H) \cap (N(u) \setminus \{v, v'\})$ and $y \in V(H) \cap (N(v) \setminus \{u, u'\})$. Then, C separates x from y : if x and y belonged to the same connected component of $G - C$, the graph G would contain a subdivision of $K_{3,3}$, contradicting the planarity assumption.

Next, we show that $\{u, u'\}$ is a pair of twins in G as well. Assume there is a vertex $a \in N(u) \setminus N(u')$. Let C_x, C_y denote the connected components of $G - C$ containing x and y , respectively, and let x' and x'' denote vertices in $V(H) \cap V(C_x)$ at distance 1 and 2 from x , respectively. Similarly we define y', y'' . To avoid an induced $S_{1,2,2}$ on $\{x, u', u, a, x', x''\}$, we see that a has a neighbor in $\{x, x', x''\}$. Therefore, $a \in C_x$. Next, we observe that a is adjacent to v , since otherwise an $S_{1,2,2}$ arises on the vertex set $\{v, u', u, a, y, y'\}$. By symmetry, we conclude that a is adjacent to v' . However, this leads to a contradictory

¹For definiteness, we do the following for each equivalence class $C \in \mathcal{C}$: we fix, once and for all, a numbering of vertices of C , and then put into F_C the first $\hat{\omega}(v_C)$ vertices of C .

$K_{3,3}$ -minor contained in the vertex set $\{a, u, u', v, v', x, x', x''\}$. A symmetric argument shows that $\{v, v'\}$ is also a pair of twins in G .

Now, we show that $\{u, u'\}$ separates x from v in G . Assume that there is a path $P = (v_1, \dots, v_r)$ in $G - \{u, u'\}$ from x to v (with $v_1 = x$ and $v_r = v$). Then $r \geq 3$ and since $\{v, v'\}$ is a pair of twins in G , v_{r-1} is adjacent to v' too. But now, a subdivision of $K_{3,3}$ arises on $V(P) \cup V(C)$, a contradiction.

Therefore, $\{u, u'\}$ is a pair of twins in G that separates a pair of vertices of H with different neighborhoods in G . As can be seen from the above proof, $\{u, u'\}$ separates x from v in \bar{G} as well. Since x and v belong to different equivalence classes of \mathcal{C} , the vertex v_{C_u} separates v_{C_x} from v_{C_v} in \hat{G} . Thus, v_{C_u} is a cut-vertex of \hat{G} and the proof is complete. \square

Corollary 2 *Let $(G', \omega) = \phi_1(G)$ and let $(\hat{G}, \hat{\omega})$ be an input to ϕ_2 such that \hat{G} is contained in a 2-connected component of G' . Then $\bar{G} = \phi_2(\hat{G}, \hat{\omega})$ is an induced subgraph of G that contains no minimal augmenting strips with inner twins.*

We are now ready to present the procedure that finds a maximum independent set in a graph $G \in X$.

Procedure ALPHA

Input: An $S_{1,2,2}$ -free planar graph $G = (V, E)$.

Output: A independent set I of G of maximum cardinality.

Step 0. (Preprocessing) Determine the connected components C^1, \dots, C^r of G . If $r > 1$, return $I = \cup_{i=1}^r \text{ALPHA}(C^i)$ and halt. Else, compute the equivalence classes $\mathcal{C} = \{C_v : v \in V\}$.

Step 1. Compute $(G', \omega) = \phi_1(G)$.

Step 2. Compute a maximum-weight independent set I' of G' . To this end, first reduce the problem to the 2-connected components of G' . To compute a maximum-weight independent set of a 2-connected component \hat{G} , with vertex weights $\hat{\omega}$, perform the following steps:

2.0. Remove the vertices of \hat{G} with non-positive weights. (Note that the reduction to 2-connected components is performed via the decomposition by clique separators [22]. During each step of this recursive procedure, some vertex weights are redefined, and they can become non-positive.)

2.1. Compute $\bar{G} = \phi_2(\hat{G}, \hat{\omega})$.

2.2. Compute a maximum independent set \bar{I} of \bar{G} (by augmentation).

2.3. Compute $\hat{I} = \{v_C : C \in \mathcal{C}, C \cap \bar{I} \neq \emptyset\}$, a maximum-weight independent set of \hat{G} .

Step 3. Return $I := \cup_{v \in I'} C_v$, a maximum independent set in G , and halt.

Using this algorithm, we can derive the main result of this section.

Theorem 4 *The MAXIMUM INDEPENDENT SET problem is polynomially solvable for $S_{1,2,2}$ -free planar graphs.*

Together with Corollary 1 this proves Theorem 3, i.e., polynomial-time solvability of the problem in the class of $S_{1,2,k}$ -free planar graphs, for any particular value of k .

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